Revision April 2007

Separable DEs

eg IVP $\frac{dy}{dt} = y^2 e^t, \quad y(0) = 3.$

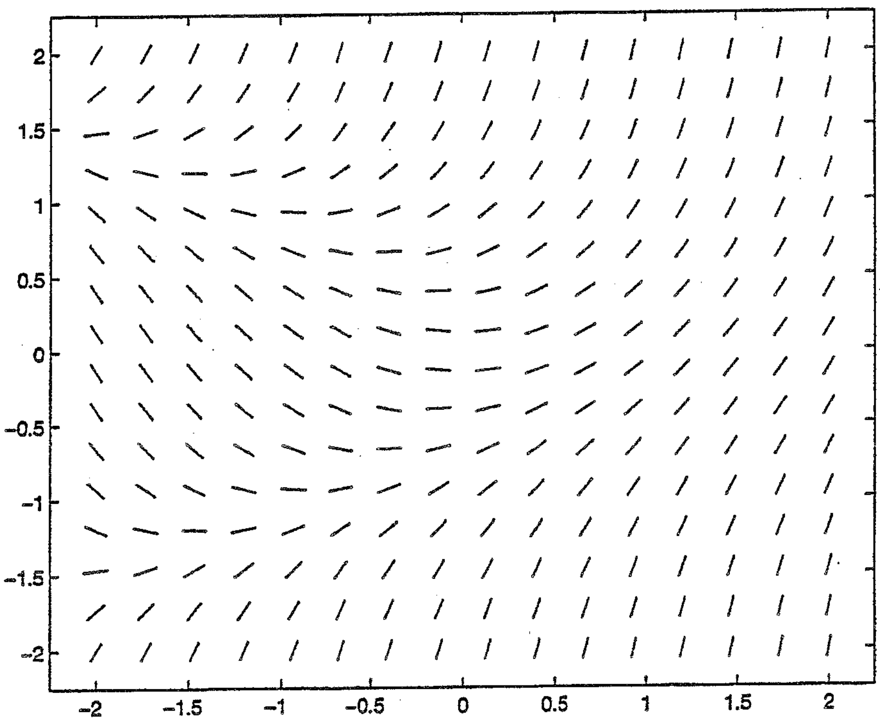
$$\int \frac{dy}{y^2} = \int e^t \, dt$$

$$-\frac{1}{y} = e^t + c$$

$$y = \frac{-1}{e^t + c}$$

$$3 = \frac{-1}{1+c} \Rightarrow c = -4/3$$

Slope fields



Numerical methods $\quad y' = f(t, y)$

Use to estimate numerical solution.

Euler

$$y_i = y_{i-1} + h f(t_{i-1}, y_{i-1})$$

Example

$y' = t + y, \; y(1) = 2$. Estimate $y(1.2)$ in 2 steps.

$h =$, $t_0 =$, $y_0 =$, $f(t,y) =$

$$y_1 = y_0 + h f(t_0, y_0)$$

$$y_2 = y_1 + h f(t_1, y_1)$$

Improved Euler

$$m_1 = f(t_i, y_i)$$

$$m_2 = f(t_{i+1}, y_i + h m_1)$$

$$y_{i+1} = y_i + \frac{h}{2}(m_1 + m_2)$$

Example $\quad y' = t + y, \; y(1) = 2$

Do 1 step to estimate $y(1.1)$ ie $h = 0.1$

$m_1 = f(1, 2) = 3$

$m_2 = f(1.1, 2 + 0.1 \times 3) = f(1.1, 2.3) = 3.4$

$y(1.1) \approx y_1 = 2 + \frac{0.1}{2}(3 + 3.4) = 2.32$

Order of numerical methods

If $|E(h)| \approx kh^p$ as $h \to 0$, then p is order of method.

Euler order is 1. $|E(h)| = k_1 h$

Halve stepsize then error is halved.

Improved Euler order is 2 $|E(h)| = k_2 h^2$

Halve stepsize, error is divided by 4.

4th order Runge-Kutta

Order is 4. $|E(h)| = k_3 h^4$
Halve stepsize, error is divided by 16.

Effective order at h

$$q = \frac{\ln|E(2h)| - \ln|E(h)|}{\ln 2}$$

$$q \to p \quad \text{as } h \to 0.$$

Error $= |y(t_n) - y_n|$

Can use exact soln if known, or a more accurate method.

Efficiency

Often measure in fn evaluations required to get desired accuracy.

Existence + Uniqueness of Solns

$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0$$

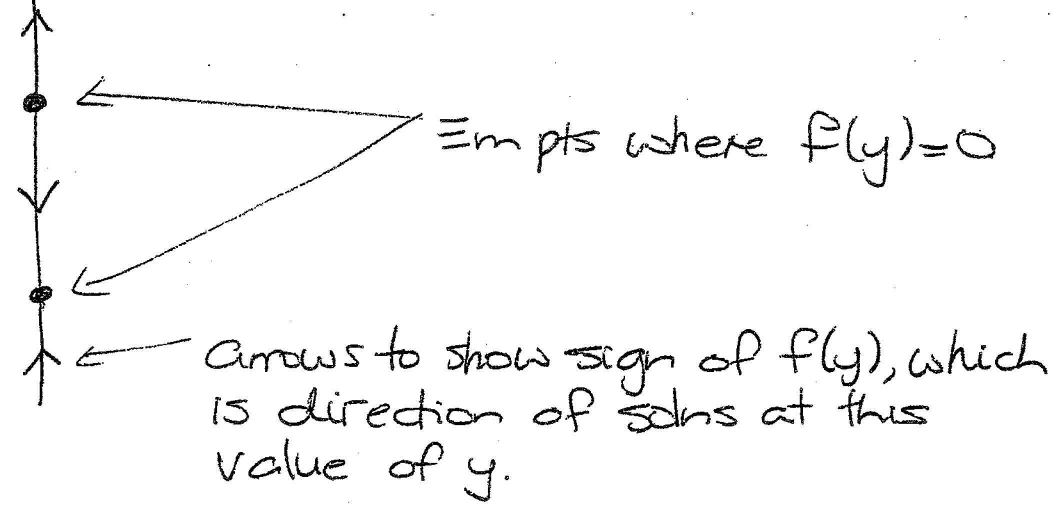
$f(t,y)$ is cts fn of t + y at (t_0, y_0)
$\Rightarrow \exists\ \varepsilon > 0$ such that a soln $y(t)$ exists on $(t_0 - \varepsilon, t_0 + \varepsilon)$.

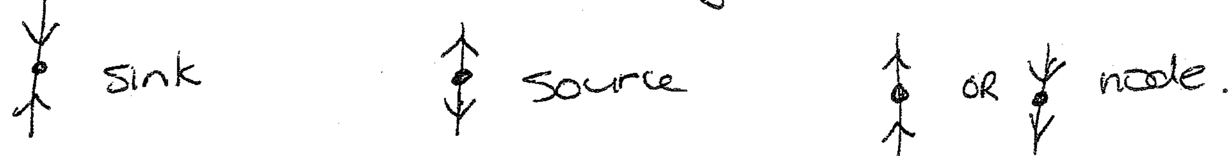
If also $\frac{\partial f}{\partial y}$ is cts fn of t + y at (t_0, y_0), the soln above is unique.

Phase Line

For autonomous DE

$$\frac{dy}{dt} = f(y)$$





Linearisation Thm

If y_0 is Em pt of $y' = f(y)$, and f and $\frac{\partial f}{\partial y}$ are cts, then

1. $\left.\frac{\partial f}{\partial y}\right|_{y_0} < 0 \Rightarrow y_0$ is sink
2. $\left.\frac{\partial f}{\partial y}\right|_{y_0} > 0 \Rightarrow y_0$ is source

If $\left.\frac{\partial f}{\partial y}\right|_{y_0} = 0$ or is undefined – no information.

Bifurcations

When small change in parameter gives *qualitative* change in solns.

Example

$$\frac{dy}{dt} = y(y+\lambda)$$

$\equiv$m pts $\quad y=0, \; y=-\lambda$

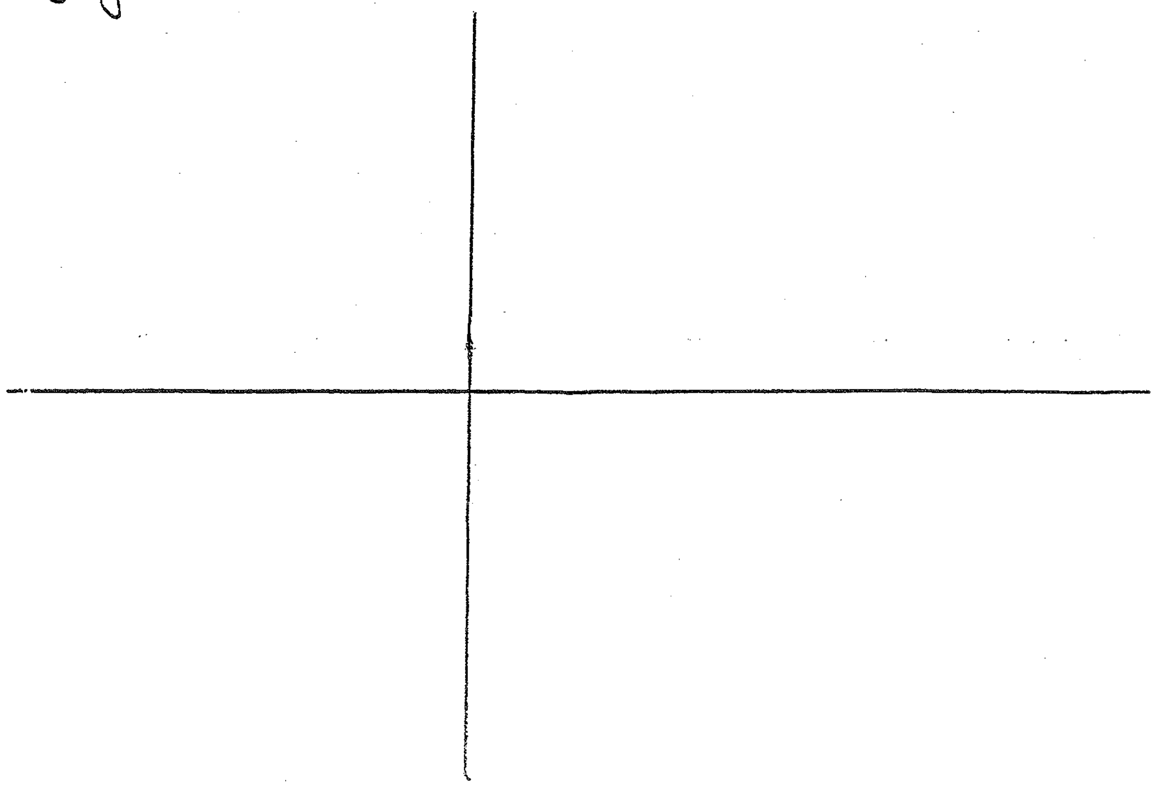
(2 $\equiv$m pts unless $\lambda=0$, then one - bifurcation at $\lambda=0$)

$f(y) = y(y+\lambda) \qquad \frac{\partial f}{\partial y} = 2y+\lambda$

$\left.\frac{\partial f}{\partial y}\right|_{y=0} = +\lambda$

$y=0$ is a source if $\lambda>0$, sink $\lambda<0$.

$\left.\frac{\partial f}{\partial y}\right|_{y=-\lambda} = -\lambda$ which is source if $\lambda<0$, sink $\lambda>0$.



Draw a bifurcation diagram for

$$\frac{dy}{dt} = y^2 + \mu$$

Linear DEs

$$\frac{dy}{dt} = g(t)y + f(t)$$

Example

$$\frac{dy}{dt} = -2y + e^{-t} \rightsquigarrow \frac{dy}{dt} + 2y = e^{-t}$$

Integrating factor $\mu(t) =$

Systems of DEs

$$\frac{dx_1}{dt} = f_1(t, x_1, \ldots x_n)$$

$$\vdots$$

$$\frac{dx_n}{dt} = f_n(t, x_1, x_2, \ldots x_n)$$

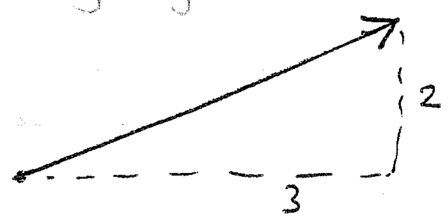
OR $$\frac{dY}{dt} = F(t, Y)$$

$$\frac{dY}{dt} = F(Y) \quad \text{is autonomous.}$$

Can draw direction field / phase portrait for autonomous D.E.
Easier if 2-D.

eg $$\frac{dx}{dt} = x + y$$
$$\frac{dy}{dt} = y + 1$$

At point $(2,1)$, for example, draw the vector $(\underset{x+y}{2+1}, \underset{y+1}{1+1}) = (3,2)$



Can sketch solns on this to make the phase portrait.

Existence + Uniqueness Thm

$$\frac{d}{dt}Y = F(t, Y), \quad Y(t_0) = Y_0$$

If F is cts and 1st partial derivatives are cts at (t_0, Y_0), then there is unique soln defined on $t_0 - \varepsilon < t < t_0 + \varepsilon$

Numerical methods

Can use Euler + other numerical methods for systems also.

eg $\quad X_1 = X_0 + hF(t_0, X_0) \quad$ (Euler)

eg. $\quad \dfrac{dX}{dt} = \begin{pmatrix} y \\ t - x \end{pmatrix}$ where $X = \begin{pmatrix} x \\ y \end{pmatrix}$

$X(1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Estimate $X(1.3)$ in 3 steps.

$h =$

$X_1 =$