1 Complex Numbers

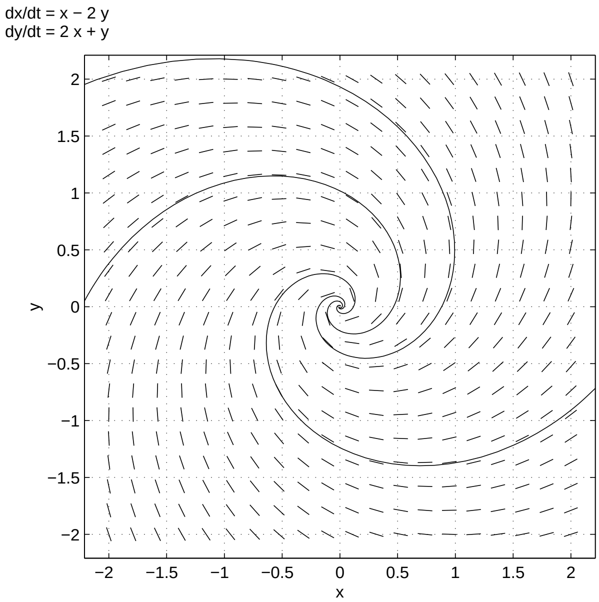
1.1 Introduction

There exist linear systems for which there are no straight-line solutions.

Example: Consider the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{Y}.$$

Slope field and some solutions



What goes wrong?

Calculate the eigenvalues:

$$0 = \det \begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 5.$$

So the quadratic formula gives:

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2}.$$

We need the square root of a negative number!

Let's suppose that we know what the square root of -1 is. We'll call it i. Then we could simplify our expression for λ:

$$\lambda = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4\sqrt{-1}}{2} = 1 \pm 2i.$$

This is an example of a complex number. Since complex numbers show up in the theory of differential equations (and in lots of other areas of mathematics), we need a good understanding of them.

1.2 Complex Numbers

Definition: An expression $a + bi$, where a and b are real numbers, is called a complex number. a is called the real part of the complex number, b is called the imaginary part.

Notation: $a = \text{Re } z, b = \text{Im } z$.

We define the following operation on complex numbers:

1. **Addition:**

$$(a + bi) + (c + di) = (a + c) + (b + d)i. \tag{1}$$

   Example: $(5 + 3i) + (6 - 7i) = 11 - 4i$.

2. **Subtraction:**

$$(a + bi) - (c + di) = (a - c) + (b - d)i. \tag{2}$$

3. **Multiplication:**

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i. \tag{3}$$

4. **Division:**

$$(a + bi)/(c + di) = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i. \tag{4}$$

Note:

1. The normal rules of algebra apply to complex numbers because they form a field under the definitions above (They allow us to use distributive, associative, commutative rules etc.)

2. Because of (1), we do not need to memorise the definitions of the operations. Just use the fact $i^2 = -1$ and apply the usual rules of algebra.
   Example:

$$\begin{aligned}(5+4i)(6-7i) &= 30+24i-35i-28i^2 \\ &= 30+24i-35i+28 \\ &= 58-11i\end{aligned}$$

For division, we may use the fact that

$$(a+bi)(a-bi) = a^2 + abi - abi + b^2 = a^2 + b^2 \text{ real.}$$

Definition: $a - bi$ is called the complex conjugate of $a + bi$.

Notation: If $z = a + bi$, then $\bar{z} = a - bi$ denotes complex conjugate of z.

Division rule: To work out $\frac{a+bi}{c+di}$, multiply numerator and denominator by conjugate of the denominator:

$$\frac{5+2i}{3-4i} = \frac{5+2i}{3-4i}\frac{3+4i}{3+4i} = \frac{15-8+6i+20i}{3^2+4^2} = \frac{7+26i}{25}$$

Since $i^2 = -1$, we can think of i as being a square root of -1 ($i = \sqrt{-1}$), another square root is $-i$.

Example:

$$m^2 + 2m + 2 = 0$$

The solutions are:

$$\begin{aligned}m &= = \frac{-2 \pm \sqrt{-4}}{2} \\ &= \frac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm i.\end{aligned}$$

So, $m_1 = -1 - i$, $m_2 = -1 + i$ are two solutions. Notice that $m_2 = \bar{m}_1$ (complex conjugate). Complex roots always occur in complex conjugate pairs if the polynomial has real coefficients.

1.3 Complex Plane (or Argand diagram)

Since complex numbers are determined by two real numbers, it is natural to plot them on the usual coordinate plane. The vertical axis is called the *imaginary axis* and the horizontal axis is called the *real axis*. The real axis consists of purely real numbers. The imaginary axis consists of points of the form bi, these are called purely imaginary numbers. Complex conjugates are mirror images of each other in the real axis.

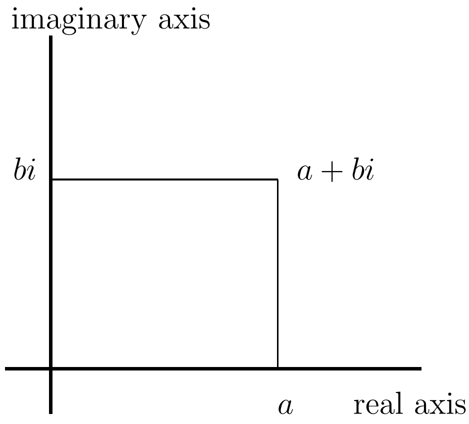


Figure 1: Argand Diagram

Definition: The absolute value (or modulus) of a complex numbers $z = a + bi$ is defined to be $|z| = \sqrt{a^2 + b^2}$. Notice that $|z|$ is the distance between the origin and the point z.

Example: $z = 3 - 4i$, $|z| = \sqrt{9 + 16} = 5$.

Notice that $|z| = \sqrt{z}$ or $|z|^2 = z\bar{z}$ because $z\bar{z} = a^2 + b^2$.

1.4 The polar form of complex numbers

Clearly $a = r\cos\theta, b = r\sin\theta$, where $r = |z|$, θ is called an *argument* of z and is denoted arg$z$.

$$z = r(\cos\theta + i\sin\theta) \text{ (polar form)}$$

Notice that if θ is an argument of z then so is $\theta + 2m\pi$, for any integer m. Sometimes it's useful to restrict arguments to be in $[0, 2\pi)$ or some other interval of length 2π. Such arguments are called *principal arguments* and are often denoted Arg z.

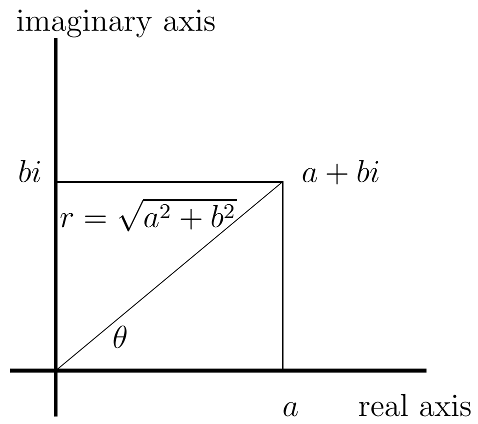


Figure 2: Polar coordinates

Multiplication of polar forms: Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

be any two complex numbers, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \end{aligned}$$

Hence,

multiplying $\Leftrightarrow$ *absolute value = product of absolute values*
argument = sum of arguments

It wasn't obvious from the definition of mutiplication of complex numbers (3), that the multiplication has such a geometric interpretation!

This in turn gives us:

De Moivre's formula

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta),$$

a very useful formula.

Example: Express $\cos 3\theta, \sin 3\theta$ in terms of $\cos\theta, \sin\theta$.

From the de Moivre's formula, we have

$$\begin{aligned}
\cos(3\theta) + i\sin(3\theta) &= (\cos(\theta) + i\sin(\theta))^3 \\
&= \cos^3(\theta) + i3\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)) \\
&= \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) + i(3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)).
\end{aligned}$$

So

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$$
$$\sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta).$$

Polar forms are sometimes useful for solving equations.

Example: Solve $z^3 = 1$.
$z = 1$ is obviously a solution. Any others? Let's write

$$z = r(\cos\theta + i\sin\theta),$$

where $r = |z| > 0$. Then

$$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$$

and therefore

$$r^3(\cos 3\theta + i\sin 3\theta) = 1$$

and

$$r^3 = 1, \text{ and } 3\theta = 0 + 2n\pi \Leftrightarrow \theta_1 = \frac{2n\pi}{3}, n = 0, 1, 2...$$

So the solutions will be

$$z = \cos 0 + i\sin 0, \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

i.e.

$$z = 1, -\frac{1}{2} + i\frac{\sqrt{3}\pi}{2}, -\frac{1}{2} - i\frac{\sqrt{3}\pi}{2}.$$

Notice that for $n = 3, 4...$, the solutions given coincide with the above solutions because of the periodicity of cos and sin.

Note *Fundamental Theorem of Algebra.* A polynomial equation of degree n has n roots (some of these may be repeated). More precisely,

$$a_n z^n + a_{n-1}z^{n-1}... + a_0 = a_n(z - z_1)(z - z_2)...(z - z_n).$$

This is true because, roughly to say, the polynomial is dominated by the term $a_n z^n$ for large enough $|z| = M$ and the mapping of the disc $|z| \leq M$ through the function $a_n z^n$ covers the disc $|z| \leq |a_n| M^n$ n times. It means the equation $a_n z^n = b$ with b a complex number $b \leq |a_n| M^n$ has n roots. The other terms in the polynomial do slightly change the mapping, however the number of roots of the whole polynomial will not change.

Derivatives of a complex-valued function of a real variable

$$f(t) = u(t) + iv(t).$$

We define

$$f'(t) = u'(t) + iv'(t).$$

Complex Exponentials

Consider the complex valued function

$$f(\theta) = \cos(\theta) + i\sin(\theta)$$

De Moivre's formula gives

$$f(\theta)^n = f(n\theta)$$

for any integer n. It shows $f(\theta)$ has a similar behavior to the real-valued exponential function $e^{a\theta}$. Moreover the multiplication rule gives

$$f(\theta_1)f(\theta_2) = f(\theta_1 + \theta_2) \tag{5}$$

which is again similar to real valued exponential function. Actually, real exponential functions can be characterized as a continuous functions satisfying (5). We expect $f(\theta)$ behaves just as real-valued exponential function.

Exercise: Show that $f'(\theta) = if(\theta)$.

This suggests $f(\theta)$ might be considered as the function $e^{a\theta}$ with $a = i$. Notice that $f(0) = 1$ so this is consistent with the real exponential. All this prompts the definition:

Definition: If θ is real,

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{Euler's formula.}$$

Euler discovered this when he was working on differential equations!

Note that we already know $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$.

Example: What is $e^{i\pi}, e^{i\pi/2}e^{i\pi/4}$?

The exponential of other complex numbers: If $z = x + iy$, we define

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Note that

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{x_1+iy_1}e^{x_2+iy_2} \\ &= e^{x_1}e^{x_2}e^{iy_1}e^{iy_2} \\ &= e^{x_1+x_2}e^{iy_1+iy_2} \\ &= e^{z_1+z_2} \end{aligned}$$

Theorem If m is a complex number, then

$$\frac{d}{dt}e^{mt} = me^{mt}.$$

Proof: Write $m = a + ib$. Then

$$\begin{aligned} e^{mt} = e^{at+ibt} & \\ &= e^{at}(\cos(bt) + i\sin(bt)) \\ &= u(t) + iv(t). \end{aligned}$$

where $u(t) = e^{at}\cos(bt), v(t) = e^{at}\sin(bt)$. Clearly,

$$\begin{aligned} \frac{du}{dt} &= ae^{at}\cos(bt) - be^{at}\sin(bt) \\ \frac{dv}{dt} &= ae^{at}\sin(bt) + be^{at}\cos(bt). \end{aligned}$$

Then

$$\frac{d}{dt}e^{mt} = ae^{at}\cos(bt) - be^{at}\sin(bt) + iae^{at}\sin(bt) + ibe^{at}\cos(bt).$$

But

$$\begin{aligned} me^{mt} &= (a+ib)e^{at}(\cos(bt) + i\sin(bt)) \\ &= ae^{at}\cos(bt) - be^{at}\sin(bt) + iae^{at}\sin(bt) + ibe^{at}\cos(bt). \end{aligned}$$

Example:

$$\frac{d}{dt}e^{(3+i)t} = (3+i)e^{(3+i)t}.$$

Note: For x real, we know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots$. This power series also converges for x complex and agrees with e^x for x complex. This can be proved using power series for $\cos x$ and $\sin x$.

Example: Find all solutions of the form $y = e^{\lambda t}$ to the differential equation

$$y''(t) + 2y'(t) + 10y(t) = 0.$$

Solution: If $y = e^{\lambda t}$ then $y' = \lambda e^{\lambda t}$ and $y'' = \lambda^2 e^{\lambda t}$. Putting these into the DE gives

$$\begin{aligned} \lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + 10 e^{\lambda t} &= 0 \\ \text{so} \quad (\lambda^2 + 2\lambda + 10)e^{\lambda t} &= 0 \\ \text{so} \quad (\lambda^2 + 2\lambda + 10) &= 0 \end{aligned}$$

The quadratic formula gives

$$\lambda = -1 \pm 3i,$$

which gives the two solutions

$$\begin{aligned} e^{(-1+3i)t} &= e^{-t}(\cos 3t + i \sin 3t), \\ e^{(-1-3i)t} &= e^{-t}(\cos -3t + i \sin -3t) = e^{-t}(\cos 3t - i \sin 3t) \end{aligned}$$

Note: We'll see later that the *general solution* of such equations can be found by taking a linear combination of the real and imaginary parts of the complex exponential solutions. So the general solution of the equation above is

$$y = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t.$$

1.5 Problems

1. Find all of the solutions to the equation $z^3 + 27 = 0$ and plot them on a graph.

2. Find all of the solutions to the equation $z^4 + 81 = 0$ and plot them on a graph.

3. Calculate $\dfrac{1+5i}{3-2i}$.

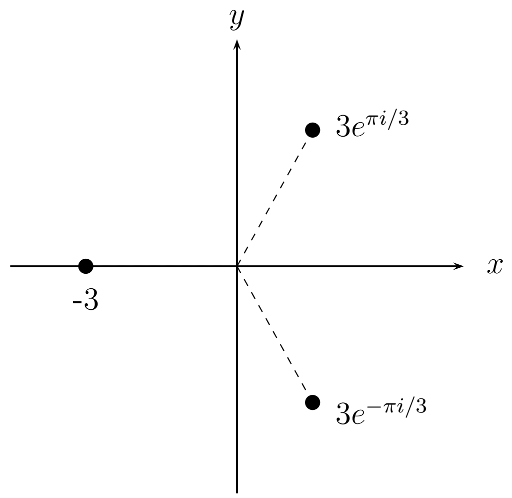
4. Calculate the polar forms of $z_1 = \sqrt{3} + i$ and $z_2 = 1 + i$ and plot z_1 and z_2 on a graph. Calculate the polar forms of $z_1 z_2$ and z_1/z_2 and show these complex numbers on the same graph. Show the arguments and absolute values (moduli) of these complex numbers on your graph.

5. Solve
$$y'' + 10y' + 29y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

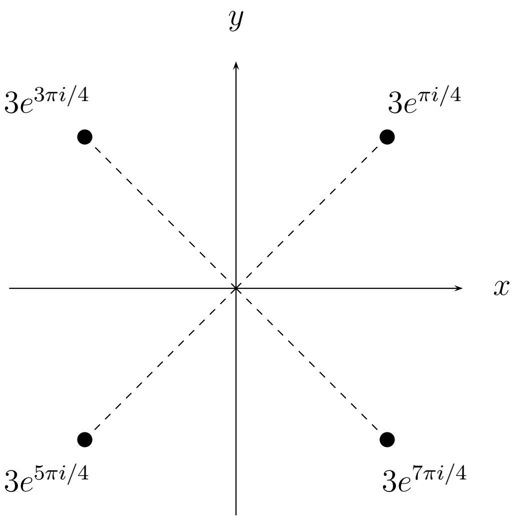
1.6 Solutions

1. $z^3 = -27$ so $|z|^3 = 27$ and thus $|z| = 3$. Hence we can write $z = 3e^{i\theta}$. Plugging this into the equation gives $e^{3i\theta} = -1$. Hence $3\theta = \pi + 2n\pi$, where n is an integer. Hence $\theta = \pi/3 + 2n\pi/3$, $n = 0, 1, 2$. There are no more solutions because starting at $n = 3$, the previous solutions reappear. Note that you could describe these solutions with negative values of θ as well (e.g. take $n = 1, -1, 0$). The important thing is to get the correct complex numbers which are

$$
\begin{aligned}
z &= -3, \quad \text{for } n = 1,\\
z &= 3e^{\pi i/3} = \frac{3}{2}(1 + i\sqrt{3}), \quad \text{for } n = 0,\\
z &= 3e^{-\pi i/3} = \frac{3}{2}(1 - i\sqrt{3}), \quad \text{for } n = -1.
\end{aligned}
$$



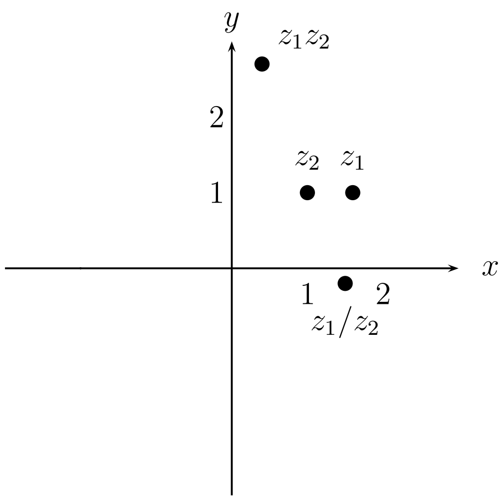
2. $z^4 = -81$ so $|z|^4 = 81$ and thus $|z| = 3$. Hence we can write $z = 3e^{i\theta}$. Plugging this into the equation gives $e^{4i\theta} = -1$. Hence $4\theta = \pi + 2n\pi$, where n is an integer. Hence $\theta = \pi/4 + n\pi/2$, $n = 0, 1, 2, 3$. There are no more solutions because starting at $n = 4$, the previous solutions reappear. Note that you could describe these solutions with negative values of θ as well (e.g. take $n = -2, -1, 0, 1$). The important thing is to get the correct complex numbers as shown on the graph below.



3.

$$\frac{1+5i}{3-2i} = \frac{(1+5i)}{(3-2i)}\frac{(3+2i)}{(3+2i)} = \frac{3-10+15i+2i}{3^2+2^2} = -\frac{7}{13} + \frac{17}{13}i.$$

4. $|z_1| = \sqrt{(3+1)} = 2$. Hence $z_1 = 2e^{i\theta_1}$, where $\theta_1 = \tan^{-1}(1/\sqrt{3}) = \pi/6$. Similarly, $z_2 = \sqrt{2}e^{i\pi/4}$. $z_1 z_2 = 2\sqrt{2}e^{i(\pi/6+\pi/4)} = 2\sqrt{2}e^{5\pi i/12}$. Also, $z_1/z_2 = \sqrt{2}e^{i(\pi/6-\pi/4)} = \sqrt{2}e^{-\pi i/12}$. Of course you could calculate these without using the polar form as well. Here is the picture:



5.

$$y'' + 10y' + 29y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

$$\lambda^2 + 10\lambda + 29 = 0,$$

$$\lambda = \frac{-10 \pm \sqrt{100 - 116}}{2}$$

$$= \frac{-10 \pm \sqrt{-16}}{2} = -5 \pm 2i.$$

Hence (see the last example before the problems)

$$\begin{aligned} y &= c_1 e^{-5t} \cos(2t) + c_2 e^{-5t} \sin(2t) \\ y' &= -5c_1 e^{-5t} \cos(2t) - 2c_1 e^{-5t} \sin(2t) - 5c_2 e^{-5t} \sin(2t) + 2c_2 e^{-5t} \cos(2t). \end{aligned}$$

We need

$$c_1 = 1,$$

$$-5c_1 + 2c_2 = 2,$$

thus

$$c_2 = \frac{7}{2}.$$

$$y = e^{-5t} \cos(2t) + \frac{7}{2} e^{-5t} \sin(2t).$$