1. (**4 marks**) For $n \in \mathbb{N}$ let P_n be the statement that $1 \cdot 2 + 2 \cdot 3 + \dots n(n+1) = \frac{1}{3}n(n+1)(n+2)$.

Base: (**3 marks**) P_1 is the statement that $1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3}$, which is true because $\frac{1 \cdot 2 \cdot 3}{3} = 2 = 1 \cdot 2$.

Inductive step: (**6 marks**) Suppose $k \in \mathbb{N}$ and P_k is true, that is,

$$1 \cdot 2 + 2 \cdot 3 + \dots k(k+1) = \frac{1}{3}k(k+1)(k+2).$$

Then

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + \dots k(k+1) + (k+1)(k+2) &= [1 \cdot 2 + 2 \cdot 3 + \dots k(k+1)] + (k+1)(k+2) \\ &= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \\ &= (k+1)(k+2)(\frac{1}{3}k + 1) \\ &= \frac{1}{3}(k+1)(k+2)(k+3), \end{aligned}$$

so $1 \cdot 2 + 2 \cdot 3 + \dots k(k+1) + (k+1)(k+2) = \frac{1}{3}(k+1)(k+2)(k+3)$, in other words P_{k+1} is true.

(**2 marks**) Hence, by induction, P_n is true for all $n \in \mathbb{N}$.

2. (**3 marks**) For $n \in \mathbb{N}$ let P_n be the statement that $W(n)$ is well-ordered.

Base: (**2 marks**) P_1 is the statement that $W(1) = \mathbb{N} \cup \{0\}$ is well-ordered, which is true as shown in the class.

Inductive step: (**4 marks**) Suppose $k \in \mathbb{N}$ and P_k is true, that is, $W(k)$ is well-ordered. Then $W(k+1) = \{-k\} \cup W(k)$. Let K be a subset of $W(k+1)$ with $K \neq \emptyset$.

If $-k \in K$, then $-k$ is its least element.

If $-k \notin K$, then $K \subseteq W(k)$ and by induction assumption, K also has a least element. It follows that P_{k+1} is true.

(**1 mark**) Hence, by induction, P_n is true for all $n \in \mathbb{N}$.

3. (**3 marks**) For $n \in \mathbb{N}$ let P_n be the statement that $7 \mid 3^{4n+1} + 4^{n+1}$.

Base: (**2 marks**) P_1 is the statement that $7 \mid 3^5 + 4^2$, which is true because $3^5 + 4^2 = 259 = 7 \cdot 37$.

Inductive step: (**8 marks**) Suppose $k \in \mathbb{N}$ and P_k is true, in other words there is some $x \in \mathbb{Z}$ such that $3^{4k+1} + 4^{k+1} = 7x$, so $3^{4k+1} = 7x - 4^{k+1}$. Then

$$\begin{aligned} 3^{4(k+1)+1} + 4^{(k+1)+1} &= 3^{4k+1+4} + 4^{k+1+1} \\ &= 3^{4k+1} \cdot 3^4 + 4^{k+1} \cdot 3 \\ &= (7x - 4^{k+1})3^4 + 4 \cdot 4^{k+1} \\ &= 81 \cdot 7x - 81 \cdot 4^{k+1} + 4 \cdot 4^{k+1} \\ &= 81 \cdot 7x - 77 \cdot 4^{k+1} \\ &= 7(81x - 11 \cdot 4^{k+1}), \end{aligned}$$

so $7 \mid 3^{4(k+1)+1} + 4^{(k+1)+1}$, in other words P_{k+1} is true.

(**2 marks**) Hence, by induction, P_n is true for all $n \in \mathbb{N}$.

4. (**7 marks**) Euclidean Algorithm gives the following results:

3598	1	0	
1603	0	1	
392	1	-2	$r_2 - 2r_1$
35	-4	9	$r_2 - 4r_3$
7	45	-101	$r_3 - 11r_4$
0	-229	514	$r_4 - 5r_5$

(**3 marks**) Thus $\gcd(3598, 1603) = 7$ and $7 = 45 \times 3598 + (-101) \times 1603$.