1. (a) (**5 marks**) Since S is bounded, there are real numbers a, b such that for any $x \in S$, $a \leq x \leq b$. Now T is a subset of S, each element y of T is also an element of S, so that $a \leq y \leq b$ and T is bounded below by a and above by b.

(b) (**4 marks**) Since T and S are bounded below and above, it follows that glb S, glb T, lub S and lub T exist.

(**8 marks**) Let L_X and U_X be the set of lower bounds and upper bounds of a set X. As shown in (a) above, $L_S \subseteq L_T$ and $U_S \subseteq U_T$. By definition, glb S is the greatest element of L_S, in particular, glb $S \in L_S$ and so glb $S \in L_T$. But glb T is the greatest element of L_T, so glb $S \leq$ glb T. Similarly, lub $S \in U_T$ and lub T is the least element of U_T, so lub $T \leq$ lub S. For any $y \in T$, glb $T \leq y \leq$ lub T. It follows that

$$\text{glb } S \leq \text{glb } T \text{lub } T \leq \text{lub } S.$$

2. (**10 marks**) Suppose c is a least element of S, where $S = (a, b)$. Then $c \in S$ and so $a < c < b$. If $z = \frac{c+a}{2}$, then

$$a = (\frac{a}{2} + \frac{a}{2}) < z = \frac{c+a}{2} < (\frac{b}{2} + \frac{b}{2}) = b,$$

so that $z \in S$. But $z < \frac{c+c}{2} = c$, so c is not a least element of S, which is impossible.

3. (a) (**10 marks**)

$$\begin{aligned} a_{n+1} - a_n &= \frac{3}{1 + 5^{-n-1}} - \frac{3}{1 + 5^{-n}} \\ &= \frac{3((1 + 5^{-n} - (1 + 5^{-n-1}))}{(1 + 5^{-n-1})(1 + 5^{-n})} \\ &= \frac{3(5^{-n} - 5^{-n-1})}{(1 + 5^{-n-1})(1 + 5^{-n})} \\ &= \frac{3 \cdot 5^{-n-1}(5 - 1)}{(1 + 5^{-n-1})(1 + 5^{-n})} \\ &= \frac{12 \cdot 5^{-n-1}}{(1 + 5^{-n-1})(1 + 5^{-n})} \\ &> 0. \end{aligned}$$

Thus $\{a_n\}$ is increasing. Since $a_1 = \frac{3}{1+5^{-1}} \leq a_n < 3$, it follows that $\{a_n\}$ is bounded.

(b) (**5 marks**) $\text{glb}\{a_n\} = a_1 = \frac{5}{2}$ and $\text{lub}\{a_n\} = 3$. In addition, $\frac{5}{2} = a_1 \in \{a_n\}$ and $3 \notin \{a_n\}$.

(c) (**8 marks**) For any $\epsilon > 0$, there is an natural number N such that $N > \frac{3}{\epsilon} - 1$. If $n \geq N$, then

$$\begin{aligned}
\left|\frac{3}{1+5^{-n}} - 3\right| &= 3\left|\frac{1}{1+5^{-n}} - 1\right| \\
&= 3\frac{5^{-n}}{1+5^{-n}} \\
&= \frac{3}{5^n + 1} \\
&< \frac{3}{1+2^n} \\
&< \frac{3}{1+n} \\
&\leq \frac{3}{1+N} \\
&\leq \frac{3}{1+(\frac{3}{\epsilon}-1)} \\
&= \epsilon.
\end{aligned}$$

By definition, $\lim_{n\to\infty} a_n = 3$.