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1(a) $y = 1 - \frac{1}{2}t + \frac{c}{t}$

(b) $y = \frac{-1}{\sqrt{5 - e^{2t}}}$

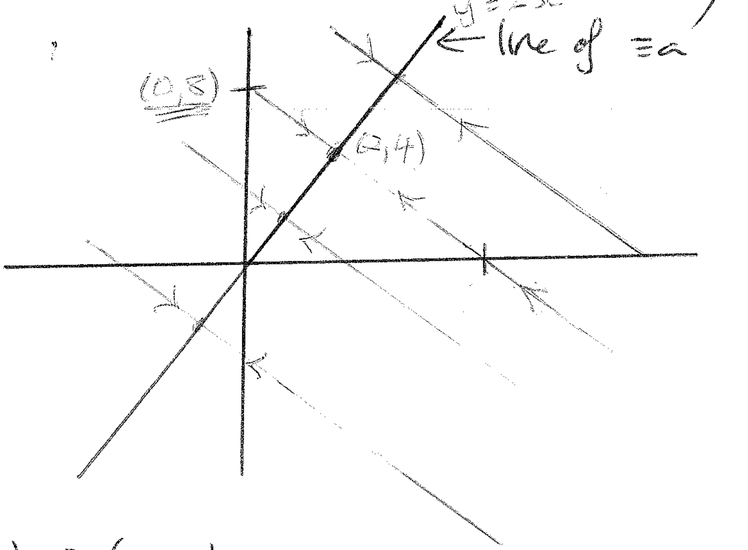
2 (a) —

(b) $y_2 = 0.547565$

(c) -

3 (a) (i) $y(t) = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{-4t}\begin{pmatrix}1\\-2\end{pmatrix}$

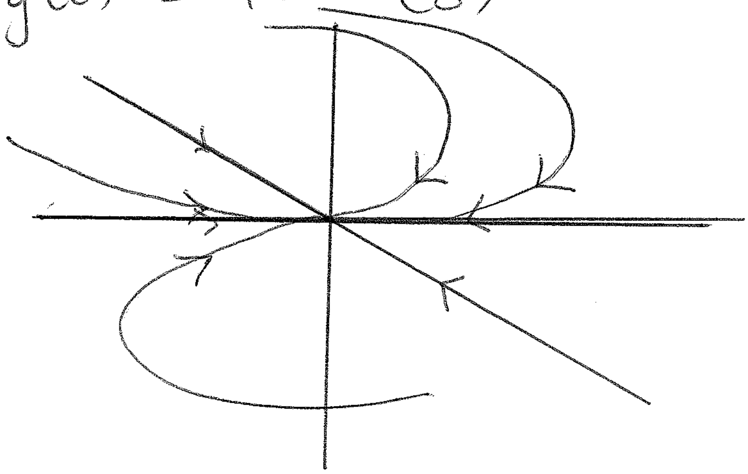
(ii)



(iii) → (2,4)

(b) (i) $y(t) = c_1 e^{-2t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{-3t}\begin{pmatrix}1\\-1\end{pmatrix}$

(ii)

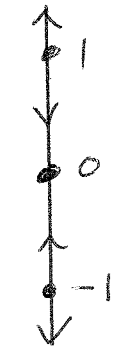


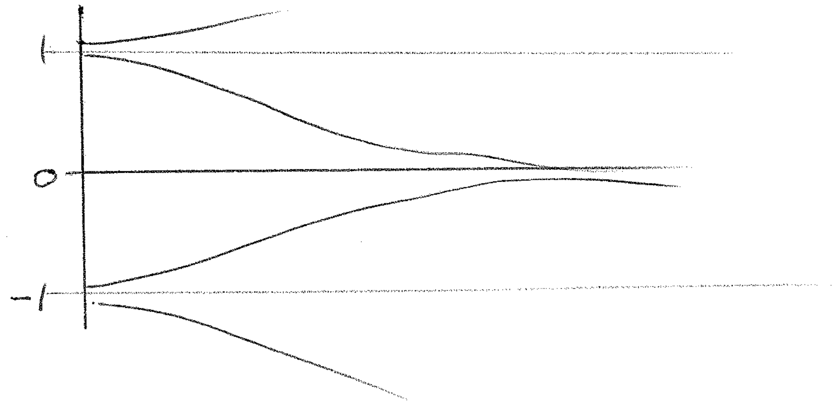
(iii) → (0,0)

4(a) $y=0$ source for $\lambda>0$, sink $\lambda<0$, source $\lambda=0$

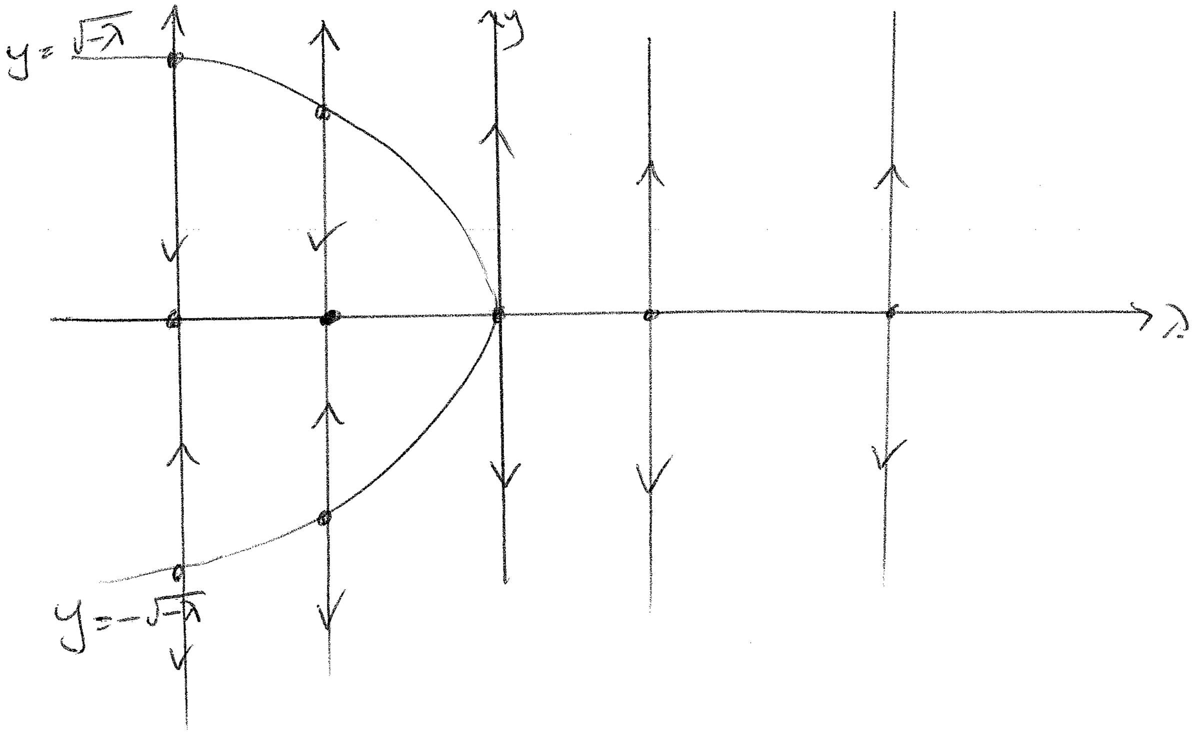
$y=\pm\sqrt{-\lambda}$ source, exists for $\lambda<0$.

(b)





(c) Bifurcation at $\lambda=0$



5(a) Competing etc

(b) $(0,0)$, $(0,17)$, $(10,0)$, $(6,8)$

(c) $$J = \begin{pmatrix} 1-.2x-.05y & -.05x \\ -.15y & 1.7-.2y-.15x \end{pmatrix}$$

$(0,0)$ real source

$(0,17)$ saddle

$(10,0)$ saddle

$(6,8)$ real sink.

260 055 ~~old~~

5(d) —

(e) Both decrease then x decrease + y increase

→ (6, 8) ie 6000 + 8000

6. (a) $y(t) = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin 2t$

$+ \frac{1}{8}\sin t - \frac{1}{8}\cos t$

(b) $\frac{1}{8}\sin t - \frac{1}{8}\cos t$

7. $y(t) = 3e^{-t} + 2te^{-t} + t - 2$

8. N/A.

~~260 05S~~

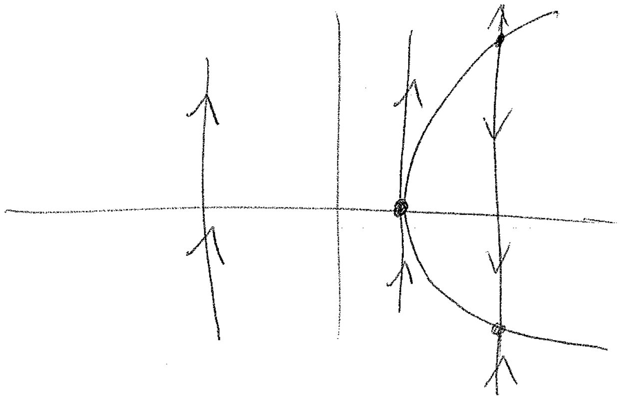
260 06F

1 (a) $y(t) = -\frac{1}{4}e^{-3t} + ce^{t}$

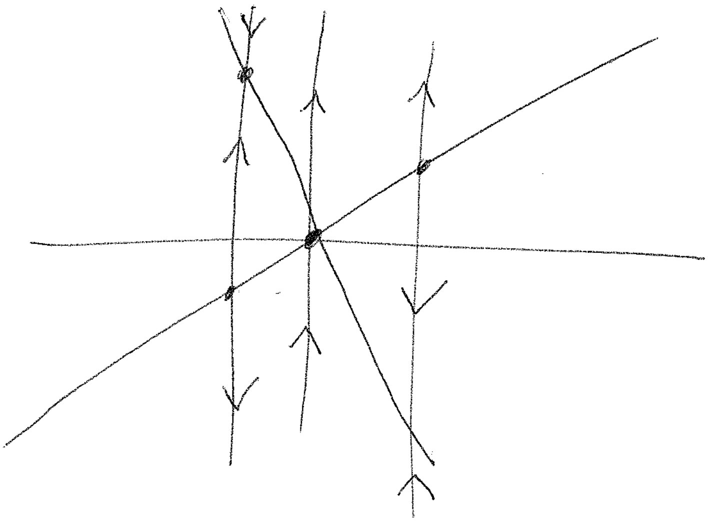
$y = -\frac{1}{4}e^{-3t} + \frac{5}{4}e^{t}$

(b) $y(t) = \frac{-1}{\frac{1}{2}t^2 + \frac{1}{3}t^3 - 1}$

2 (a) Bifurcation at 1



(b) Bifurcation at 0



3 (a) —

(b) $h=1$ 1.75, $h=0.5$ 1.7347

(c) $h=1$ 1.8×10^{-2}, $h=0.5$ 2.6×10^{-3}

(d) 6.6×10^{-4}

4. (a) $Y(t) = c_1 e^{t}\begin{pmatrix}-2\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}$

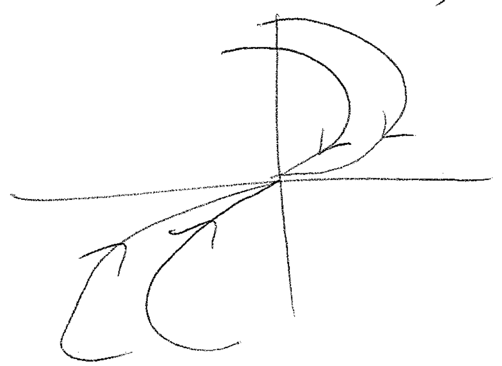
saddle $\to \infty$ as $t \to \infty$ Draw your own

(b) $Y(t) = c_1 e^{t}\begin{pmatrix}\cos t\\-\sin t\end{pmatrix} + c_2 e^{t}\begin{pmatrix}\sin t\\\cos t\end{pmatrix}$

Spiral source clockwise (draw it)

$\to \infty$ as $t \to \infty$

(c) $Y(t) = c_1 e^{-t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{-t}\left(t\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right)$



5. (a) Em solns are $(0,a)$ and $(1-a,1)$

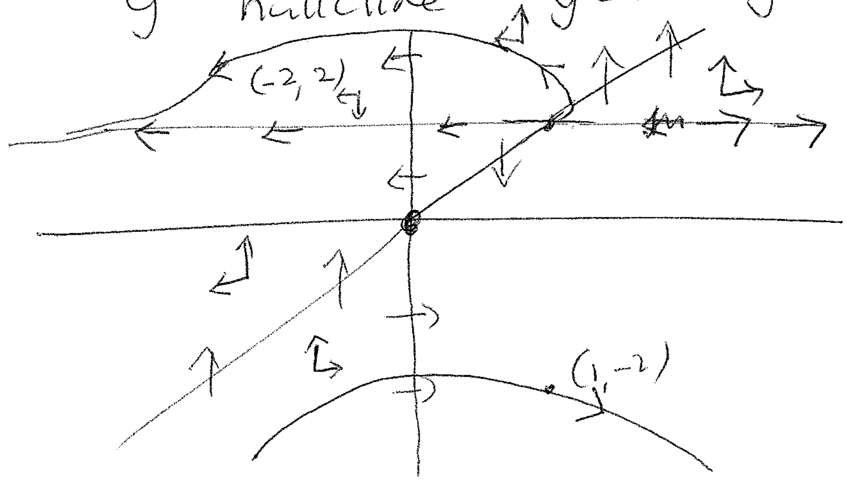
$(0,a)$
- $a<1$ saddle
- $a=1$ line of Em
- $1<a<\frac{5}{4}$ real source
- $a=\frac{5}{4}$ (repeated root) source
- $a>\frac{5}{4}$ (complex) spiral source

(b) $(1-a,1) = (1,1)$

source (repeated root $\lambda = 1$)

x nullcline $y=x$

y nullcline $y=1$ or ~~y~~ $x=0$



6. (a) $y(t) = c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4}$

(b) $y(t) = -e^{t}\sin t + e^{t}$

$\cancel{7.\ y(t) = 3e^{-t} + 2te^{-t} + t + 2}$

260 06F

1. (a) $y' = y + e^{-3t}$ $\qquad y(0) = 1$

$y' - y = e^{-3t}$

$\mu(t) = e^{-t}$

$e^{-t}y' - e^{-t}y = e^{-4t}$

$\frac{d}{dt}(e^{-t}y) = e^{-4t}$

$e^{-t}y = -\frac{1}{4}e^{-4t} + c$

$y = -\frac{1}{4}e^{-3t} + ce^{t}$

$y(0) = 1$

$1 = -\frac{1}{4} + c \Rightarrow c = \frac{5}{4}$

$y(t) = -\frac{1}{4}e^{-3t} + \frac{5}{4}e^{t}$

(b) $\frac{dy}{dt} = y^2 t(1 + t)$ $\qquad y(0) = 1$

$\int \frac{dy}{y^2} = \int (t + t^2)\,dt$

$-\frac{1}{y} = \frac{1}{2}t^2 + \frac{1}{3}t^3 + c$

$y = \frac{-1}{\frac{1}{2}t^2 + \frac{1}{3}t^3 + c}$

$1 = \frac{-1}{c} \Rightarrow c = -1$

$y = \frac{-1}{\frac{1}{2}t^2 + \frac{1}{3}t^3 - 1}$

2(a) $y^2 - \mu + 1 = 0$

$y = \pm\sqrt{\mu - 1}$ provided $\mu \geq 1$ Bifurcation at $\mu = 1$

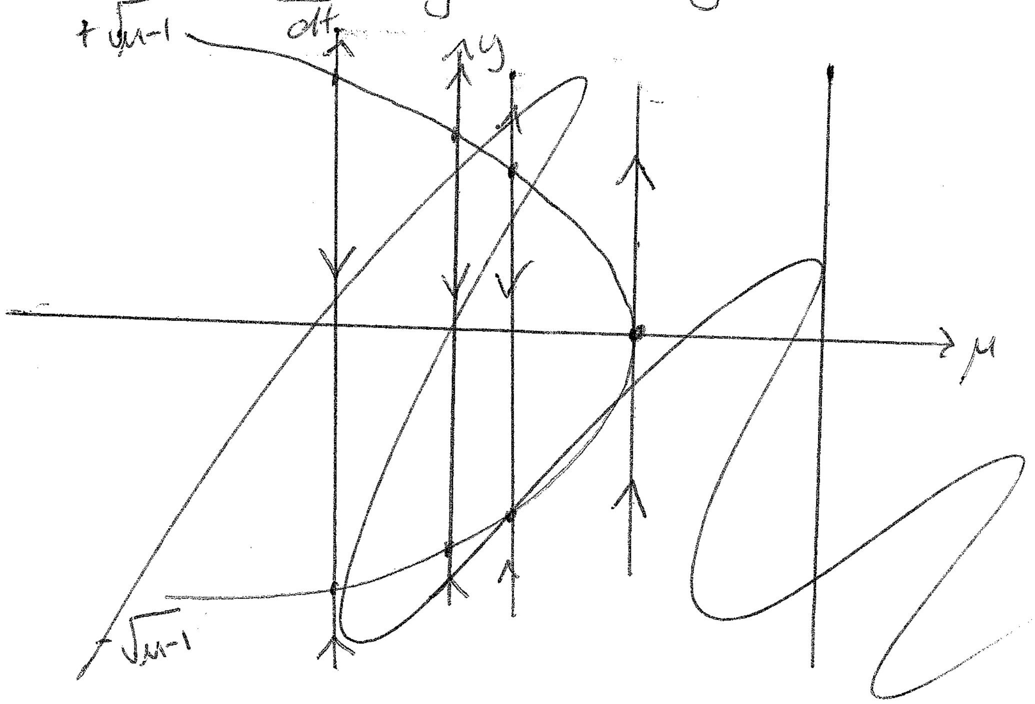
$\frac{\partial f}{\partial y} = 2y$

$\left.\frac{\partial f}{\partial y}\right|_{+\sqrt{\mu-1}} > 0$ $\mu > 1$ source

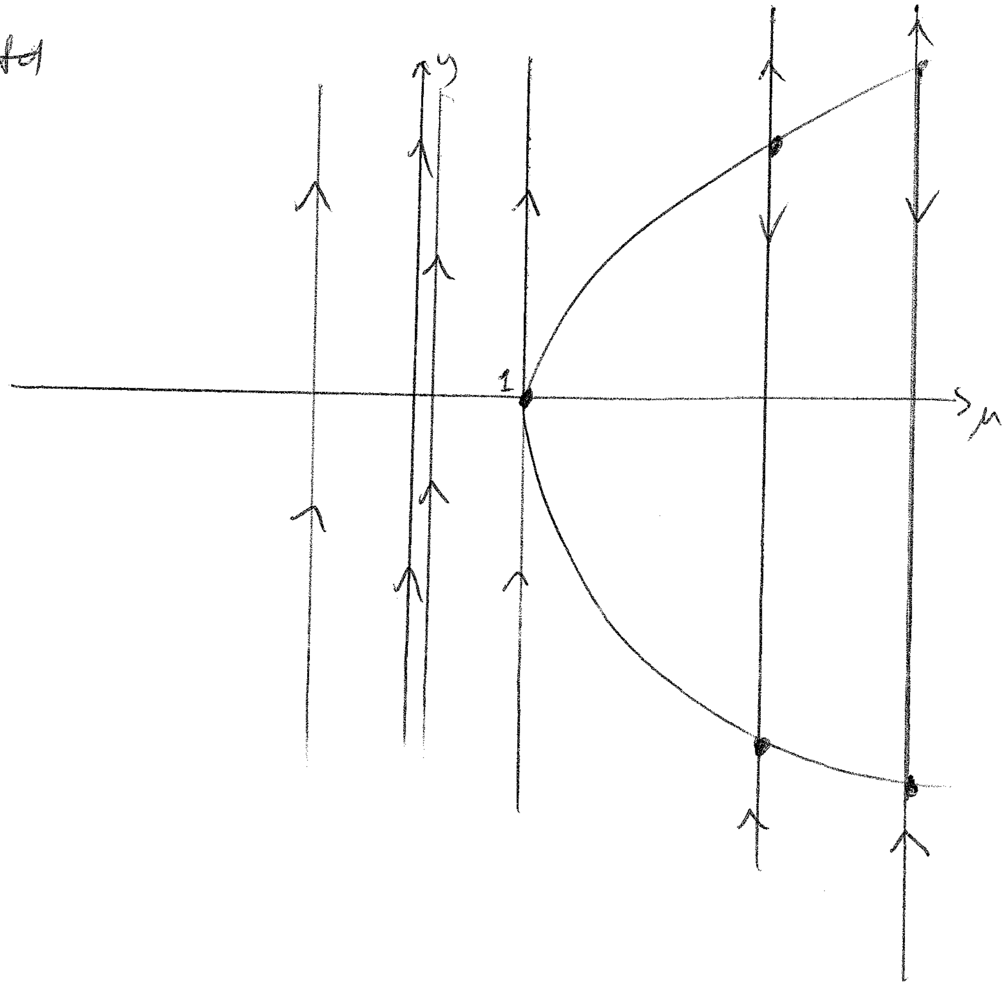
$\left.\frac{\partial f}{\partial y}\right|_{-\sqrt{\mu-1}} < 0$ $\mu > 1$ sink

For $\mu = 1$, $y = 0$ is $\equiv$m.

$\frac{dy}{dt} = y^2 \geq 0 \;\; \forall y$ $\therefore$ node



2(a) ctd



(b) $y' = y^2 + \mu y - 2\mu^2$

$y^2 + \mu y - 2\mu^2 = 0$

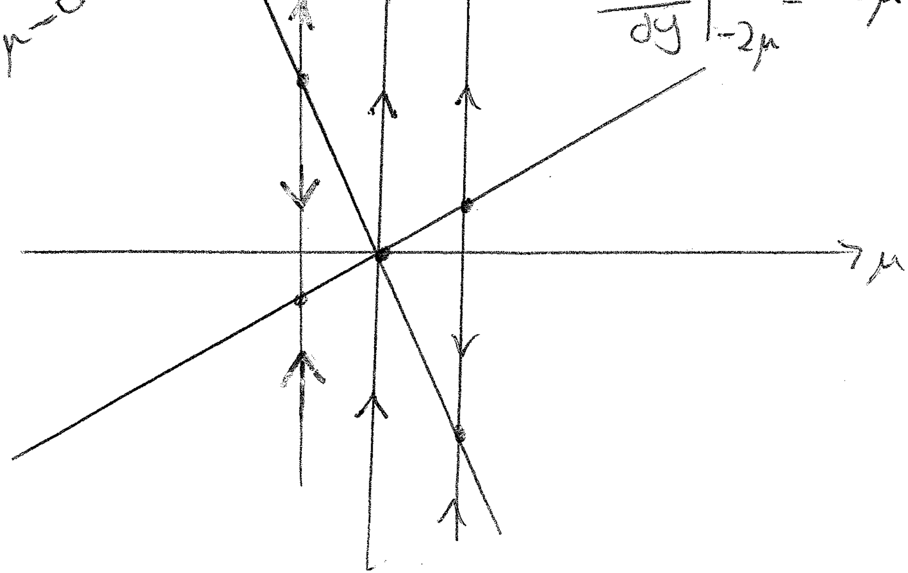
$$y = \frac{-\mu \pm \sqrt{\mu^2 + 8\mu^2}}{2} = \frac{-\mu \pm 3\mu}{2}$$

$$= \mu, -2\mu$$

$\frac{\partial f}{\partial y} = 2y + \mu \qquad \left.\frac{\partial f}{\partial y}\right|_{\mu} = 3\mu$ source for $\mu > 0$, sink $\mu < 0$

$\left.\frac{\partial f}{\partial y}\right|_{-2\mu} = -3\mu$ sink for $\mu > 0$, source for $\mu < 0$.

Bifurcation at $\mu = 0$



$f(y)\big|_{\mu=0} = y^2 \geq 0$

node.

3. (a) $\frac{dy}{dt} = -\frac{t}{y}$ $y(0) = 2$

$$\int y\,dy = -\int t\,dt$$

$$\frac{1}{2}y^2 = -\frac{1}{2}t^2 + c$$

$y(0)=2$ $2 = c$

$$y^2 = -t^2 + 4$$

$$y = \sqrt{-t^2+4}$$

Since $y(0) = 2$, use $+\sqrt{\ }$

(b) (i) $t_0=0, t=1$ $y_0 = 2$ $t_0 = 0$, $h=1$, $t_1 = 1$ $f(t,y) = -\frac{t}{y}$

$$m_1 = f(t_0, y_0) = f(0,2) = 0$$

$$m_2 = f(t_0+h, y_0 + hm_1)$$

$$= f(1, 2 + 1\times 0) = f(1,2) = -\frac{1}{2}.$$

$$y_1 = y_0 + \frac{h}{2}(m_1 + m_2)$$

$$= 2 + \frac{1}{2}\left(0 - \frac{1}{2}\right) = \underline{\underline{1\tfrac{3}{4}}}$$

(ii) $t_0 = 0$, $y_0 = 2$, $h = 0.5$.

$$m_1 = f(0,2) = 0$$

$$m_2 = f(0.5, 2 + 0.5\times 0) = f(.5, 2) = -.25$$

$$y_1 = 2 + \frac{.5}{2}(0 - .25) = 1.9375$$

$$m_1 = f(0.5, 1.9375) = -.2581$$

$$m_2 = f(1, 1.9375 + .5\times -.2581)$$

$$= f(1, 1.8085) = -.5530$$

$$y_2 = y_1 + \frac{h}{2}(m_1 + m_2)$$

$$= \cancel{1.75}\ 1.9375 + \frac{0.5}{2}(-.2581 - .5530)$$

$$= \cancel{1.5472}\ 1.7347.$$

(c) ~~Error, h=1~~

Exact soln $= \sqrt{-1^2+4} = \sqrt{3}$

<u>Error h=1</u>

$1.75 - \sqrt{3} \approx 1.8 \times 10^{-2}$

<u>Error h=.5</u>

$1.7347 - \sqrt{3} = 2.6 \times 10^{-3}$

(d) $E(h) \approx Ch^2$

$E\left(\frac{h}{2}\right) \approx C\left(\frac{h}{2}\right)^2 = \frac{Ch^2}{4}.$

$\therefore$ Est error at $h = .25$ as 6.6×10^{-4}

4. (a) $\begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$

$\det \begin{pmatrix} 3-\lambda & 4 \\ -2 & -3-\lambda \end{pmatrix} = (3-\lambda)(-3-\lambda) + 8$

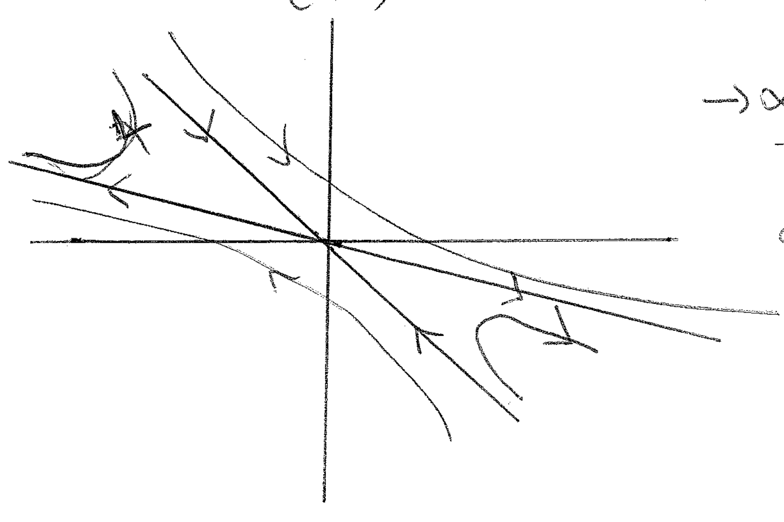
$= \lambda^2 - 1$

$\lambda = \pm 1$

<u>$\lambda = 1$</u> $\begin{pmatrix} 2 & 4 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $2u + 4v = 0$, $u = -2v$ $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

<u>$\lambda = -1$</u> $\begin{pmatrix} 4 & 4 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $u = -v$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$y(t) = c_1 e^{t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$



$\to \infty$ as $t \to \infty$ for $c_1 \neq 0$

$c_1 = 0 \Rightarrow$ origin

4 (b) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2+1 = \lambda^2-2\lambda+2$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= 1 \pm i$$

~~[illegible]~~

$\underline{\lambda = 1+i}$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0 \qquad iu = v \qquad \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$e^{(1+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^t(\cos t + i\sin t)\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} e^t\cos t + ie^t\sin t \\ -e^t\sin t + ie^t\cos t \end{pmatrix}$$

$$Y(t) = c_1e^t\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2e^t\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

spiral source

$\to \infty$ oscillating as $t \to \infty$.

(c) $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ eigenvalues are $-1, -1$

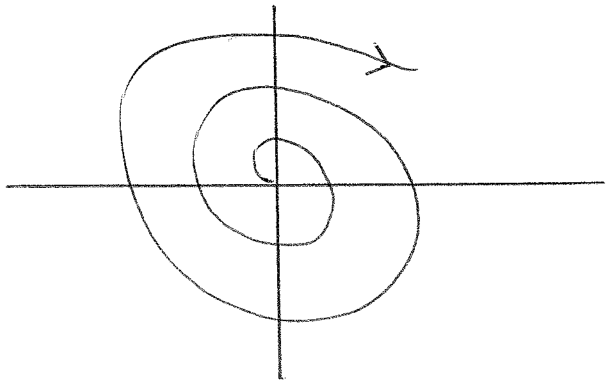
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0 \Rightarrow v = 0 \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Gen elv $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad v_1 = 1$

$$y(t) = c_1e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2e^{-t}\left(t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

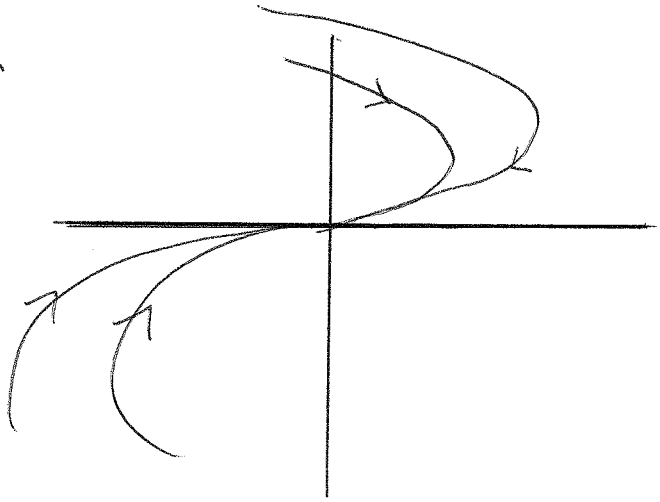
4 (b) again

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



(c) again

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



5.

$$x - y + a = 0$$
$$(y-1)x = 0$$

$x = 0 \Rightarrow y = a$ $\qquad (0, a)$

$y = 1 \Rightarrow x = 1-a$ $\qquad (1-a, 1)$

$$J = \begin{pmatrix} 1 & -1 \\ y-1 & x \end{pmatrix}$$

$$J(0, a) = \begin{pmatrix} 1 & -1 \\ a-1 & 0 \end{pmatrix}$$

$$\det\begin{pmatrix} 1-\lambda & -1 \\ a-1 & -\lambda \end{pmatrix} = -\lambda(1-\lambda) + a-1 = \lambda^2 - \lambda + a - 1$$

$$\lambda = \frac{1 \pm \sqrt{5-4a}}{2}$$

For $a < \phi$, saddle.
$a = \phi$, ~~line of =a~~
$\phi < a < \frac{5}{4}$ real source
$a = \frac{5}{4}$ repeated root source
$a > \frac{5}{4}$ complex source.

(b)

Em solns $(0,0)$ and $(1,1)$

(i) $J = \begin{pmatrix} 1 & -1 \\ y-1 & x \end{pmatrix}$ $J(0,0) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

cannot tell from lin thm

$J(1,1) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ $\lambda = 1, 1$ Repeated root source.

$\det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2$

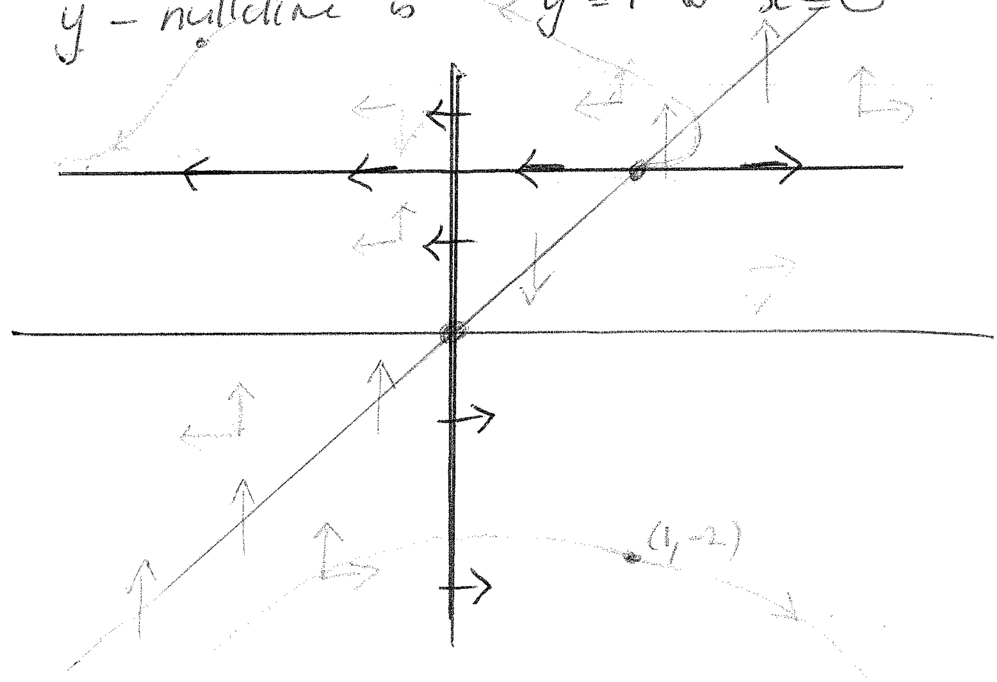
$\lambda = \dfrac{2 \pm \sqrt{4-8}}{2}$

complex source spiral.

(ii) ~~Null~~

x-null cline is $x - y = 0$ ie $y = x$ /

y - nullcline is $y = 1$ or $x = 0$



6. (a) $\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+2)(\lambda+1) = 0$

$\lambda = -1, -2$

$c_1 e^{-t} + c_2 e^{-2t}$.

UC set $= \{t^2, t, 1\}$

$y_p = k_1 t^2 + k_2 t + k_3$

$y_p' = 2k_1 t + k_2$

$y_p'' = 2k_1$

6(a) ctd

LHS $= 2k_1 + 6k_1 t + 3k_2 + 2k_1 t^2 + 2k_2 t + 2k_3$

$= 2k_1 t^2 + (6k_1 + 2k_2)t + 2k_1 + 3k_2 + 2k_3$

RHS $= t^2$

$2k_1 = 1 \Rightarrow k_1 = \frac{1}{2}$

$6k_1 + 2k_2 = 0 \Rightarrow 3 + 2k_2 = 0 \Rightarrow k_2 = -\frac{3}{2}$

$2k_1 + 3k_2 + 2k_3 = 0 \Rightarrow k_3 = -\frac{1}{2}(1 - \frac{9}{2}) = \frac{7}{4}$

$y_p = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4}$

$y(t) = c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4}$.

(b) $\lambda^2 - 2\lambda + 2 = 0$

$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$c_1 e^t \cos t + c_2 e^t \sin t$

UC set $\{e^t\}$

$y_p = ke^t \quad y_p' = ke^t \quad y_p'' = ke^t$

LHS $= ke^t - 2ke^t + 2ke^t = ke^t$

RHS $= e^t \Rightarrow k = 1$

$y(t) = \cancel{c_1 e^{-t} + c_2 e}$

$= c_1 e^t \cos t + c_2 e^t \sin t + e^t$

~~$y'(t)$~~ $y(0) = 1 \qquad c_1 + 1 = 1 \Rightarrow c_1 = 0$

$y(t) = c_2 e^t \sin t + e^t$

$y' = c_2 e^t \sin t + c_2 e^t \cos t + e^t$

$y'(0) = c_2 + 1 = 0 \Rightarrow c_2 = -1$

$y(t) = -e^t \sin t + e^t$