THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2011
Campus: City

MATHEMATICS
Accelerated Mathematics

(Time allowed: TWO hours)

NOTE: Attempt all 6 questions. The questions are NOT all of equal value. There is a total of 100 marks.

SHOW ALL WORKING. Unsupported answers may receive no marks.

Restricted calculators only.

1. (a) For the following system of linear equations:

$$\begin{cases} x + 2y \quad - w \quad = 4 \\ 2x + 4y + z + w = 7 \\ 2x + 4y - 3z + w = -1 \end{cases}$$

(i) write down the **augmented matrix** of the system;

(ii) use row operations to find the **reduced row echelon form** of the matrix in (i);

(iii) give the **general solution** of the system.

[12 marks]

2. In this question you will use the points $P(1, -1, 3)$, $Q(-1, 0, 2)$, $R(2, -1, 4)$ and $S(2, 1, -2)$.

(a) Find the angle θ between the vectors PQ and PR.

(b) Find a parametric vector equation of the line through the points P and Q.

(c) Give the Cartesian equation of the plane through the point S that contains the line L with parametric equations $\begin{cases} x = 1 - t \\ y = -1 + 3t, \ t \in \mathrm{IR} . \\ z = -2 + 2t \end{cases}$

(d) Do the line in (b) and the plane in (c) meet? If so, find their intersection, and if not, explain why not.

[16 marks]

3. (a) Let $B = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

(i) Calculate, using co-factors, the determinant of B.

Use the result in (i) to find the determinants of

(ii) B^T; (iii) B^{-1} ; (iv) $2B^2$.

(b) Find the volume of the parallelepiped generated by the row vectors of the matrix B given in (a) above.

(c) Prove that for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in IR^3, $\mathbf{u} \bullet \mathbf{v} \times \mathbf{w} = \mathbf{w} \bullet \mathbf{u} \times \mathbf{v}$.

[14 marks]

4. (a) Let $T_1: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which rotates vectors by $\frac{\pi}{3}$ about the origin, and $T_2: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which reflects vectors in the line $y = -x$.

(i) Find the standard matrices C_1 and C_2 of T_1 and T_2.

(ii) Hence or otherwise find $T_3(\mathbf{v})$ where $\mathbf{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ and $T_3: \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which first rotates vectors by $\frac{\pi}{3}$ about the origin then reflects vectors in the line $y = -x$.

(b) Find the standard matrix of $S: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $S(\mathbf{x}) = \mathbf{x}\times\mathbf{u}$, where $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$.

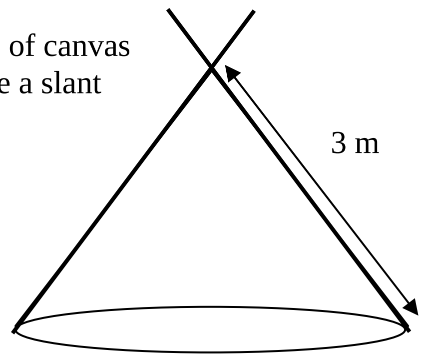
[12 marks]

5. (a) State the Extreme Value Theorem for a function $f: [a, b] \to \mathbb{R}$.

(b) An outdoor programme has several circular pieces of canvas which it can use to make conical tepees which have a slant height of 3 metres.

Find the vertical height and floor area of the tepee with the greatest volume.

You must justify your answer.



(c) (i) Give the definition of the derivative of a function $f: A \to \mathbb{R}$ at $a \in A$.

(ii) Use the definition of the derivative to find $g'(\frac{\pi}{4})$ where $g(x) = \cos(x)$.

(d) Find $h'(x)$ where

(i) $h(x) = x^{\frac{2}{x}}$, $x > 0$; (ii) $h(x) = \int_1^{x^3+1} \sin^2\left(\frac{\pi t}{4}\right) dt$.

[26 marks]

6. (a) A tank initially contains 10 litres of water with 1 gram of salt dissolved in it. A pipe then carries 2 litres of water per minute into the tank. The water coming into the tank contains 1 gram of salt per litre. A second pipe meanwhile carries 1 litre per minute of the mixture out of the tank.

The rate of change of salt within the tank can be modelled, until the tank is full, as

$$\frac{ds}{dt} = 2 - \frac{s}{10+t} \ .$$

(i) Define the variables used in this model, specifying the units involved. You do NOT need to justify the differential equation given above.

(ii) Write down the initial condition in this initial value problem.

(iii) Solve the initial value problem.

(iv) Determine how much salt is in the tank after 10 minutes.

(b) Evaluate the following integrals:

(i) $\displaystyle\int \frac{\ln(x^2)}{x^2}\, dx$ (ii) $\displaystyle\int_0^{\pi/4} \cos(x)\sin(2x)\, dx$.

[20 marks]