1. (a) (**6 marks**)

Since $g \circ f = \{(1, a), (2, d), (3, b)\}$ and since a, b, d are distinct, it follows that $g \circ f$ is one-to-one. Since $f(1) = 4, f(2) = 6$ and $f(3) = 8$, it follows that f is also one-to-one. But g is not one to one since $g(5) = g(8) = b$.

(b) (**9 marks**)

Suppose $w \circ h$ is one-to-one and $h(x_1) = h(x_2)$ for some $x_1, x_2 \in X$. Then $w \circ h(x_1) = w(h(x_1)) = w(h(x_2)) = w \circ h(x_2)$. But $w \circ h$ is one-to-one, so $x_1 = x_2$. We have proved that if $h(x_1) = h(x_2)$ for some $x_1, x_2 \in X$, then $x_1 = x_2$, that is, h is one-to-one.

2. (a) (**4 marks**)

Since $f(0) = 0$ and $f(4) = 16$, it follows that $f(A_1) = f((0, 4]) = (0, 16]$.

$f(A_2) = f(\{-2, -1, 1, 3, 10\}) = \{1, 4, 9, 100\}$.

$A_1 \cap A_2 = \{1, 3\}$ and $f(A_1 \cap A_2) = f(\{1, 3\}) = \{1, 9\}$

(b) (**3 marks**)

$f(A_1) \cap f(A_2) = (0, 16] \cap \{1, 4, 9, 100\} = \{1, 4, 9\}$, so $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. But $4 \notin f(A_1 \cap A_2)$, so $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$. Thus $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.

(c) (**5 marks**)

$f^{-1}(B_1) = f^{-1}(\{-1, 1, 4, 100\}) = \{-10, -1, 1, -2, 2, 10\}$.

$f^{-1}(B_2) = f^{-1}((0, 16)) = (-4, 0) \cup (0, 4)$.

$B_1 \cap B_2 = \{1, 4\}$ and $f^{-1}(B_1 \cap B_2) = f^{-1}(\{1, 4\}) = \{-1, 1, -2, 2\}$.

(d) (**3 marks**)

$f^{-1}(B_1) \cap f^{-1}(B_2) = \{-10, -1, 1, -2, 2, 10\} \cap ((-4, 0) \cup (0, 4)) = \{-1, 1, -2, 2\}$, so $f^{-1}(B_1) \cap f^{-1}(B_2) = f^{-1}(B_1 \cap B_2)$.

3. (a) (**12 marks**)

(i) (**5 marks**)

$x \in h^{-1}(Y_1 \cap Y_2) \iff h(x) \in Y_1 \cap Y_2 \iff h(x) \in Y_1 \wedge h(x) \in Y_2 \iff x \in h^{-1}(Y_1) \cap h^{-1}(Y_2)$, that is,

$$h^{-1}(Y_1 \cap Y_2) = h^{-1}(Y_1) \cap h^{-1}(Y_2).$$

(ii) (**7 marks**)

Suppose h is on-to-one and $A, B \in \mathcal{P}(X)$. If $A \subseteq B$ and $y \in h(A)$, then $y = h(x)$ for some $x \in A$. Since $A \subseteq B$, $x \in B$ and $y = h(x) \in h(B)$. We have shown that $A \subseteq B \implies h(A) \subseteq h(B)$.

Conversely, if $h(A) \subseteq h(B)$ and $x \in A$, then $h(x) \in h(A)$ and so $h(x) \in h(B)$. Thus $h(x) = h(b)$ for some $b \in B$. Since h is one-to-one, it follows that $x = b$ and hence $x \in B$. We have shown that $h(A) \subseteq h(B) \implies A \subseteq B$.

Therefore $A \subseteq B \iff h(A) \subseteq h(B)$ and $h : \mathcal{P}(X) \to \mathcal{P}(Y)$ is order preserving.

(b) (**8 marks**)

First we show that $f : A \to B$ is **one-to-one**.

Suppose $f(\frac{1}{n}) = f(\frac{1}{m})$ for some $\frac{1}{n}, \frac{1}{m} \in A$. Then $n, m \in \mathbb{N}$ and $-n = f(\frac{1}{n}) = f(\frac{1}{m}) = -m$, so $n = m$ and f is one-to-one.

Then we show that $f : A \to B$ is **onto**.

Suppose $b \in B$. Then $b = -m$ for some $m \in \mathbb{N}$, so $\frac{1}{m} \in A$. Since $f(\frac{1}{m}) = -m = b$, it follows that f is onto.

Therefore f is bijective.

Finally we show that f is **order preserving**.

Suppose $a_1, a_2 \in A$, so that $a_1 = \frac{1}{n_1}$, $a_2 = \frac{1}{n_2}$ for some $n_1, n_2 \in \mathbb{N}$. Thus $a_1 \leq a_2 \iff \frac{1}{n_1} \leq \frac{1}{n_2} \iff n_2 \leq n_1 \iff -n_1 \leq -n_2 \iff f(\frac{1}{n_1}) \leq f(\frac{1}{n_2}) \iff f(a_1) \leq f(a_2)$.
Since f is an order preserving bijection, it is an order isomorphism.