DEPARTMENT OF MATHEMATICS

MATHS 255	Assignment 2 Solutions	Due: 1 August, 2001

1. Let A,B,C be sets. Prove that $(A \cap B) \cup C = A \cap (B \cup C) \quad \Leftrightarrow \quad C \subseteq A$.

First assume $(A \cap B) \cup C = A \cap (B \cup C)$.
$x \in C \Rightarrow x \in (A \cap B) \cup C \Rightarrow x \in A \cap (B \cup C) \quad \Rightarrow x \in A.$ Hence, $C \subseteq A$. Conversely, assume $C \subseteq A$. We show (i)$(A \cap B) \cup C \subseteq A \cap (B \cup C)$ and (ii) $(A \cap B) \cup C \supseteq A \cap (B \cup C)$. For (i), $(A \cap B) \cup C \subseteq (A \cap B) \cup A \subseteq A \cup A = A,$ and $(A \cap B) \cup C \subseteq B \cup C$. Hence $(A \cap B) \cup C \subseteq A \cap (B \cup C)$. For (ii), $A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \subseteq (A \cap B) \cup C$.[Notice that we did not need to use the assumption $C \subseteq A$ to prove (ii).]

Given sets A,B we define a set operation Δ (called "symmetric difference') as follows:

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

2. If $A_1 = \{\varnothing\}$, $A_2 = A_1 \cup \{A_1\}$, $A_3 = A_2 \cup \{A_2\}$, then list the elements of A_2, A_3, and $A_2 \Delta A_3$.

$A_2 = \{\varnothing, \{\varnothing\}\}$ (two elements), $A_3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ (three elements), $A_2 \Delta A_3 = A_3 \setminus A_2$ (since $A_2 \subseteq A_3$) $=$ $\{\{\varnothing, \{\varnothing\}\}\}$ (one element).

3. Find a way to write $A \cup B$ and $A \setminus B$ using only the set operations $\cap$ and Δ.
   [Hint: If $X \cap Y = \varnothing$ then $X \cup Y = X \Delta Y$.]

   $A \cup B = (A \Delta B) \Delta (A \cap B)$ (This can be seen by separating $A \cup B$ into the two disjoint parts $X = A \Delta B,\ Y = A \cap B$.)
   $A \setminus B = A \Delta (A \cap B).$
   [There are other correct answers, but probably none more simple than these.]

4. Indicate whether each of the following relations on the given set is reflexive, symmetric, antisymmetric, transitive. Explain each answer.

   i. $A = \{p\colon\ p$ is a guest at a particular party$\}$. $x \rho y$ iff x knows the name of y.

Reflexive (everyone knows their own name, at least at the beginning of the party)
Not symmetric (usually some guests are better known than others.)

Not antisymmetric (usually at least one pair of quests know each other.)

Not transitive (a friend of a friend is not necessarily a friend.)

ii. $A = \{p\colon\ p$ is a person in New Zealand$\}$. $x\rho y$ iff x is at least as tall as y.

Reflexive (everyone is at least as tall as theirself, i.e. for all p, height of $p \geq$ height of p.)

Not symmetric (some people are strictly taller than others.)

Not antisymmetric (different people sometimes have the same height.)

Transitive (ht(x) $\geq$ ht(y) and ht(y) $\geq$ ht(z) implies ht(x) $\geq$ ht(z).)

iii. $A = P(\mathbf{N})$. $x\rho y$ iff $x \subseteq y$.

Reflexive ($x \subseteq x$)

Not symmetric (e.g. $\varnothing \subseteq \{1\}$, but not the other way around)

Antisymmetric ($x \subseteq y$ and $y \subseteq x$ implies $x = y$.)

Transitive ($x \subseteq y$ and $y \subseteq z$ implies $x \subseteq z$.)

iv. $A = \{a, b, c\}$ (distinct elements). $\rho = \{(a,a),(a,b),(b,c),(a,c),(c,a)\}$.

Not reflexive ($(b,b) \notin \rho$)

Not symmetric ($(b,c) \in \rho$ but $(c,b) \notin \rho$)

Not antisymmetric ($(c,a) \in \rho$ and $(a,c) \in \rho$ but $a \neq$ c .)

Not transitive ($(c,a) \in \rho$ and $(a,c) \in \rho$ but $(c,c) \notin \rho$.)

v. $A = \{\ l\colon\ l$ is a line in the Cartesian plane$\}$. $x\rho y$ iff x and y are either parallel or identical lines.

Reflexive (Every line is identical to itself)

Symmetric (If x is identical to y, then y is identical to x, and if x crosses y then y crosses x)

Not antisymmetric (There are plenty of distinct lines which are parallel to each other)

Transitive (If x and y are parallel or identical and y and z are parallel or identical then x and z are parallel or identical.)