Maths 260 Lecture 14

Topics for today
Direction fields and solutions
Equilibrium solutions
Using the tool *pplane* from Matlab

Reading for this lecture
BDH Section 2.2

Suggested exercises
BDH Section 2.2: 1st ed. 1, 3, 13, 17-20, 21-24, 29
2nd ed. 1, 3, 11, 13-16, 19, 27

Reading for next lecture
BDH Section 2.4

Today's handout
Lecture 14 notes

Section 2.2 Direction Fields

Directions fields are the analogue for systems of equations of slope fields.

Consider a system of two autonomous DEs:

$$\begin{aligned}\frac{dx}{dt} &= f(x, y)\\ \frac{dy}{dt} &= g(x, y)\end{aligned}$$

Write

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

and

$$\mathbf{V}(\mathbf{Y}) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$$

Then the system written in vector form is

$$\frac{d\mathbf{Y}}{dt} = \mathbf{V}(\mathbf{Y})$$

$\mathbf{V}(\mathbf{Y})$ is known as a **vector field** i.e., it is a function that assigns a vector to each point of the x, y-plane.

Example

$$\begin{aligned}\frac{dx}{dt} &= 0.5x - 0.4xy \\ \frac{dy}{dt} &= -y + 0.2xy\end{aligned}$$

Sometimes write

$$\mathbf{Y} = (x, y)$$

and

$$\mathbf{V} = (f(x, y), g(x, y))$$

Plotting vector fields:

At point $\mathbf{Y_0} = (x_0, y_0)$ in the x, y-plane draw the vector $\mathbf{V}(\mathbf{Y_0})$ with the base of vector at $\mathbf{Y_0}$ and with arrow showing direction of vector.

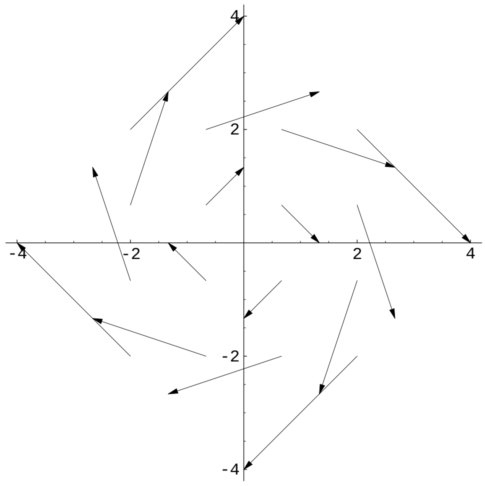
Example:

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x\end{aligned}$$

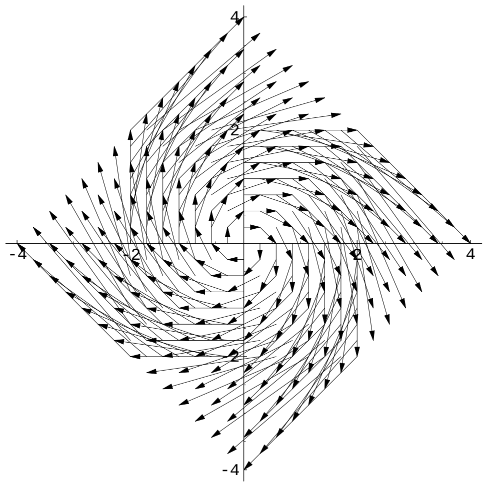
Problem with plotting vector fields: vectors can cross, which makes a big mess. For example, for the system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x \end{aligned}$$

Vector field with selected vectors:



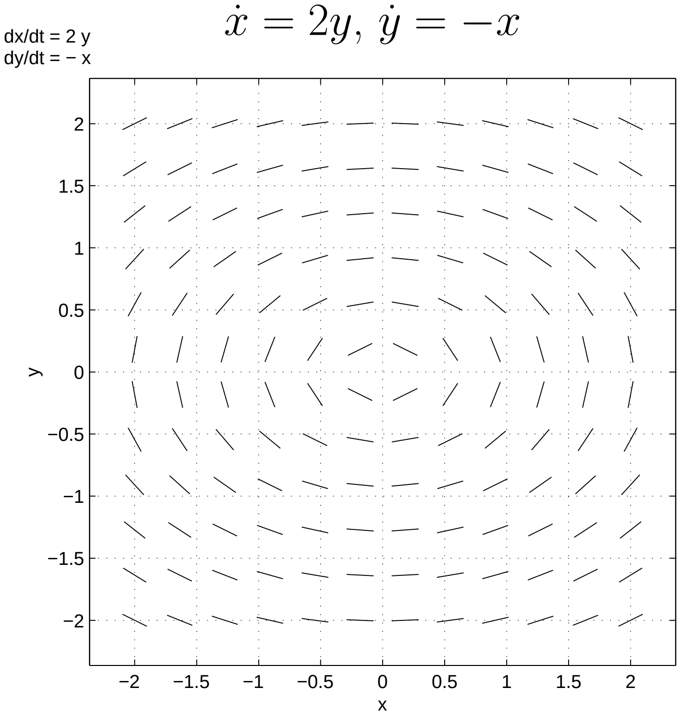
Vector field with more vectors:



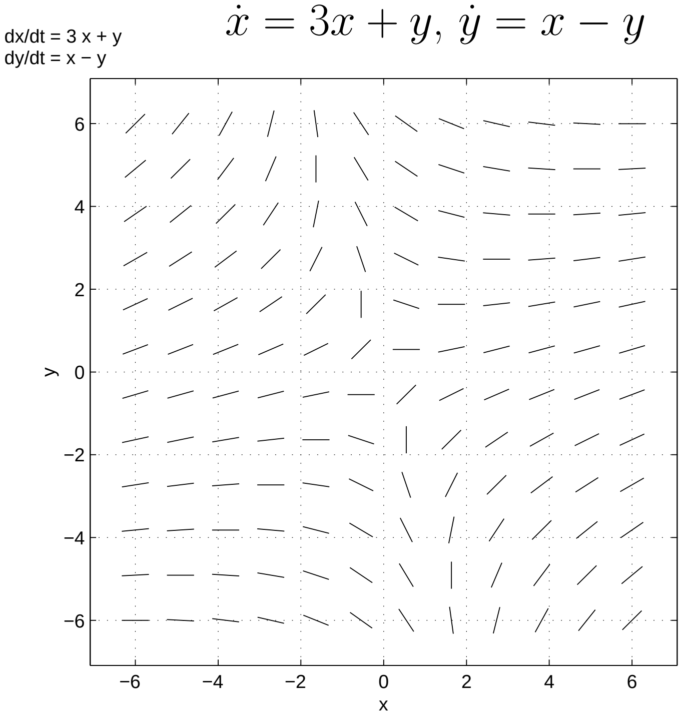
Avoid these problems by plotting **direction field**, i.e., vectors with same direction as in vector field but scaled to a uniform length. Arrowheads may or may not be shown.

The following pictures show the direction fields for some systems.

$\dot{x} = 2y,\ \dot{y} = -x$



$\dot{x} = 3x + y,\ \dot{y} = x - y$



We can use *pplane* from *Matlab* to plot direction fields. See the lab handout for details on using *pplane*.

Sketching solutions to systems:

Consider a system of DEs

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}), \quad \mathbf{Y} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{1}$$

A solution is a vector of functions $\mathbf{Y}(t)$ and corresponds to a curve in phase space, parametrised by time (i.e., vary t to move along curve).

The vector

$$\left.\frac{d\mathbf{Y}}{dt}\right|_{t=t_0}$$

is tangent to curve of $\mathbf{Y}(t)$ at $t = t_0$. Thus, equation (1) says that vectors in direction field are tangent to solutions of DE.

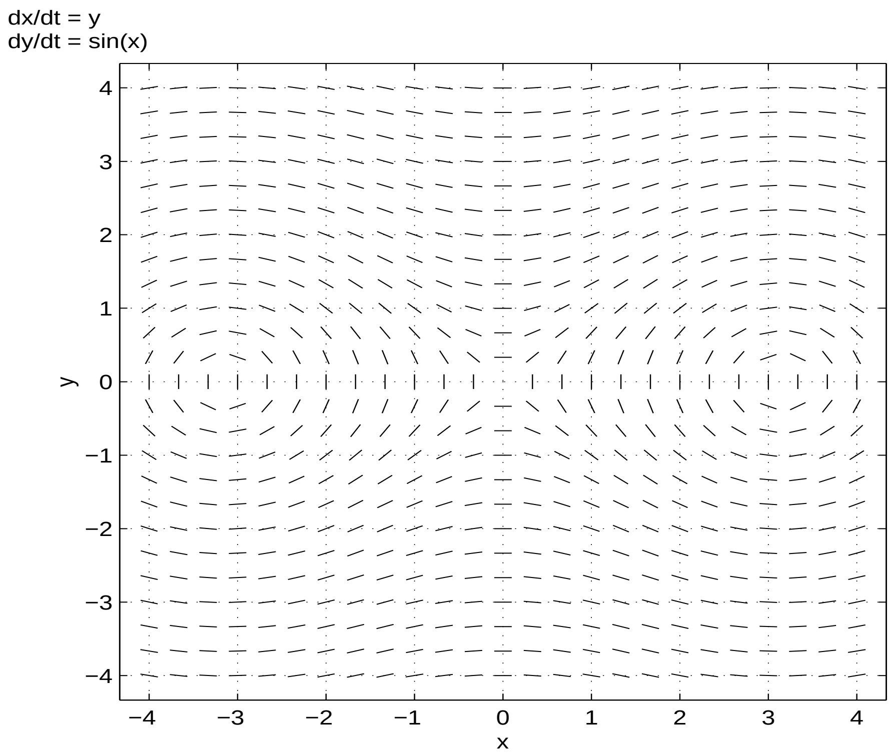
So to sketch solution curves to DE (1),

1. plot direction field, then
2. starting at some initial point, sketch a smooth curve that follows vectors in direction field.

Example: Sketch some representative solutions for the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \sin(x) \end{aligned}$$

The direction field is given below.



Equilibrium solutions

The point $\mathbf{Y}_0$ is an *equilibrium point* for the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{V}(\mathbf{Y})$$

if $\mathbf{V}(\mathbf{Y}_0) = 0$. If $\mathbf{Y}_0$ is an equilibrium point, then the constant function $\mathbf{Y}(t) = \mathbf{Y}_0$ is a solution of the system.

Example 1

$$\begin{aligned} \dot{x} &= 2x + y \\ \dot{y} &= 2y + x \end{aligned}$$

Example 2

$$\begin{aligned} \dot{x} &= x + y \\ \dot{y} &= y(2 - x) \end{aligned}$$

Behaviour of solutions near equilibria can be observed with *pplane*. Note that

1. direction of vectors in direction field changes dramatically near an equilibirum point, and

2. solutions passing near an equilibrium go very slowly (because all components of vector field $\rightarrow 0$ near an equilibrium).