Maths 260 Lecture 16

Topic for today
Solutions to some special systems
Linear systems - some properties

Reading for this lecture
BDH Section 2.3, pp 175–178 (1st ed) 185–188 (2nd ed); Section 3.1

Suggested exercises
BDH Section 2.3: 5, 7, 9; Section 3.1: 5, 7, 9, 24, 27, 29

Reading for next lecture
BDH Section 3.1 (again)

Section 2.4 Analytic methods for some special systems

Some very special systems of DEs **decouple**, i.e., the rate of change of one or more of the dependent variables depends only on its own value.

Example 1

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= -2y \end{aligned}$$

Example 2

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= 2x - y \end{aligned}$$

Sometimes can find analytic solutions to systems that decouple.

Example 1 again: Wish to find and plot solutions to:

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= -2y \end{aligned}$$

Can solve each equation separately. Find that $(x(t), y(t)) = (c_1 e^t, c_2 e^{-2t})$ is a solution for all choices of c_1 and c_2. The values of c_1 and c_2 are determined by the initial conditions: $c_1 = x(0)$ and $c_2 = y(0)$.

Plotting solutions:

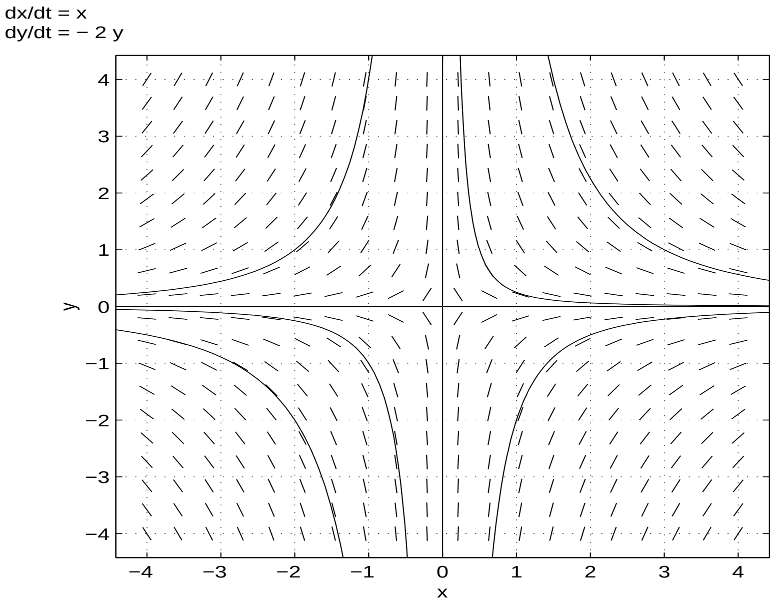
e.g., $x(0) = 1$, $y(0) = 1$,

Plotting in phase space:

Plotting some other solutions

$x(0)$	$y(0)$	c_1	c_2	$x(t)$	$y(t)$
1	-1				
-2	1				
0	1				
1	0				

Compare phase plane plot above with numerical solutions from Matlab:



Example 2 again:

Wish to find and plot solutions to:

$$\begin{aligned}\dot{x} &= x, \\ \dot{y} &= 2x - y\end{aligned}$$

Solve first equation to get $x = c_1 e^t$. Then substitute this expression for x into second equation to get

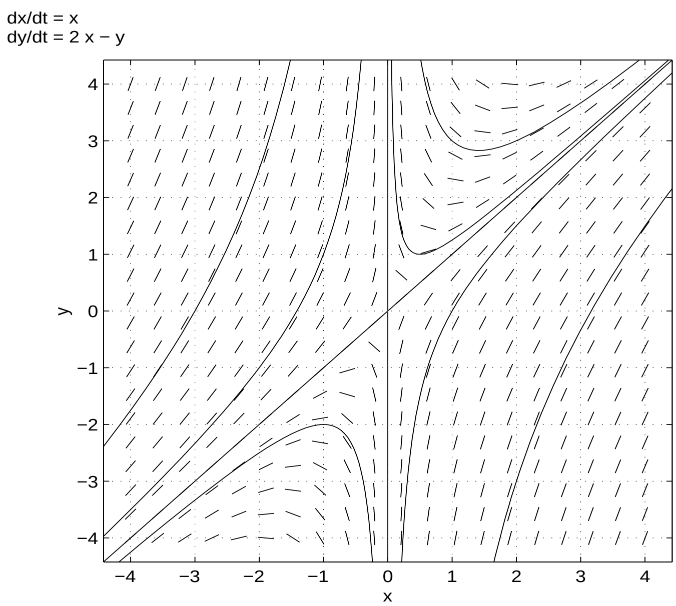
$$\dot{y} = 2c_1 e^t - y$$

So $(x(t), y(t)) = (c_1 e^t, c_1 e^t + c_2 e^{-t})$ is a solution for all choices of c_1 and c_2.

Plotting solutions in phase space

$x(0)$	$y(0)$	c_1	c_2	$x(t)$	$y(t)$
1	1				
0	1				
1	0				
0.25	1				

Compare phase plane plot above with numerical solutions from Matlab:



In both examples, system could be solved. Found that some solutions gave straight-line solution curves in phase portrait but (from Matlab) most solution curves not straight lines.

Section 2.5 Linear Systems

Linear systems are an important class of systems of DEs, partly because some important models are linear but also because we can use linear systems to help understand nonlinear systems.

A **linear system** is a system of DEs where the dependent variables only appear to the first power. Mostly interested in systems that can be written as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where $\mathbf{Y}$ is a vector:

$$\mathbf{Y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

and $\mathbf{A}$ is a matrix of constants:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix}$$

Example

$$\begin{aligned} \frac{dx}{dt} &= 2x - z \\ \frac{dy}{dt} &= -x - z \\ \frac{dz}{dt} &= x + y \end{aligned}$$

Can write this system as

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where

$$\mathbf{Y} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

$\mathbf{A}$ is called the coefficient matrix. The number of dependent variables is called the dimension of the system.

Some properties of linear systems

1. **Equilibrium solutions**

Want to find equilibrium solutions of

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

i.e., find $\mathbf{Y_0}$ such that $\mathbf{AY_0} = \mathbf{0}$. From linear algebra, know that if $\det \mathbf{A} \neq 0$, then the only solution of $\mathbf{AY_0} = \mathbf{0}$ is $\mathbf{Y_0} = \mathbf{0}$ (called the trivial solution). Thus, if $\det \mathbf{A}$ $\neq 0$, then $\mathbf{Y}(t) = \mathbf{0}$ is the only equilibrium solution to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}.$$

2. **Linearity Principle**

If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are both solutions to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

then so is $k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$ for any constants k_1 and k_2. The function $k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t)$ is called a **linear combination** of $\mathbf{Y_1}$ and $\mathbf{Y_2}$.

Example 2 again

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= 2x - y \end{aligned}$$

Found earlier that

$$\mathbf{Y}_1(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}, \quad \mathbf{Y}_2(t) = \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}$$

are (straightline) solutions of this system and that all solutions can be written as

$$\mathbf{Y}(t) = \begin{pmatrix} c_1 e^t \\ c_1 e^t + c_2 e^{-t} \end{pmatrix} = c_1\mathbf{Y}_1 + c_2\mathbf{Y_2},$$

i.e., as a linear combination of $\mathbf{Y}_1$ and $\mathbf{Y}_2$.

Important ideas for today
Solutions for systems that decouple
Straight line solutions
Linear systems - linearity principle