THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2002
Campus: City

MATHEMATICS

Principles of Mathematics

(Time allowed: THREE hours)

NOTE: This is an OPEN BOOK examination.
Answer ALL **EIGHT** questions. All questions carry equal marks.

1. Let P be the statement that if x and y are both odd integers, then xy is an odd integer.

(a) Write down the negation of P. (2 marks)

(b) Write down the contrapositive of P. (2 marks)

(c) Write down the converse of P. (2 marks)

(d) Use a **direct proof** to show that P is true. (5 marks)

(e) Use a **proof by contradiction** to show that P is true. (4 marks)

(f) Determine whether or not the converse is true. Explain fully with reasons. (5 marks)

2. (a) Let $\mathbf{0} = (0,0,0)$, $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$ be vectors of vector space $\mathbb{R}^3$, and $A = \{\mathbf{0}, \mathbf{i}, -\mathbf{i}, \mathbf{j}, -\mathbf{j}, \mathbf{k}, -\mathbf{k}\}$. Let $*$ be the cross product $\times$ of $\mathbb{R}^3$, that is,

$$(a_1, a_2, a_3) * (b_1, b_2, b_3) = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

(i) Show $*$ is a binary operation of A, and construct its Cayley table. (8 marks)

(ii) Show $*$ is not associative. (6 marks)

(b) Let $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$ be the group of all symmetries of a square under composition $\circ$, and let $S = \{R_0, H, V, D\}$ be a subset of D_4.

(i) Show $(S, \circ)$ has an identity. (2 marks)

(ii) Show each element of S has an inverse in S. (2 marks)

(iii) Show S is *not* a subgroup of D_4. (2 marks)

3. (a) Prove by mathematical induction that, for every natural number $n \in \mathbb{N}$, $9^n - 8n - 1$ is divisible by 64.

(12 marks)

(b) Let $|$ be the operation of division on the set of natural numbers $\mathbb{N}$, and $a, b \in \mathbb{N}$. Show that the set $\{a, b\}$ has a least upper bound in the poset $(\mathbb{N}, |)$. Find the least upper bound of the set $\{21, 24\}$.

(8 marks)

4. (a) Let $f(x)$ and $g(x)$ be polynomials in $\mathbb{Z}_5[x]$ defined by

$$f(x) = x^4 + 2x^3 + 4x^2 + 2x + 3, \quad g(x) = 4x^3 + 2x^2 + 4x + 2.$$

Here for simplicity, we denote $\bar{a}$ by a for $\bar{a} \in \mathbb{Z}_5$. Find quotient $q(x)$ and remainder $r(x)$ when $f(x)$ is divided by $g(x)$. Find also a greatest common divisor of $f(x)$ and $g(x)$.

(12 marks)

(b) Let $\mathbb{R}[x]$ be the set of all polynomials with real coefficients. For $f, g \in \mathbb{R}[x]$, define $f \sim g$ if $f' = g'$, where f' is the derivative of f. Show $\sim$ is an equivalence relation. Find the equivalence class $[x^2]$ containing x^2.

(8 marks)

5. (a) Let $\mathrm{Mat}_2(\mathbb{R})$ be the group of all 2×2 matrices with entries from the set of real numbers $\mathbb{R}$ under matrix addition. Let Tr be the function from the group $\mathrm{Mat}_2(\mathbb{R})$ to the group $\mathbb{R}$ defined by

$$Tr\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + d.$$

Show Tr is a group homomorphism, and determine with reasons whether or not Tr is a group isomorphism.

(8 marks)

(b) Let $U(10) = \{1, 3, 7, 9\}$ be the group under the operation of multiplication modulo 10. Write out the Cayley table of $(U(10), \cdot_{10})$ and show $(U(10), \cdot_{10})$ is isomorphic to the group $(\mathbb{Z}_4, +_4)$.

(12 marks)

6. (a) Find all integers $x \in \mathbb{Z}$ such that

$$35x \equiv 14 \pmod{42}.$$

(7 marks)

(b) Let n be a natural number with $n \geq 2$.

(i) If n is a prime, then show that

$$(n-1)! \equiv -1 \pmod{n}.$$

(9 marks)

(ii) Suppose n is not a prime. Show by an example that there is some integer n with $n \geq 2$ such that

$$(n-1)! \not\equiv -1 \pmod{n}.$$

(4 marks)

7. (a) Let $\{a_n\}_{n=1}^{\infty}$ be the sequence such that $a_n = \dfrac{3^n}{1+3^{2n}}$.

(i) Show that $\{a_n\}_{n=1}^{\infty}$ is monotone and bounded.

(8 marks)

(ii) Find the greatest lower bound and the least upper bound of the set $\{a_n\}_{n=1}^{\infty}$ and determine whether or not either is an element of $\{a_n\}_{n=1}^{\infty}$. Find also the limit $\lim\limits_{n\to\infty} a_n$.

(5 marks)

(b) Let S and T be two non-empty sets of real numbers such that $x \leq y$ for all $x \in S$ and $y \in T$. Let lub S be the least upper bound of S and glb T be the greatest lower bound of T. Show that both lub S and glb T exist and

$$\text{lub } S \leq \text{glb } T.$$

(7 marks)

8. Let f and g be continuous functions from $\mathbb{R}$ to itself. Define

$$h(x) = \begin{cases} f(x) & \text{if } x \geq 0, \\ g(x) & \text{if } x < 0. \end{cases}$$

(a) Suppose $f(0) = g(0)$. Prove from first principles that $h(x)$ is continuous at 0.

(10 marks)

(b) Suppose $f(0) \neq g(0)$. Prove from first principles that $h(x)$ is *not* continuous at 0.

(10 marks)