Maths 260 Lecture 17

Topics for today
Linear independence of vectors and solutions
The general solution

Reading for this lecture
BDH Section 3.1

Suggested exercises
BDH Section 3.1, 31, 33, 34, 35

Reading for next lecture
BDH Section 3.2

Section 2.5 Linear Systems (continued)

Linear Independence of Vectors: Two vectors in the plane are **linearly independent** if neither vector is a multiple of the other, i.e., if they do not both lie on the same line through the origin.

e.g. $v_1 = (1, 1)$, $v_2 = (2, -1)$ are linearly independent.

e.g. $v_1 = (1, 1)$ and $v_3 = (-2, -2)$ are linearly dependent.

Important result: If two vectors (x_1, y_1) and (x_2, y_2) are linearly independent vectors in the plane, then for any other vector (x_0, y_0) there exists k_1, k_2 such that

$$k_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Important consequence of this result: Consider the DE

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where $\mathbf{A}$ is a 2×2 matrix. If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions to the system with $\mathbf{Y}_1(0)$ and $\mathbf{Y}_2(0)$ being linearly independent vectors, then for any initial condition

$$\mathbf{Y}(0) = (x_0, y_0)$$

we can find constants k_1 and k_2 so that $k_1\mathbf{Y}(t) + k_2\mathbf{Y}(t)$ is the solution to IVP

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

that is, every solution can be expressed as a linear combination of $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.

Note: If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions with $\mathbf{Y}_1(0)$ and $\mathbf{Y}_2(0)$ linearly independent, then $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent vectors for all t. In this case we say that $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent solutions.

Summary: For the system

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}$$

if we can find two linearly independent solutions we can write down the general solution and hence solve any IVP arising from this DE.

These results can be generalized to higher dimensions:

A set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$ is **linearly dependent** if there are constants $c_1, c_2, \ldots, c_m$ (not all zero) such that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_m\mathbf{v_m} = \mathbf{0} \tag{1}$$

If all the c_i are zero whenever equation (**??**) is satisfied, the set of vectors is linearly independent.

Checking linear independence of solution vectors in higher dimensions

(This method works in two dimensions also, but the earlier method is quicker in this case.)

If $\mathbf{Y}_1(t), \mathbf{Y}_2(t), ..., \mathbf{Y}_m(t)$ are solutions for the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where $\mathbf{A}$ is an $m \times m$ matrix, then the set of solution vectors $\{\mathbf{Y_1}, ..., \mathbf{Y_m}\}$ is linearly independent if and only if

$$W(\mathbf{Y}_1, ..., \mathbf{Y}_m)(t) = \det(\mathbf{Y}_1 \ ... \ \mathbf{Y}_m) \neq 0$$

for all t.

Notes:

1. This test (the **Wronskian** test) does not work if the vectors are not all solution vectors for the same system.
2. It turns out that the Wronskian is either identically zero (i.e., $W(t) = 0$ for all t) or $W(t) \neq 0$ for any t. Therefore, only need to calculate the Wronskian at one value of t, say $t = 0$.
3. We need m solution vectors to do the test.

Example: The vectors

$$\begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} e^{5t}, \quad \begin{pmatrix} 2t + \frac{1}{2} \\ -\frac{1}{2} \\ -t - 1 \end{pmatrix} e^{5t}$$

are all solution vectors to some system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}.$$

Are they linearly independent?

Example: Show that the vectors

$$e^{2t}\begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}, \quad e^{-4t}\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \quad e^{-2t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are linearly independent solutions to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & -2 \end{pmatrix}.$$

Hence find the solution to the IVP

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

Main result: If $\mathbf{Y}_1(t)$, $\mathbf{Y}_2(t)$,..., $\mathbf{Y}_m(t)$, are linearly independent solution vectors to the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

where $\mathbf{A}$ is an $m \times m$ matrix, then the general solution to the system is

$$\mathbf{Y}(t) = c_1\mathbf{Y}_1(t) + c_2\mathbf{Y}_2(t) + \cdots + c_m\mathbf{Y}_m(t)$$

where $c_1,\ c_2,\ \ldots, c_m$ are arbitrary constants, that is, every solution to the system can be written in this form by appropriate choice of c_1, c_2,..., c_m.