Department of Mathematics

Maths 260 Assignment 4 - Sample Solutions

June 1, 2007 Due: 22 May 2007

Students should hand their assignments in at the Student Resource Centre in the basement of the Mathematics/Physics Building. Your completed assignment should be handed in to the appropriate box outside the Student Resource Centre **before** 4pm on the date due. Late assignments or assignments placed in the wrong box will not be accepted. Your assignment **must** be accompanied by a blue Mathematics Department coversheet. Copies of the coversheet are available from a box next to the Student Resource Centre.

1. (9 marks)

(a) The characteristic polynomial for the matrix is

$$\lambda^2 + \lambda + 4$$

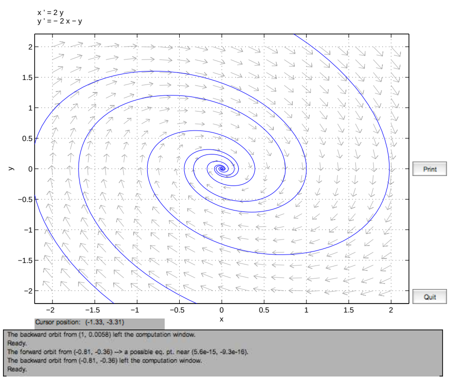
and the zeros are

$$\frac{-1 \pm \sqrt{15}i}{2}.$$

Therefore the origin is a spiral sink since the real part of the eigenvalue is negative.

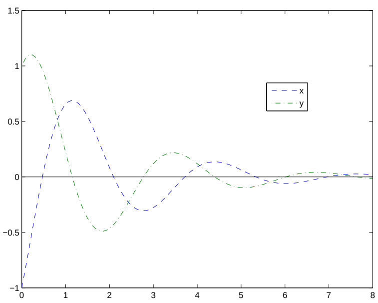
(b) Find the direction at any point by multiplying a position vector by the matrix. For example, at (1,0), the direction is (0,-2). By sketching this you can see it is clockwise.

(c) The phase portrait should be drawn by hand, not by `pplane`.



(d) The imaginary part is $\frac{\sqrt{15}}{2}$, so the sin and cos values will be return every 2π, so the period is $4\pi/\sqrt{15}$.

(e) The graphs are:



Graphs to be drawn by hand and showing that the solutions are dying away.

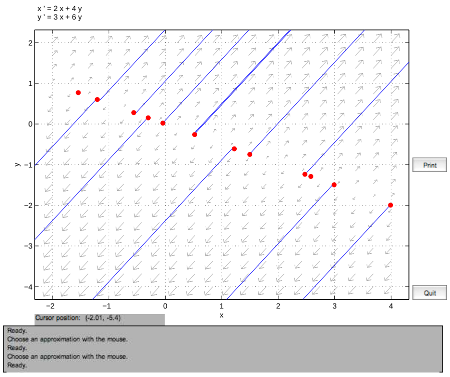
2. (7 marks) The characteristic polynomial for the matrix is

$$\lambda^2 - 8\lambda$$

and the zeros are 8 and 0. The eigenvector corresponding to eigenvalue 8 is $(2,3)^T$, and to eigenvalue 0 is $(-2,1)^T$. Thus the general solution is

$$Y(t) = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{8t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

The phase portrait should be drawn by hand, not by `pplane`. Note that there is a line of equilibrium solutions, and the solutions tend away from this towards infinity.



3. (7 marks)

   (a) The characteristic polynomial for the matrix is

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2,$$

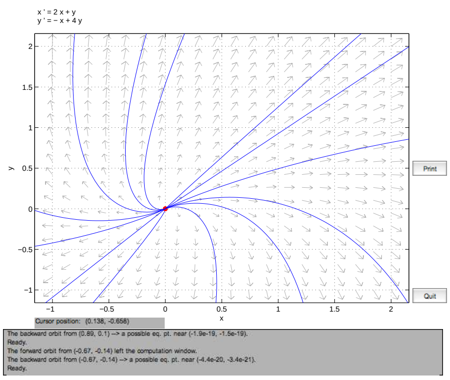
and there is a repeated zero at 3. There is only one eigenvector which is $(1,1)^T$, so we need a generalised eigenvector. Solve

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This means that $-v_1 + v_2 = 1$. Choose, say, $v_1 = 0$, then $v_2 = 1$, and a generalised eigenvector is $(0,1)^T$. General solution is

$$Y(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \left(t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

The phase portrait should be drawn by hand, not by `pplane`.



(b)

$$Y(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so we need $c_1 = 1, c_2 = 2$ and so the required solution is

$$Y(t) = e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2te^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

4. (12 marks) The characteristic polynomial for the matrix is

$$\lambda^2 - (a+1)\lambda + a + 3.$$

There will be a zero eigenvalue if $a = -3$. The solutions will tend towards a line of equilibria.

The eigenvalues will be complex if $(a+1)^2 - 4(a+3) = a^2 - 2a - 11 < 0$, which is when $1 - 2\sqrt{3} < a < 1 + 2\sqrt{3}$. The real part is $a + 1$ so if $a > -1$, there will be a spiral source, for $a = -1$ a centre, and for $a < -1$ a spiral sink.

The eigenvalues will be repeated if $a = 1 - 2\sqrt{3}$, both negative (sink), or if $a = 1 + 2\sqrt{3}$, both positive (source). Each will have just one straight line solution.

For $a > 1 + 2\sqrt{3}$, $a + 3$ and $a + 1$ will both be positive, and hence there will be a real source.

For $-3 < a < 1 - 2\sqrt{3}$, $a + 3 > 0$ and $a + 1 < 0$, so there are two negative eigenvalues and there is a real sink.

For $a < 3$, there is a saddle.

5. (15 marks)

   (a) At an equilibrium, $y = a$ and $x = \sqrt{y}$. If $a < 0$, then there is no square root so no equilibrium.

   (b) One equilibrium solution of $x = y = 0$.

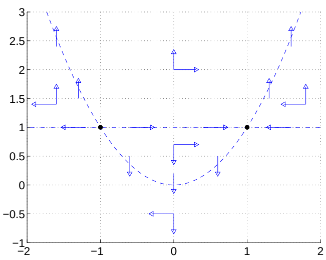
   (c) For $a = 1$:

      i. There are equilibrium solutions at $(-1, 1)$ and $(1, 1)$. The Jacobian is given by

$$J = \begin{pmatrix} -2x & 1 \\ 0 & 1 \end{pmatrix}.$$

      At $(-1, 1)$, the eigenvalues will be 2 and 1 so it will be a source. At $(1, 1)$, the eigenvalues will be -2 and 1 so it will be a saddle.

      ii. The x-nullcline is $y = x^2$, and the y-nullcline is $y = 1$. Show on diagram and put in arrows for the regions.



iii. The phase portrait should be drawn by hand, not by `pplane`.