Maths 260 Assignment 2 Sample Solutions

April 19, 2007 Due:

Total Marks for Assignment 2: 40

1. Note that

$$f(t,y) = 1 - t + 4y.$$

(a) There are two steps to do so $h = 0.1$. Also $t_0 = 0$ and $y_0 = 1$.
Step 1

$$\begin{aligned} m_1 &= f(t_0, y_0) = 5 \\ m_2 &= f(t_0 + h, y_0 + hm_1) \\ &= f(-0.1, 1.5) = 6.9 \\ y_1 &= y_0 + \frac{h}{2}(m_1 + m_2) \\ &= 1 + \frac{0.1}{2}(5 + 6.9) = 1.595 \end{aligned}$$

$$t_1 = 0.1$$

Step 2

$$\begin{aligned} m_1 &= f(0.1, 1.595) = 7.28 \\ m_2 &= f(0.2, 2.323) = 10.092 \\ y_2 &= 1.595 + \frac{.1}{2}(7.28 + 10.092) \\ &= 2.4636 \\ t_2 &= 0.2 \end{aligned}$$

Therefore $y(0.2) \approx 2.4636$.

(b) There is one step to do so $h = 0.2$. Also $t_0 = 0$ and $y_0 = 1$ *Step 1*

$$\begin{aligned} m_1 &= f(t_0, y_0) = 5 \\ m_2 &= f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}m_1) \\ &= f(0.1, 1.5) = 6.9 \\ m_3 &= f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}m_2) \\ &= f(0.1, 1.69) = 7.66 \\ m_4 &= f(t_0 + h, y_0 + hm_3) \\ &= f(0.2, 2.532) = 10.928 \\ y_1 &= y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4) \\ &= 1 + \frac{0.2}{6}(5 + 2 \times 6.9 + 2 \times 7.66 + 10.928) \\ &= 2.5016 \end{aligned}$$

Therefore $y(0.2) \approx 2.5016$.

Total marks for Q1: 6

2. This is a linear differential equation so write it as

$$\frac{dy}{dt} - 4y = 1 - t.$$

The integrating factor is $\mu(t) = exp(\int(-4)dt) = e^{-4t}$.

$$\begin{aligned}
e^{-4t}\frac{dy}{dt} - 4e^{-4t}y &= e^{-4t}(1-t) \\
\frac{d}{dt}(ye^{-4t}) &= e^{-4t}(1-t) \\
ye^{-4t} &= \int e^{-4t}(1-t)dt \\
&= -\frac{3}{16}e^{-4t} + \frac{1}{4}te^{-4t} + c \\
y &= -\frac{3}{16} + \frac{1}{4}t + ce^{4t} \\
1 &= y(0) \\
&= -\frac{3}{16} + c \\
c &= \frac{19}{16} \\
y(t) &= -\frac{3}{16} + \frac{1}{4}t + \frac{19}{16}e^{4t} \\
y(0.2) &= 2.5053
\end{aligned}$$

Therefore the error in 1(a) is 4.2×10^{-2} and in 1(b) it is 3.7×10^{-3}. The error for the Runge-Kutta method is smaller even though the stepsize was twice as large. Each method required 4 function evaluations so the Runge-Kutta is more efficient.

Total marks for Q2: 7

3. (a) The formula for effective order at stepsize h is

$$q = \frac{\ln|E(2h)| - \ln|E(h)|}{\ln 2}.$$

i. At $h = 0.125$,

$$q = \frac{\ln|E(0.25)| - \ln|E(0.125)|}{\ln 2} = \frac{\ln 1.45 \times 10^{-3} - \ln 2 \times 10^{-4}}{\ln 2} = 2.858$$

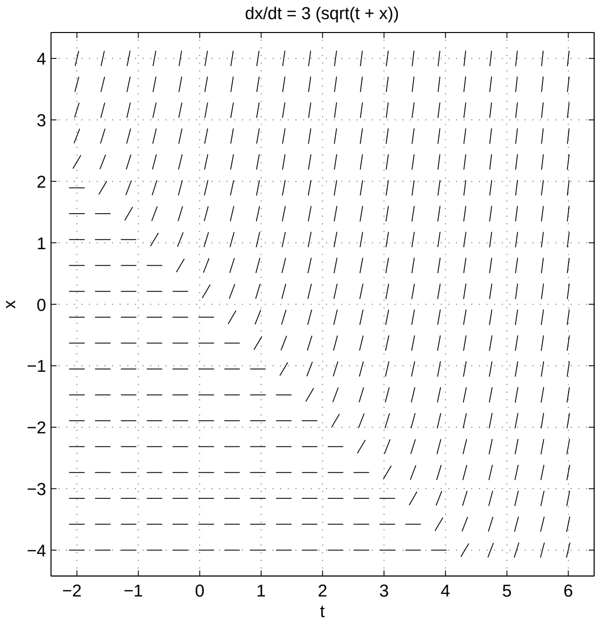
ii. At $h = 0.03125$

$$q = \frac{\ln|E(0.0625)| - \ln|E(0.03125)|}{\ln 2} = \frac{\ln 2.63 \times 10^{-5} - \ln 3.37 \times 10^{-6}}{\ln 2} = 2.9642$$

(b) The order is probably 3 because as $h \to 0$, q appears to be approaching 3.

Total marks for Q3: 3

4. (a) The direction field is

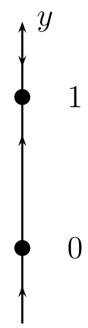


(b) $f(t, y) = 3(t + y)^{1/2}$ which is defined only if $t + y \geq 0$. The existence theorem says that a solution through (t_0, y_0) will exist if f is continuous (and defined) at (t_0, y_0), so a solution exists if $t_0 + y_0 > 0$. We can see on the direction field that for $y < -t$ that the slope marks do not exist (they look a bit like zero slope but they do not exist).

(c) $\frac{\partial f}{\partial y} = 3/(2(t + y)^{1/2})$, which exists and is continuous at all (t_0, y_0) where $t_0 + y_0 > 0$. Again this is what we see on the direction field.

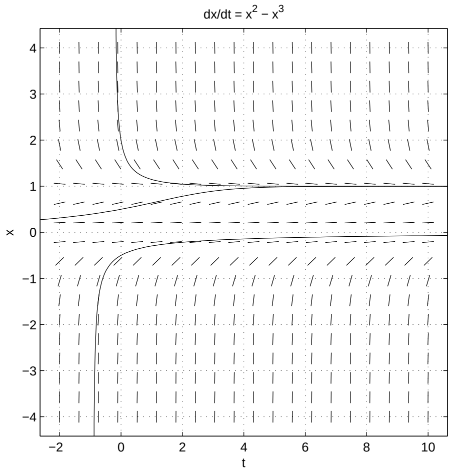
Total marks for Q4: 7

5. (a) The equilibrium solutions are when $y^2 - y^3 = 0$ ie $y^2(1 - y) = 0$ ie at $y = 0$ and $y = 1$. The phase line is



(b) $y = 0$ is a node and $y = 1$ is a sink.

(c) The solutions are shown below on a direction field, but the students should just sketch them by hand.

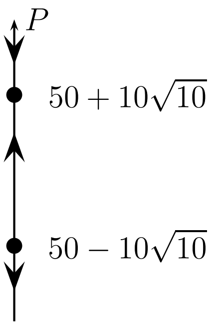


Total marks for Q5: 6

6. (a) The differential equation to model the population is

$$\frac{dP}{dt} = 0.1P(1 - \frac{P}{100}) - \alpha.$$

(b) The equilibrium solutions are $50 \pm 10\sqrt{10}$. The phase line will be This means that the

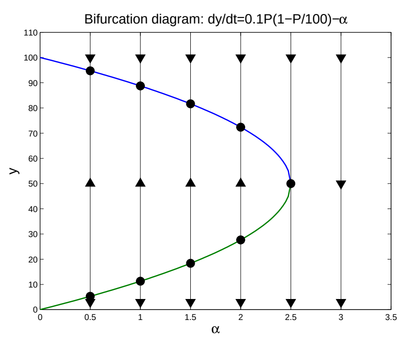


minimum number required is $50 - 10\sqrt{10}$ which means about 18,400 fish.

(c) The curve for the bifurcation diagram will be

$$\alpha = 0.1P(1 - \frac{P}{100})$$

which will be a parabola on its side centred about $P = 50$.



Total marks for Q6: 7

7. This is a linear differential equation and the integrating factor will be e^{at}. Multiply through by this. For $a \neq \lambda$,

$$\begin{aligned} e^{at}\frac{dy}{dt} + ae^{at}y &= be^{at}e^{-\lambda t} \\ \frac{d}{dt}(e^{at}y) &= be^{(a-\lambda)t} \\ e^{at}y &= \frac{b}{a-\lambda}e^{(a-\lambda)t} + c \\ y &= \frac{b}{a-\lambda}e^{-\lambda t} + ce^{-at} \end{aligned}$$

Since λ and a are positive, $y \to 0$ as $t \to \infty$.
When $a = \lambda$, then we get

$$\begin{aligned} \frac{d}{dt}(e^{at}y) &= b \\ e^{at}y &= bt + c \\ y &= (bt + c)e^{-at} \end{aligned}$$

Some plots are shown below. It appears that $y \to 0$ as $t \to \infty$ just as when $a \neq \lambda$.
Total marks for Q7: 4