Tuesday: Existence proofs and counterexamples

Existence proofs [1.9]

To prove something of the form “there is an \(x\) such that \(A(x)\), we do two steps:

- produce a suitable value of \(x\) (like pulling a rabbit from a hat)
- show that that particular value of \(x\) does what is claimed.

**Example 1.** Show that there is some \(x \in \mathbb{R}\) such that \(x^2 + 12x - 85 = 0\).

**Proof.** Let \(x = 5\). Then \(x^2 + 12x - 85 = 5^2 + 12 \cdot 5 - 85 = 25 + 60 - 85 = 0\), as required. \(\square\)

Uniqueness proofs [1.10]

To prove that there is at most one \(x\) with the property \(A(x)\), we suppose that we have two objects \(x\) and \(y\) with \(A(x)\) and \(A(y)\), and deduce that \(x = y\).

**Lemma 2.** If \(x, y \in \mathbb{R}\) with \(x^2 + xy + y^2 = 0\) then \(x = y = 0\).

**Proof.** Exercise. Hint: \(x^2 + xy + y^2 = \frac{1}{4}(x + y)^2 + \frac{1}{4}(x - y)^2\). \(\square\)

**Example 3.** Cube roots are unique, in other words if \(r\) is a real number then there is at most one \(x \in \mathbb{R}\) with \(x^3 = r\).

**Proof.** Suppose that \(x, y \in \mathbb{R}\) with \(x^3 = r\) and \(y^3 = r\). Then \(x^3 - y^3 = r - r = 0\), and \(x^3 - y^3 = (x - y)(x^2 + xy + y^2)\). Now, if \(a, b \in \mathbb{R}\) with \(ab = 0\) then \(a = 0\) or \(b = 0\), so \(x - y = 0\) or \(x^2 + xy + y^2 = 0\). Now if \(x - y = 0\) then \(x = y\), and if \(x^2 + xy + y^2 = 0\) then \(x = y = 0\), by the Lemma. So \(\square\)

Examples and counterexamples [1.11]

Remember when we want to prove an implication \(A(x) \implies B(x)\), we are really proving the statement \((\forall x)(A(x) \implies B(x))\). To show that the implication is not a theorem, we are proving \(\sim(\forall x)(A(x) \implies B(x))\), i.e. \((\exists x)(A(x) \land \sim B(x))\). So what we have to do is give an existence proof. Again, we find an object \(x\) and then demonstrate that it has the properties \(A(x)\) and \(\sim B(x)\). Such an object is called a counterexample to the implication \(A(x) \implies B(x)\).

Example: Exercise 1.11.1
Wednesday: Sets, subsets, set equality

Sets and Set notation [2.1]

A \textit{set} is a collection of objects. We write \(x \in A\) if the object \(x\) is in the set, otherwise \(x \notin A\). We can specify a set in three ways:

- enumerate the elements, e.g. \(X = \{1, 2, 3\}, Y = \{1, 3, 5, \ldots, 17\}, N = \{1, 2, 3, \ldots\}\).
- use set builder notation, e.g. \(X = \{x \in N : 1 \leq x \leq 3\}, Y = \{n \in N : n \text{ is odd and } 1 \leq n \leq 17\}, N = \{x : x \text{ is a natural number}\}\).
- use an indexing set, e.g. \(Y = \{2n - 1 : n \in \{1, 2, \ldots, 9\}\}\).

Some sets are so important they have their own names, e.g. \(N, Z, R, Q\) and intervals such as \([a,b], [a,b), (a,b)\) and \((-\infty, b)\). One other set with a name: the \textit{empty set} \(\emptyset\).

Subsets [2.2]

A \textit{subset} of a set \(A\) is a set \(S\) with the property that every element of \(S\) is also an element of \(A\). We write \(S \subseteq A\).

Examples: \(N \subseteq Z, Q \subseteq R\). For any set \(X\), \(\emptyset \subseteq X\) and \(X \subseteq X\).

Important: do not mix up \(x \in A\) and \(x \subseteq A\).

Notice that \(S \subseteq A\) is an implication: “if \(x \in S\) then \(x \in A\)”.

Exercise 2.2.4.

A \textit{proper subset} of a set \(A\) is a set \(S\) with \(S \subseteq A\) and \(S \neq A\). We will sometimes write \(S \subset A\) in this case. Warning: some books use \(S \subset A\) to mean \(S\) is a subset of \(A\), not necessarily a proper subset of \(S\).

To say that two sets \(A\) and \(B\) are equal is to say that they have exactly the same elements, i.e. that \(A \subseteq B\) and \(B \subseteq A\). So to prove that two sets are equal, we have to prove two implications.

Example: to show that \(\{x \in \mathbb{R} : x^2 + 12x - 85 = 0\} = \{5, -17\}\) we have to prove two implications:

- if \(x \in \mathbb{R}\) with \(x^2 + 12x - 85 = 0\) then \(x = 5\) or \(x = -17\); and
- if \(x = 5\) or \(x = -17\) then \(x \in \mathbb{R}\) with \(x^2 + 12x - 85 = 0\).

Thursday: Set operations

Complement, intersection and union [2.3]

Given a set \(U\) (which we call a \textit{universal set}) and a set \(S \subseteq U\), we define the \textit{complement} of \(S\) in \(U\) to be \(S^C_U\). If \(U\) is fixed and understood, we may simply write \(S^C\) and refer to the \textit{complement} of \(S\).
Example (exercise 2.3.2 and 2.3.3). Put \( S = [-5, 2] \), \( U = [-5, 5] \). Find \( S_C \) and \( S_C^C \).

Definition: if \( A \) and \( B \) are sets then the **intersection** of \( A \) and \( B \) is \( A \cap B = \{ x : x \in A \land x \in B \} \) and the **union** of \( A \) and \( B \) is \( \{ x : x \in A \lor x \in B \} \).

Example (exercise 2.3.5): let \( A = \{ a, b, c, d, e, f, g \} \), \( B = \{ a, e, i, o, u \} \). Find \( A \cap B \) and \( A \cup B \).

We may use **Venn diagrams** to illustrate these.

**Set identities [2.4]**

Recall that to show that two sets are equal we have to prove two implications.

**Example 4.** Let \( A \) and \( B \) be sets. Show that \( A \cap (A \cup B) = A \).

**Proof.** Let \( x \in A \cap (A \cup B) \). Then \( x \in A \) and \( x \in A \cup B \). So \( x \in A \).

Conversely, let \( y \in A \). Then \( y \in A \cap (A \cup B) \).

**Example (Theorem 2.4.2):** for any sets \( A, B \) and \( C \) we have \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \).

**Set operations with indexing sets**

Suppose we have a set \( \Lambda \), and for each \( \alpha \in \Lambda \) we have a set \( U_\alpha \). Then we may form the union of all these sets and (provided \( \Lambda \neq \emptyset \)) the intersection of all these sets. We define the union to be

\[
\bigcup_{\alpha \in \Lambda} U_\alpha = \{ x : x \in U_\alpha \text{ for at least one } \alpha \in \Lambda \}
\]

and the intersection to be

\[
\bigcap_{\alpha \in \Lambda} U_\alpha = \{ x : x \in U_\alpha \text{ for every } \alpha \in \Lambda \}.
\]

Example: for each \( n \in N \) let \( I_n = [0, \frac{1}{n}] \). Find \( \bigcap_{n \in N} I_n \) and \( \bigcup_{n \in N} I_n \).

Example: find \( \bigcap_{n \in \mathbb{Z}} [n, n+1] \) and \( \bigcup_{n \in \mathbb{N}} [n, n+1] \).

**Friday: The power set**

Exercise: list all the subsets of \( \{1, 2, 3\} \).

The collection of all subsets of a set \( A \) is called the **power set** of \( A \), written \( \mathcal{P}(A) \). So we have \( S \in \mathcal{P}(A) \) if and only if \( S \subseteq A \).

**Example 5 (Theorem 2.5.4).** Show that if \( A \) and \( B \) are sets then \( A \subseteq B \) if and only if \( \mathcal{P}(A) \subseteq \mathcal{P}(B) \).

**Example 6 (Theorem 2.5.5).** Let \( A \) and \( B \) be sets. Show that \( \mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B) \).

**Example 7.** Let \( A \) and \( B \) be sets. Show that \( \mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B) \). Find an example of sets \( A \) and \( B \) such that \( \mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B) \).