Maths 260 Lecture 12

Topic for today
Linear differential equations

Reading for this lecture
BDH Section 1.8

Suggested exercises
BDH Section 1.8: 1, 3, 9, 13

Reading for next lecture
BDH Section 2.1

Today's handouts
Lecture 12 notes

Section 1.8: Linear Differential Equations

A first order DE is **linear** if it can be written in the form

$$\frac{dy}{dt} = g(t)y + f(t)$$

where $g(t)$ and $f(t)$ are arbitrary functions of t.

Examples:

1. $\dfrac{dy}{dt} = y\cos t + t^2$
2. $y\dfrac{dy}{dt} = ty^2 + ty$
3. $\left(t^2+1\right)\dfrac{dy}{dt} + 2ty - 1 = 0$
4. $\dfrac{dy}{dt} = ty\,(1-y)$ is nonlinear.

Linear means that the dependent variable y appears in the equation only to the first power.

Finding Solutions to linear DEs
First rewrite the DE as

$$\frac{dy}{dt} + a\left(t\right)y = f\left(t\right)$$

(where $a\left(t\right) = -g\left(t\right)$).

A clever trick:
Multiply through by $\mu(t)$, an unknown, non-zero function which will be determined later. We have

$$\mu\left(t\right)\frac{dy}{dt} + \mu\left(t\right)a\left(t\right)y = \mu\left(t\right)f\left(t\right)$$

Now assume we can pick $\mu(t)$ so that

$$\mu\left(t\right)\frac{dy}{dt} + \mu\left(t\right)a\left(t\right)y = \frac{d}{dt}\left(\mu\left(t\right)y\right)$$

So

$$\frac{d}{dt}\left(\mu\left(t\right)y\right) = \mu\left(t\right)f\left(t\right)$$

Integrating both sides with respect to t:

$$\mu(t)y(t) = \int \mu(t)f(t)dt$$

$$\Rightarrow y(t) = \frac{1}{\mu(t)}\int \mu(t)f(t)dt$$

If we can find such a $\mu(t)$ and do the integration then we can find $y(t)$. The function $\mu(t)$ is called an integrating factor.

Finding the integrating factor, $\mu(t)$.

We want $\mu\left(t\right)$ such that

$$\begin{aligned}\mu(t)\frac{dy}{dt} + \mu(t)a(t)y &= \frac{d}{dt}(\mu(t)y)\\ &= \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t)\end{aligned}$$

After cancelling terms, this is:

$$\mu(t)a(t)y = \frac{d\mu}{dt}y$$

$$\Rightarrow \frac{d\mu}{dt} = \mu(t)a(t)$$

This is a separable DE for μ. Solve it:

$$\int \frac{d\mu}{\mu} = \int a(t)dt$$

$$\Rightarrow \ln|\mu| = \int a(t)dt$$

$$\Rightarrow \mu(t) = \pm\exp(\textstyle\int a(t)dt)$$

Different choice of the constant of integration will give different μ, but all choices give a valid integrating factor. Pick the easiest.

Summary of method

To find a solution to

$$\frac{dy}{dt} + a(t)y = f(t)$$

find the integrating factor:

$$\mu(t) = \exp\left(\int a\left(t\right) dt\right)$$

Then the solution is

$$y(t) = \frac{1}{\mu(t)} \int \mu(t) f(t) dt$$

Example 1: Find a one-parameter family of solutions to

$$\frac{dy}{dt} = \frac{y}{t} + t^4$$

It's interesting to graph solutions for various values of the arbitrary constant.

Notice that in this case you can't always solve initial value problems with initial condition $y(t_0) = y_0$ and that when you can, the solution isn't unique. Is this what you expect (check the Existence and Uniqueness theorems).

Example 2: Find a solution to the IVP

$$\frac{dy}{dt} = -2y - 3t, \quad y(0) = \frac{1}{2}.$$

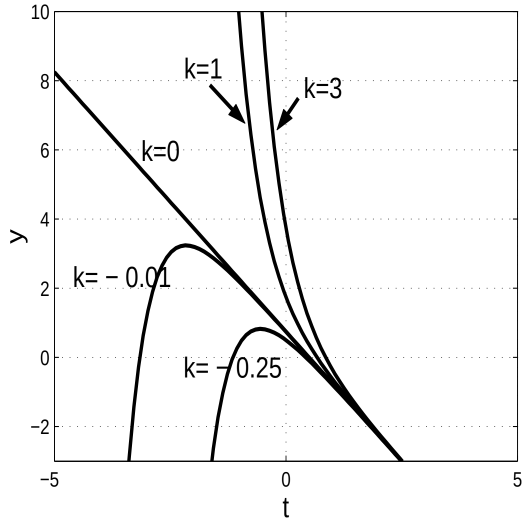
We find

$$\frac{dy}{dt} = -2y - 3t$$

has a one-parameter family solutions

$$y(t) = -\frac{3}{2}t + \frac{3}{4} + ke^{-2t}.$$

The choice $y(0) = 1/2$ determines $k = -1/4$. Some solutions to the DE including the solution to the IVP are plotted below.



Example 3: Find a solution to the DE

$$\frac{dy}{dt} = 1 + 2ty$$