Maths 260 Part 2: First order systems of differential equations

Maths 260 Lecture 13

Topic for today
Introduction to systems of differential equations

Reading for this lecture
BDH Section 2.1

Suggested exercises
BDH Section 2.1: 1–4, 9, 10

Reading for next lecture
BDH Section 2.2

Today's handout
Lecture 13 notes

Chapter 2: First Order systems of differential equations

Section 2.1: Introduction

DEs that contain more than one dependent variable are known as systems of DEs.

Examples:

1.

$$\frac{dx}{dt} = -2x + 3y$$
$$\frac{dy}{dt} = -2y$$

2.

$$\frac{dx}{dt} = 10(y - x)$$
$$\frac{dy}{dt} = 28x - y - xz$$
$$\frac{dz}{dt} = -\frac{8}{3}z + xy$$

3.

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = x - x^3 - y + \mu\cos(t)$$

The purpose of today's lecture is to introduce some important ideas for the study of systems of DEs, but with formal definitions and other details mostly left to later lectures.

Mostly interested in systems of first order DEs. Write these in standard form:

$$\begin{aligned}
\frac{dx_1}{dt} &= f_1(t, x_1, x_2, \ldots, x_n) \\
\frac{dx_2}{dt} &= f_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}$$

The notation

$$\frac{dx_1}{dt} = \dot{x_1}, \quad \frac{dx_2}{dt} = \dot{x_2}$$

is often used.

Solutions to systems of DEs

A **solution** to a system of n first order equations is a set of n functions that satisfy the differential equations.

Example: Determine which of the following pairs of functions is a solution to the system

$$\begin{aligned}
\frac{dx}{dt} &= -2x + 3y, \\
\frac{dy}{dt} &= -2y.
\end{aligned}$$

1. $x(t) = -3te^{-2t}, \quad y(t) = -e^{-2t}$;
2. $x(t) = 3e^{-2t}, \quad y(t) = 0$;
3. $x(t) = 3e^{-2t} + te^{-2t}, \quad y(t) = -e^{-2t}$.

Example: Model of two populations (predator/prey)

Let $R(t)$ = no. of prey ("Rabbits") in 1000's, let $F(t)$ = no. of predators ("Foxes") in 1000's. A possible model of change in the two populations is given by

$$\dot{R} = 0.4R - 0.1RF, \tag{1}$$
$$\dot{F} = -0.5F + 0.1RF, \tag{2}$$

with $R \geq 0$, $F \geq 0$.

Physical significance of terms on rhs

- The term $0.4R$ in (1) gives unlimited growth of prey if no predators exist.
- The term $-0.5F$ in (2) gives exponential decay in predator population if no prey exist.
- The term $-0.1RF$ in (1) models the negative effect on prey population of 'interactions' between prey and predators (i.e., predators eat prey and prey population decreases).
- The term $0.1RF$ in (2) models the positive effect on predator population of interactions between prey and predators (i.e., predators eat prey and predator population increases).

(... as long as prey have plenty to eat themselves: the model is limited in this respect!)

Solutions to the predator/prey system - Some special cases:

1. The pair of constant functions $R(t) = 0$, $F(t) = 0$ is an **equilibrium** solution.

2. Rewriting system as:

$$\begin{aligned}\dot{R} &= R\,(0.4 - 0.1F)\,,\\ \dot{F} &= F\,(0.1R - 0.5)\,,\end{aligned}$$

we see that $(R(t), F(t)) = (5, 4)$ is an equilibrium solution.

Physically, this tells us that a prey population of 5000 and a predator population of 4000 is perfectly balanced; neither population increases or decreases over time.

3. If $F(t) = 0$, then $\dot{F} = 0$ for all time, regardless of behaviour of R. However, if $\dot{F} = 0$, then $\dot{R} = 0.4R$ so $R(t) = R_0 e^{0.4t}$ is a solution, i.e., if there are no predators, the prey population grows exponentially.

4. Similarly, if $R(t) = 0$, then $\dot{R} = 0$ for all time, regardless of behaviour of F. However, if $\dot{R} = 0$, then $\dot{F} = -0.5F$ so $F(t) = F_0 e^{-0.5F}$ is a solution, i.e. if there are no prey, the predator population decreases exponentially.

Apart from special cases, don't (yet) have analytic or qualitative methods to investigate solutions to this DE. Continue to study this system using numerical methods (software on book CD). Details of numerical methods for systems are in a later lecture.

Representing solutions graphically

Can plot graphs of R and F as functions of t.

From the software on the book CD we see:

- If $R(0) = 5$, $F(0) = 4$, get equilibrium solutions as expected.
- If $R(0) = 0$, $F(0) > 0$, or $R(0) > 0$, $F(0) = 0$, get exponentially decreasing or increasing solutions as expected.
- All other solutions with $R(0) > 0$ and $F(0) > 0$ are periodic (same period for R and F).

Can plot solutions in **phase space**, i.e., for given t, plot the point $(R(t), F(t))$ in $(R - F)$-space.

As t varies, point will move and sweep out a curve in $(R - F)$ space. This curve is the **solution curve** in phase space.

From the software we see that

1. equilibrium solutions correspond to a single point in the phase space, i.e., solution curve is just a single point.

2. periodic solutions correspond to a closed curve in phase space.

Use arrow to show direction that move along solution curve as time increases.

Phase space (here: phase plane) is the higher dimensional equivalent of phase line. Solutions drawn in phase space don't show explicit values of t, just how the dependent variables change as t changes.

Different solution curves plotted on the same phase space picture give the **phase portrait** of the system. For example, the phase portrait for the predator/prey system is:

The aim of this section of Maths 260 is to develop qualitative, analytic and numerical methods for getting information about systems of differential equations.