Maths 260 Lecture 15

Topic for today
Numerical methods for systems
Existence and Uniqueness Theorem for systems

Reading for this lecture
BDH Section 2.4

Suggested exercises
BDH Section 2.4: 7, 8, 9, 10

Reading for next lecture
BDH Section 2.3, pp 175–178 (1st ed) 185–188 (2nd ed); Section 3.1

Today's handout
Lecture 15 notes
Tutorial 5 questions

Section 2.3 Numerical Methods for Systems

Numerical methods used for first order equations can be generalized to systems of first order equations.

Example: Euler's Method for systems

Given the IVP

$$\begin{aligned} \frac{dx}{dt} &= f(t,x,y), \\ \frac{dy}{dt} &= g(t,x,y), \end{aligned}$$

with $x(t_0) = x_0, y(t_0) = y_0$, then Euler's Method calculates the approximate solution at $t_1 = t_0 + h$ to be

$$\begin{aligned} x(t_0+h) &\approx x_0 + hf(t_0,x_0,y_0), \\ y(t_0+h) &\approx y_0 + hg(t_0,x_0,y_0) \end{aligned}$$

Can repeat to find approximation after n steps.

Example: Use Euler's method with $h = 0.1$ to calculate an approximate solution at $t = 0.2$ to the IVP

$$\begin{aligned} \frac{dx}{dt} &= t + y \\ \frac{dy}{dt} &= y^2 - x \end{aligned}$$

when $x(0) = 1, y(0) = 0$.

Vector Form of Euler's Method

Let

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{F}(t, \mathbf{X}) = \begin{pmatrix} f_1(t, x_1, x_2, \ldots, x_n) \\ f_2(t, x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(t, x_1, x_2, \ldots, x_n) \end{pmatrix}, \quad \mathbf{X}_0 = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix}.$$

Then the Euler approximation to the solution of the IVP

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(t, \mathbf{X}), \quad \mathbf{X}(t_0) = \mathbf{X}_0$$

at $t_0 + h$ is

$$\mathbf{X}(t_0 + h) \approx \mathbf{X}_0 + h\mathbf{F}(t_0, \mathbf{X}_0)$$

It can be proved that Euler's method for systems is first order, i.e., the error in the ith component of $\mathbf{X}$ is

$$|E_i(h)| \approx k_i h$$

in the limit of small h, where k_i is a constant. Thus, halving stepsize will approximately halve the error in the estimated value of each component in $\mathbf{X}$.
Improved Euler and 4th order Runge-Kutta methods also generalise to systems and are order 2 and 4 respectively.

Existence and Uniqueness Theorem for systems

Consider the IVP

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(t, \mathbf{Y}), \quad \mathbf{Y}(t_0) = \mathbf{Y}_0.$$

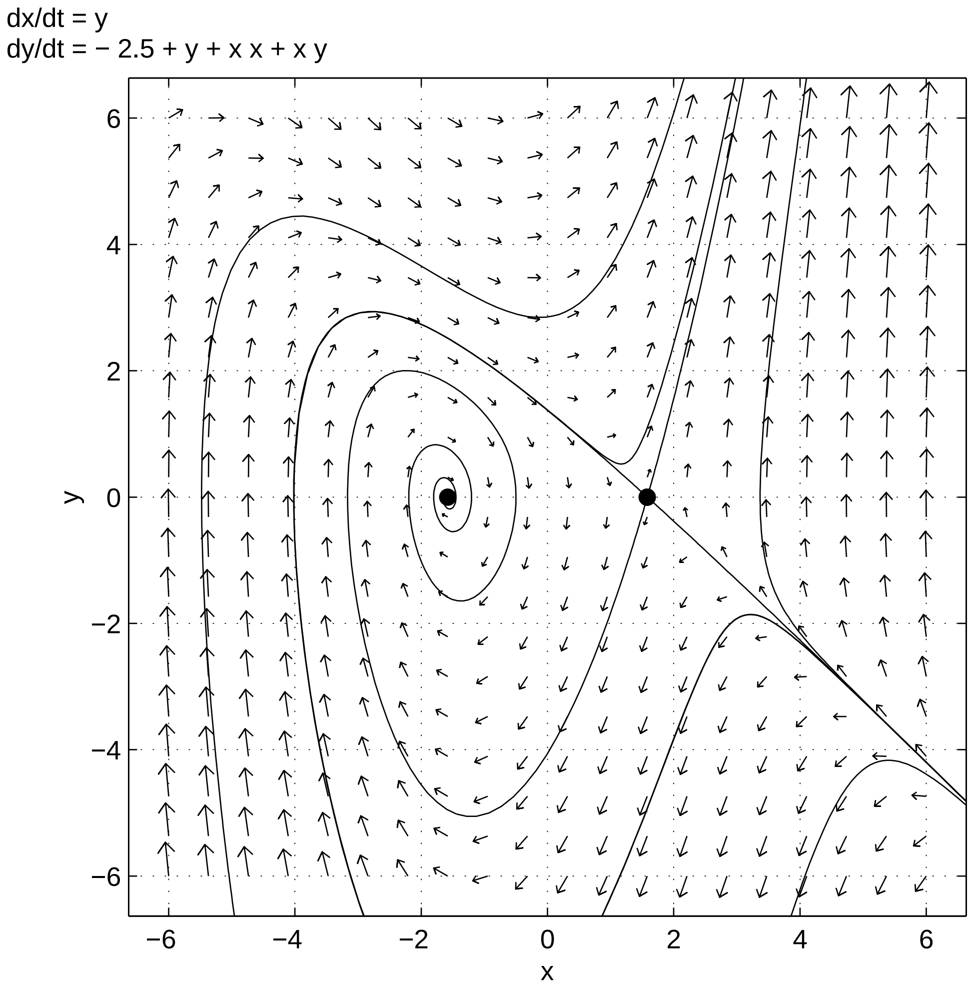
If $\mathbf{F}$ is continuous and has continuous first partial derivatives then there is an $\epsilon > 0$ and a function $\mathbf{Y}(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ such that $\mathbf{Y}(t)$ is a solution to the IVP. For t in this interval, the solution is unique.

Interpretation of EU Theorem: If a system of equations is 'nice' enough, a solution to an IVP exists and is unique. In particular, two different solutions cannot start at the same t at the same point in phase space.

For autonomous systems, two different solutions that start at the same place in phase space but at different times will correspond to the same solution curve, i.e., solution curves cannot meet or cross in phase space.

Example: The phase portrait for the following differential equation is given below. It looks as though different solution curves meet/cross but EU Theorem ensures they do not. No such guarantee exists for solution curves of non-autonomous systems; solution curves for non-autonomous systems frequently cross in phase space.

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -2.5 + y + x^2 + xy\end{aligned}$$



Important ideas from today

Numerical methods work for systems of DEs in a similar way as for single equations.

'Nice' IVPs have unique solutions.

Solution curves for autonomous systems do not cross or meet in phase space.