Maths 260 Lecture 10

Topic for today
Bifurcations

Reading for this lecture
BDH Section 1.7

Suggested exercises
BDH Section 1.7: 1, 3, 9

Reading for next lecture
BDH Section 1.7 (again)

Today's handout
Lecture 10 notes

Section 1.7: Bifurcations

Many DE models contain parameters, i.e., quantities that do not depend on the independent variable but may take on different values.

We are interested in how the behaviour of solutions (especially the long term behaviour) changes as parameters are changed. For instance,

1. What are the solutions like over a range of parameter values?

2. How good is our model if we only know the parameter value roughly?

A small change in the value of a parameter usually results in a small change in solutions. A **bifurcation** occurs when a small change in parameter gives a qualitative change in the behaviour of solutions. We look at autonomous equations that depend on one parameter, i.e.,

$$\frac{dy}{dt} = f_\mu(y).$$

This is a one-parameter family of DEs - we get one DE for each choice of the parameter μ.

Example: Consider the DE

$$\frac{dy}{dt} = f_h(y) = y(1-y) - h$$

Compare the phase lines at $h = 0$ and $h = 1$:

We see that there must be a bifurcation at some value of h in the interval $(0, 1)$. We now find and classify equilibria as a function of h and hence locate the bifurcation value of h.

Bifurcation Diagrams

A bifurcation diagram is a picture in the $\mu - y$ plane of the phase lines near a bifurcation value. It highlights the changes that the phase lines undergo as the parameter passes through the bifurcation value.

Procedure for drawing a bifurcation diagram

1. Draw μ and y axes and label them.
2. Plot curves showing position of equilibria as μ varies.
3. Sketch representative phase lines, including at least one for each of $\mu < \mu_0$, $\mu = \mu_0$, $\mu > \mu_0$ where μ_0 is a bifurcation value.
4. Label any significant values of μ and y, including bifurcation values.

Example: Draw the bifurcation diagram for the one-parameter family

$$\frac{dy}{dt} = y(1-y) - h,$$

where h is the bifurcation parameter.

Example: For the family of equations

$$\frac{dy}{dt} = \mu + y^2$$

find the value(s) of μ where a bifurcation occurs and plot the bifurcation diagram.

Important ideas from today

A bifurcation occurs when a small change in parameter gives a qualitative change in the behaviour of solutions.

A bifurcation diagram is a picture which summarises the qualitative changes in behaviour that occur near a bifurcation.