Department of Mathematics

Maths 260 Assignment 3 - Sample Solutions

April 26, 2007 Due: 8 May 2007

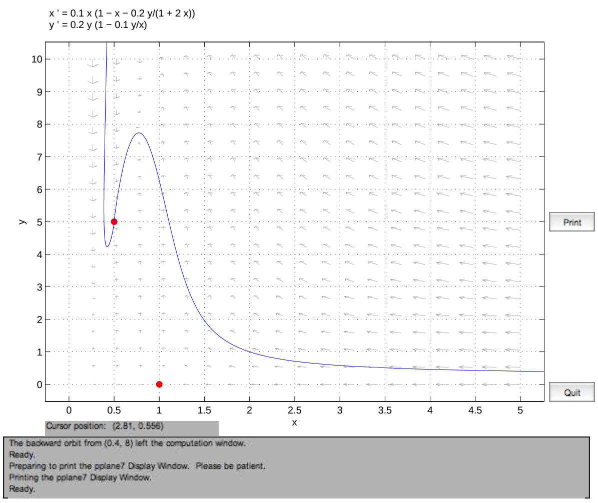
1. (10 marks)

(a) Note that $R = 0$ is not possible because of the R in the denominator of the second equation. However if we set $H = 0$, we get $R = 1$ from the first equation so (1,0) is an equilibrium solution. Now find values that make the brackets zero. From the second equation we get $R = 0.1H$. Substituting this into the bracket of the first equation and setting it to zero gives

$$\begin{aligned} 1 - 0.1H - \frac{0.2H}{1 + 0.2H} &= 0 \\ (1 - 0.1H)(1 + 0.2H) - 0.2H &= 0 \\ -0.02H^2 - 0.1H + 1 &= 0 \\ H &= \frac{0.1 \pm \sqrt{(-0.1)^2 + 0.08}}{-0.04} \\ &= \frac{0.1 \pm 0.3}{-0.04} \\ &= -10, 5. \end{aligned}$$

So the two equilibrium solutions for the population are (1,0) and (0.5,5).

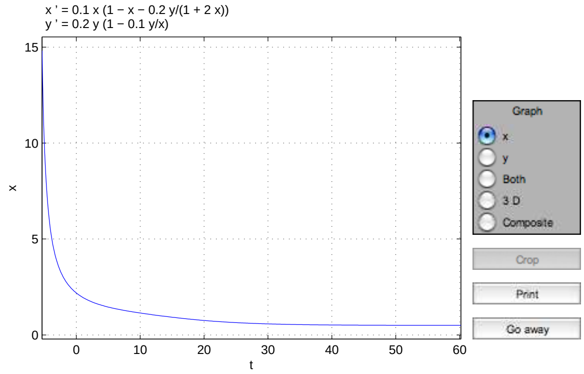
(b) The phase portrait is shown below. I have included the two initial conditions in (c) but they are not required for this.

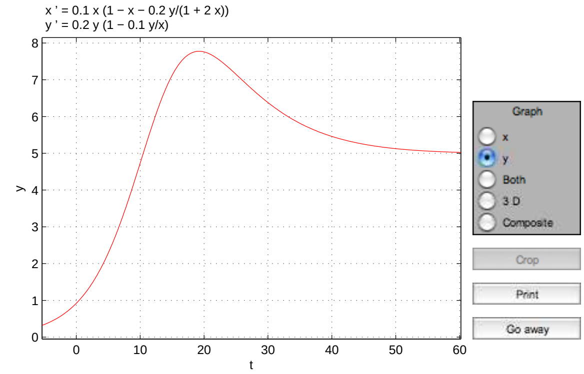


Marks: 2 marks for `pplane` portrait.

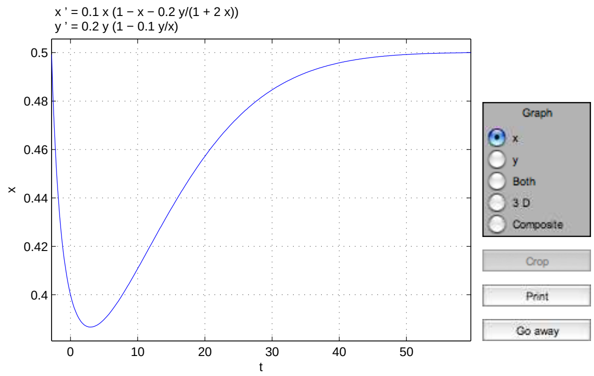
(c) The graphs below are shown from `pplane` but the students may either sketch by hand or show the `pplane` output.

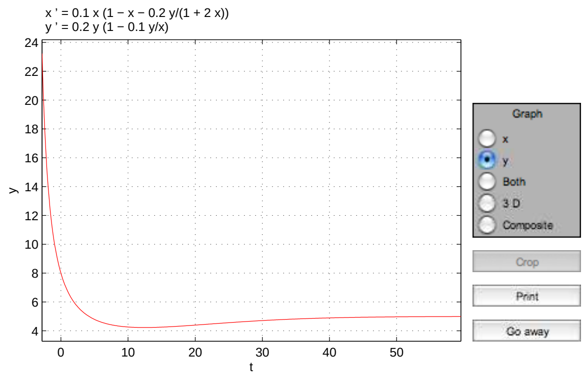
i. 2000 rabbits and 1000 hawks





ii. 400 rabbits and 8000 hawks





2. (6 marks) First solve the second equation

$$
\begin{aligned}
\frac{dy}{dt} &= 2ty \\
\int \frac{dy}{y} &= \int 2t dt \\
\ln|y| &= t^2 + c \\
y &= Ke^{t^2} \qquad \text{where } K \text{ can be any real number (allows for missing solution).}
\end{aligned}
$$

Substitute into the first equation

$$
\begin{aligned}
\frac{dx}{dt} &= -\frac{x}{t} + Ke^{t^2} \\
\frac{dx}{dt} + \frac{x}{t} &= Ke^{t^2}.
\end{aligned}
$$

The integrating factor is $\mu = \exp(\int \frac{1}{t} dt) = \exp(\ln|t|) = t$. Multiply by this.

$$
\begin{aligned}
t\frac{dx}{dt} + x &= Kte^{t^2} \\
\frac{d}{dt}(tx) &= Kte^{t^2} \\
tx &= \frac{1}{2}Ke^{t^2} + c \\
x &= \frac{1}{2t}Ke^{t^2} + \frac{c}{t}
\end{aligned}
$$

So the general solution is $x = \frac{1}{2t}Ke^{t^2} + \frac{c}{t}, \quad y = Ke^{t^2}$.

3. (10 marks) The system can be written as

$$\frac{dY}{dt} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} Y = AY.$$

First find the eigenvalues of the matrix A.

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)(-4-\lambda) + 6 \\ &= \lambda^2 + 3\lambda + 2) \\ &= (\lambda+2)(\lambda+1). \end{aligned}$$

Therefore the eigenvalues are -1 and -2. Next find the eigenvectors. For $\lambda = -2$,

$$\begin{aligned} \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= 0 \\ 3v_1 - 2v_2 &= 0 \\ v_1 &= \frac{2}{3} v_2 \\ v &= \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad \text{or some multiple,} \end{aligned}$$

and for $\lambda = -1$,

$$\begin{aligned} \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= 0 \\ v_1 - v_2 &= 0 \\ v_1 &= v_2 \\ v &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \text{or some multiple.} \end{aligned}$$

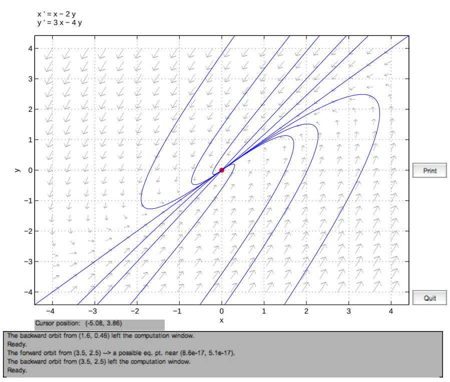
Therefore the general solution is

$$\begin{aligned} x(t) &= 2c_1 e^{-2t} + c_2 e^{-t} \\ y(t) &= 3c_1 e^{-2t} + c_2 e^{-t} \end{aligned}$$

or

$$Y(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The phase portrait for the system should be sketched by hand (showing arrows). For my (electronic) convenience it is shown from `pplane`.



4. (9 marks) First find the eigenvalues of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 4 & -1 \end{pmatrix}.$$

Now $\det(A - \lambda) = -(\lambda - 1)(\lambda^2 - 2\lambda + 5)$, so the eigenvalues are 1 and $(2 \pm \sqrt{4-20})/2 = 1 \pm 2i$. Next the eigenvectors. For $\lambda = 1$, we get

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0.$$

It can be seen that v_1 can have any value and that v_2 and v_3 must both be zero. Therefore the eigenvector is $(1, 0, 0)^T$ or any multiple. For $\lambda = 1 + 2i$,

$$\begin{pmatrix} -2i & 0 & 0 \\ 0 & 2-2i & -2 \\ 0 & 4 & -2-2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0.$$

This gives $v_1 = 0$ and $v_2 = (1 + i)/2$, so the eigenvector is $(0, 1 + i, 2)^T$.
To find the general solution we will combine the straight line solution from $\lambda = 1$ with the real-valued solutions from the complex conjugate pair of eigenvalues. The latter real-valued solutions are the real and imaginary parts of

$$e^{(1+2i)t} \begin{pmatrix} 0 \\ 1+i \\ 2 \end{pmatrix} = e^t(\cos 2t + i \sin 2t) \begin{pmatrix} 0 \\ 1+i \\ 2 \end{pmatrix}.$$

The real part is $e^t(0, \cos 2t - \sin 2t, 2\cos 2t)^T$ and the imaginary part is $e^t(0, \cos 2t + \sin 2t, 2\sin 2t)^T$. Therefore the real-valued solution to the system is

$$Y = c_1\, e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\, e^t \begin{pmatrix} 0 \\ \cos 2t - \sin 2t \\ 2\cos 2t \end{pmatrix} + c_3\, e^t \begin{pmatrix} 0 \\ \cos 2t + \sin 2t \\ 2\sin 2t \end{pmatrix}.$$