Is It Ever Safe to Vote Strategically?

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Abstract: We extend the Gibbard-Satterthwaite theorem in the following way. We prove that an onto, non-dictatorial social choice rule which is employed to choose one of at least three alternatives is safely manipulable. This means that on occasion a voter will have an incentive to make a strategic vote and know that he will not be worse off regardless of how other voters with similar preference orders would vote, sincerely or not.

Keywords: Social choice rule, strategic overshooting, strategic undershooting, safe manipulation, unsafe manipulation.

1 Introduction

In this paper we will demonstrate why a voter might be reluctant to act on a known incentive to vote strategically. To be specific, we will say that a voter has an incentive to vote strategically if, either alone or in coalition with other voters having the same preferences, he or she can manipulate the voting system under use. We will then distinguish between safe and unsafe strategic votes. Take a voter having an incentive to vote strategically. Suppose that if the voter acts on that incentive then he or she could realise a gain or could realise a loss, depending on which other voters with the same preferences and having the same incentive to vote strategically also act on that incentive (ceteris paribus). Absent the ability to co-ordinate with others, the voter described obviously has a disincentive to vote strategically: the possibility of making matters worse rather than better. We will say that the voter in question has an incentive to make an unsafe strategic vote. Alternatively, if a voter has an incentive to vote strategically, and by acting on that incentive could not realise a loss no matter which other voters with the same preferences and the same incentive also make that particular strategic vote (cet. par.), then we will say the strategic vote in question is safe for this voter.

We will be interested solely in situations in which a group of individuals is using a social choice rule to reach a collective decision. We will further restrict our attention to situations in which the number of individuals and the number of possible decisions are both finite. Our investigation has two aims. The first is to show by example that, in environments in which voters cannot co-ordinate, the distinction between safe and unsafe strategic votes is worth making. The second and more substantial aim is to extend an implication of the Gibbard-Satterthwaite (1973, 1975 respectively) theorem. Gibbard and Satterthwaite established that, under every non-dictatorial social choice rule, a voter can have an incentive to vote strategically. Their theorem implies the following:

> Suppose an onto, non-dictatorial social choice rule is employed to choose one of at least three alternatives. Then on occasion a voter will have an incentive to vote strategically.

We will prove the following extension:

> Suppose an onto, non-dictatorial social choice rule is employed to choose one of at least three alternatives. Then on occasion a voter will have an incentive to make a safe strategic vote.

It has already been shown that the fact a voter has an incentive to vote strategically does not imply the voter will act on it. For example, even if a voter can vote strategically, the costs of acquiring the necessary knowledge may deter a voter from doing so. This observation is a consequence of the results of research into the computational complexity of voting strategically, see e.g. Bartholdi *et al* (1989), Bartholdi and Orlin (1991) and, more recently, Conitzer and Sandholm (2003, 2006) and Conitzer *et al* (2007). In addition, it is obvious that a voter having an incentive to vote strategically but finding voting 'dishonestly' distasteful would have also have a disincentive. But in such a case there is little to analyse from a social choice point of view.

To the best of our knowledge, the distinction between safe and unsafe strategic votes, as we define them, was first made (albeit in the context of parliament choosing rules) in Slinko and White (2006). Parikh and Pacuit (2005) use the expression 'safe vote', but to mean a vote that is strategically superior to abstention.

This paper is organised as follows. Section 2 sets out a model of social choice. Section 3 gives five different examples of unsafe strategic votes. The remaining sections work on extending the Gibbard-Satterthwaite theorem. Five preliminary propositions appear in Section 4. Sections 5 and 6 then present the main results. Section 5 deals with the three alternative case, Section 6 with the four-or-more-alternatives case. Section 7 concludes.

2 The model

There are a number of procedures by which a group of individuals may make a collective choice. Utilising a social choice rule is one such procedure. We will formally define a social choice rule below, but essentially many of them work as follows. Each individual is asked to complete an identical ballot paper. On his or her ballot paper, the individual must strictly rank all of the alternatives. Ballot papers are completed in private, or simultaneously, or both. Once all ballot papers are completed and submitted, the social choice rule responds with a single member of the set of alternatives which is then declared the winner.

We wish to study safe and unsafe strategic voting. For that purpose we now construct a formal model of a social choice rule.

Let $\mathcal{A} = \{A, B, C, ...\}$ be a finite set containing at least three elements (henceforth *alternatives*). Each member of a non-empty finite set of individuals $[n] = \{1, 2, \dots, n\}$ (henceforth *voters*) can strictly rank these alternatives. A *profile* is a function with domain the set of voters and range the set of all possible strict rankings of the alternatives. Given a particular set of voters with specific preferences, the *profile of sincere preferences* maps each voter to his or her sincere ranking of the alternatives. If a profile R maps two (or more) voters to the same ranking then we will say that these voters *have identical preferences at* R. A profile is *completely agreed* if all voters have identical preferences at R.

There are a number of ways to represent a profile. A profile is most commonly represented by an n-tuple $R = (R_1, \dots, R_n)$ of linear orders, where R_i is the image of the ith voter. If there are m alternatives, we may index all $m!$ linear orders by integers from 1 to $m!$ and represent R as an $m!$-tuple of disjoint sets X_i, $i = 1, 2, \dots, m!$ such that $X_1 \cup X_2 \cup \ldots \cup X_{m!} = [n]$, where X_i is the set of voters whose sincere preference is the ith linear order.

Suppose, for example, that the number of alternatives is three. Then we can index the six possible linear orders as follows:

1	2	3	4	5	6
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

A profile now can be represented by a 6-tuple of sets $R = (X_1, X_2, X_3, X_4, X_5, X_6)$, where X_i, $i = 1, \dots, 6$ is the set of voters mapped to the ith linear order. When voters' identities are unknown, we know only a 6-tuple $(n_1, n_2, n_3, n_4, n_5, n_6)$, where $n_i = |X_i|$. Such a table is called a *voting situation*, see Berg and Lepelley (1994, page 135). Finally, suppose the number of voters present is two and the number of alternatives is three. Then the profile that maps voter one to the preference order A preferred to B preferred to C, and voter two to the order A preferred to

C preferred to B, can be denoted (ABC, ACB); similarly with all other profiles. The manner in which we will present a profile shall be determined by the particular aspect of the profile that we wish to highlight.

The following notation will be useful. Let R be a profile, V any set of voters with identical preferences at R, and L a preference order over $\mathcal{A}$ other than the one representing the sincere preferences of the voters in V. Then $R_{-V}(L)$ shall denote the profile obtained from R by replacing every R_i such that $i \in V$ with L, ceteris paribus. $R_{-V}(L)$ can be read informally as "the profile R, except that all the preferences of the voters in V have been switched to L".

Again let V be any set of voters with identical preferences. We define a relation on the set of alternatives as follows: if X and Y are two elements of $\mathcal{A}$, then let

$$X \succ_V Y$$

denote that voters in V rank X strictly above Y, and

$$X \succeq_V Y$$

denote that voters in V rank X no lower than Y.

Given a set of alternatives $\mathcal{A}$ and a set of voters, a *social choice rule* F is a mapping from the set of all possible profiles to $\mathcal{A}$.[1] Voter i is a *dictator* for a voting rule F if he or she would not desire to change the value of F at any profile. A social choice rule is *weakly unanimous* if it selects every voter's favorite alternative choice at all completely agreed profiles. Note that a weakly unanimous social choice rule is necessarily onto. A social choice rule F is *antagonistic* if there exists a profile at which every voter claims to rank a particular alternative $X \in \mathcal{A}$ last, yet the value of F at that profile is precisely X.

We assume each voter knows the preferences of all other voters, knows the voting intentions only of him or herself, and always votes when given the opportunity. We will use the following terminology: say there are three alternatives present, and a voter prefers A to B to C. Then we will say that the voter is 'an ABC type'. If a voter claims to prefer A to B to C, we will say that the voter 'votes ABC'. Analogously with other preference relations, and when more than three alternatives are present.

For the remainder of this section let us fix the set of alternatives $\mathcal{A}$, a set of voters $[n]$, a profile of sincere preferences R, and a social choice rule F.

We can now move on to the more interesting definitions. The motivation for the following one is as follows. If the number of voters is large, then the probability of someone being pivotal, i.e. capable of changing the value of F on her own, is very small so classical individual manipulability does not make much sense.

[1] It might seem more natural to set the domain of F to be the set of all possible profiles for all possible sets of voters, but this would make subsequent sections of this paper unnecessarily complicated.

Definition 1 (An incentive to vote strategically)**.** *Fix a voter i, and define V to be the set of all voters with preferences identical to those of i at R. If there exists a linear order $L \neq R_i$ over $\mathcal{A}$, and a subset $V_1 \subseteq V$ containing i such that*

$$F(R_{-V_1}(L)) \succ_V F(R)$$

then we will say that, at R, voter i has an incentive to vote strategically*.*

This is the key concept of the paper. We note that to have incentive to vote strategically does not mean that the voter is pivotal. What this voter can hope for is that there will be a sufficient number of like-minded voters with the same incentive who will make a strategic move. We make the classical social choice assumption that voters know sincere preferences of others but cannot know their voting intentions. We also assume the absence of co-ordination of any sort.

In some circumstances (we will present several examples in the next section) a voter may hesitate to act on an incentive to vote strategically. One reason for hesitation would be this: in attempting to manipulate, the voter could realise a gain or could realise a loss depending on which other voters with the same preferences and having the same incentive also act on that incentive. We now describe such circumstances formally.

Definition 2 (Strategic overshooting)**.** *Fix a voter i, and define V to be the set of all voters with preferences identical to those of i at R. Suppose that there exist two sets V_1 and V_2 such that $i \in V_1 \subset V_2 \subseteq V$,*[2] *and a linear order $L \neq R_i$ such that:*

- *every voter in V_2 has an incentive to strategically vote L, and*
- $F(R_{-V_1}(L)) \succ_V F(R) \succ_V F(R_{-V_2}(L))$.

Then voter i can strategically overshoot *at R.*

In the circumstances described in Definition 2, each voter in V_2 has an incentive to vote strategically. Each voter in V_2 also has a disincentive: the prospect of making the outcome worse rather than better. If the ability to co-ordinate is absent then members of V_1 - of which i is one - will be uncertain how to proceed.[3]

If F is anonymous then overshooting occurs when too many like-minded voters act strategically; by contrast, undershooting, the subject of our next definition, occurs when too few like-minded voters act strategically. More generally, undershooting occurs when the roles of the sets V_1 and V_2 are reversed.

[2]Please note that throughout this paper $\subset$ denotes 'is a proper subset of' while $\subseteq$ denotes 'is a subset of'.

[3]Slinko and White (2006) suggested that in this case voters will act in accord with their attitude towards uncertainty.

Definition 3 (Strategic undershooting)**.** *Fix a voter i, and define V to be the set of all voters with preferences identical to those of i at R. Suppose that there exist two sets V_1 and V_2 such that $i \in V_1 \subset V_2 \subseteq V$, and a linear order $L \neq R_i$ such that*

- *every voter in V_2 has an incentive to strategically vote L, and*
- *$F(R_{-V_2}(L)) \succ_V F(R) \succ_V F(R_{-V_1}(L))$.*

Then voter i can strategically undershoot *at R. The first indented condition is implied by the second, but is included to emphasise the symmetry between the definitions of over and undershooting.*

In the circumstances described in Definition 3, each voter in V_1 has both an incentive and a disincentive to strategically vote L. These voters will, therefore, be uncertain how to proceed. Note that if F is not anonymous then it may be that, in the scenario described, voting L is safe for members of $V_2 - V_1$. Members of V_1 will still be unsure how members of $V_2 - V_1$ will behave - these latter voters might be strategically inclined, but they might be inherently honest (or inherently daft) - and voting L will remain unsafe for V_1.

Motivated by Saari (1994), we note that strategic over and undershooting can have nice geometric interpretations. Suppose, for example, that F is a scoring social choice rule. Profiles can then be represented by points in Euclidean space, and the space of profiles can be divided up into regions in which F takes the same value (see Saari (1994) for the details). If a set of voters can over or undershoot then, acting in unison, they must be able to 'shift' a profile all the way across some region. The prospect of over or undershooting thus becomes more likely when the profile of sincere preferences is close to a two (or three, or four, ...) way tie. We will not explore the geometry further here, but do consider it would be worth future investigation.

Definition 4 (Unsafe strategic vote)**.** *Suppose that voter i can strategically over or undershoot at the profile R. Then we will say that voter i can make an* unsafe strategic vote *at R.*

Definition 5 (Safe strategic vote)**.** *Fix a voter i, and a profile R. Suppose that there exists a linear order $L \neq R_i$ such that*

- *at R, voter i has an incentive to strategically vote L; and*
- *voter i cannot strategically overshoot or strategically undershoot at R with a vote of L.*

Then voter i can make a safe strategic vote *at R.*

It will prove useful to classify a certain kind of potential strategic move as an escape. Suppose preferences are such that voter i ranks no element of $\mathcal{A}$ lower than X. Further suppose that R is the profile of sincere preferences (over $\mathcal{A}$), and $F(R) = X$. If, at R, voter i has an incentive to vote strategically then voter i will be said to be able to *escape* at R. Notice that at R voter i

cannot make an unsafe strategic vote, so being able to escape implies being able to make a safe strategic vote. The concept of escaping will appear often during the proofs in later sections.

The model is now complete. The next section presents examples of safe and unsafe manipulation.

3 Examples of safe and unsafe manipulation

Given any scoring social choice rule other than plurality, and a set of voters sufficiently large (more than 50, say), it is easy to create examples of strategic overshooting.

Example 1 (Strategic overshooting, escaping under the Borda rule)**.** *Suppose 94 voters are using the Borda rule to choose one of three alternatives and that the corresponding voting situation of sincere preferences is* $(17, 15, 18, 16, 14, 14)$*. If all voters vote sincerely then A would score 96, B 99, and C 87; B would win. If between four and eight ABC types vote ACB, ceteris paribus, then A would win. If 10 or more ABC types vote ACB, ceteris paribus, then C would win. So the given voting situation of sincere preferences is prone to unsafe manipulation. This voting situation is also prone to safe manipulation: if 13 or more ACB voters vote CAB, ceteris paribus, then C will win, and the manipulators will have made an escape.*

It is also easy to use scoring social choice rules (other than plurality) to create profiles that are unsafely but not safely manipulable. However, the examples that are readily apparent exhibit an absence of several voter types, and have a somewhat contrived feel.

Example 2 (Strategic overshooting under scoring social choice rules: a profile that is unsafely but not safely manipulable)**.** *Suppose 80 voters are using a scoring social choice rule to choose one of three alternatives. Suppose that a first place ranking on a ballot is worth three points, while a second place ranking is worth one point. Let the number of different voter types present at the profile of sincere preferences be given by the following table:*

Preference order	*Number of voters*
ABC	*30*
ACB	*0*
BAC	*20*
BCA	*0*
CAB	*0*
CBA	*30*

If all voters are honest then A scores 110, B 120, and C 90; B wins. Those that rank B or C highest have no incentive to act strategically. Consider the voters of type ABC. If between

11 and 19 of them state they are of type ACB, ceteris paribus, then A will win. If more than 21 of them state they are of type ACB, ceteris paribus, then C will win. Voters of type ABC have no other way to manipulate the vote. Thus the profile of sincere preferences described is 'completely' unsafe.

Example 3 below presents an example of strategic undershooting.

Example 3 (Strategic undershooting, escaping under the Borda rule)**.** *Suppose 41 voters are using the Borda rule to select one of five alternatives. Let the number of different voter types present at the profile of sincere preferences be given by the following table:*

Preference order	*Number of voters*
$ADEBC$	*0*
$BCADE$	*15*
$CABED$	*14*
$CEDBA$	*2*
$DABEC$	*10*

This example does not depend on there being only a small number of different voter types present, but such a profile eases computations. When all voters vote honestly, A scores 102, B 110, C 109, D 45, and E 21; B wins. If between 2 and 6 $DABEC$ types vote $ADEBC$, ceteris paribus, then C wins. If 8 or more $DABEC$ types vote $ADEBC$, ceteris paribus, then A wins. So $DABEC$ voters can strategically undershoot at the profile of sincere preferences. Note that by changing, say, four voters from being of type $DABEC$ to being of type $ADEBC$ we could create a profile at which the former type could escape.

Strategic undershooting and overshooting are also possible under plurality. The examples that we have do, however, rely upon the tie-breaking procedure adopted. We give them in the Appendix 2.

Strategic overshooting and undershooting do not only occur under scoring social choice rules.

Example 4 (Strategic overshooting under plurality with a run-off)**.** *Suppose 23 voters are using a plurality with a run-off social choice rule to choose one of three alternatives. Let the number of different voter types present at the profile of sincere preferences be given by the following table:*

Preference order	*Number of voters*
ABC	*4*
ACB	*6*
BAC	*7*
BCA	*0*
CAB	*0*
CBA	*6*

If all voters vote sincerely then B beats A 13-10 in the run-off. If 2 voters of type ABC vote for C in the first round, ceteris paribus, then A beats C 17-6 in the run-off. If 4 or more voters of type ABC vote for C in the first round, ceteris paribus, then C beats B 12-11 in the run-off. Thus the profile of sincere preferences described is unsafely manipulable.

To construct an example of an unsafe manipulation under a non-anonymous social choice rule is trivially simple. We now turn to showing that all onto, non-dictatorial social choice rules are safely manipulable. To avoid trivialities, we will assume throughout that more than one voter is present.

4 Preliminary propositions

We need the following definition. Let F be a social choice rule, R a profile, L a preference order over the alternatives, and V the entire set of voters mapped by R to some preference order other than L. Then $V_1 \subset V$ will be classified as *L-inferior* if and only if $F(R_{-V}(L)) \succ_V F(R_{-V_1}(L))$. We allow the possibility that $V_1 = \emptyset$.

Proposition 1. *Let F be a social choice rule, R a profile, L a preference order over the alternatives, and V the entire set of voters mapped by R to some preference order other than L. If V has an L-inferior subset then F is safely manipulable.*

Proof. Let V_1 be a maximal element of the set of L-inferior subsets of V partially ordered by inclusion. We claim that were $R_{-V_1}(L)$ a profile of sincere preferences then it would be safely manipulable by the voters in $V \setminus V_1$. At the profile $R_{-V_1}(L)$, the voters in $V \setminus V_1$ are the sole voters present with their particular preferences. Furthermore, if $V_2 \subseteq V \setminus V_1$ one has

$$F((R_{-V_1}(L))_{-V_2}(L)) = F(R_{-(V_1 \cup V_2)}(L)) \succeq_V F(R_{-V}(L)) \succ_V F(R_{-V_1}(L)).$$

So the claim is proven. □

Proposition 2. *Let F be a social choice rule. Suppose that voter i can strategically undershoot at R with a vote of $L \neq R_i$. Suppose that voter i cannot strategically overshoot at R with a vote of L. Then F is safely manipulable.*

Proof. Let V be the set of all voters having preferences identical to i at R. If V has an L-inferior subset then we may use Proposition 1. So suppose V has no L-inferior subset. Then $F(R_{-V_1}(L)) \succeq_V F(R_{-V}(L))$ for all $V_1 \subset V$. There are now two cases to deal with. Both cases lead to a contradiction. Firstly assume $F(R_{-V}(L)) \succeq_V F(R)$. Then $F(R_{-V_1}(L)) \succeq_V F(R)$ for all $V_1 \subseteq V$ and i cannot undershoot by voting L at R. This is a contradiction. Secondly assume $F(R) \succ_V F(R_{-V}(L))$. If there exists a non-empty $V_1 \subset V$ such that $F(R_{-V_1}(L)) \succ_V F(R)$ then we have a case of overshooting. If there does not exist such a V_1 then, at R, voter i has no incentive to strategically vote L. Both possibilities contradict earlier assumptions. □

Proposition 2 has the following consequence. Suppose we wish to show that a social choice rule F is safely manipulable. Suppose we know that, under F, voter i can manipulate unsafely at the profile R. Then we may assume that i can strategically overshoot at R. For if i can only strategically undershoot at R we may directly apply Proposition 2 and be done.

The following proposition will prove extremely useful.

Proposition 3. *Let F be a social choice rule. Suppose that, at R, voter i can unsafely manipulate by strategically voting $L \neq R_i$. Let V be the set of voters with preferences identical to voter i at R. If $F(R_{-V}(L)) \succeq_V F(R)$ then F is safely manipulable.*

Proof. Suppose that $F(R_{-V}(L)) \succeq_V F(R)$. Since R is unsafely manipulable, there must exist a nonempty $V_1 \subset V$ such that $F(R) \succ_V F(R_{-V_1}(L))$. We now have $F(R_{-V}(L)) \succeq_V F(R) \succ_V F(R_{-V_1}(L))$. So V_1 is an L-inferior subset of V. Then by Proposition 1, F is safely manipulable. □

Proposition 4. *If a social choice rule F is non-contsant and antagonistic then it is safely manipulable.*

Proof. There exists a configuration of preferences such that (i) R is the profile of sincere preferences, (ii) every voter considers (without loss of generality) A to be the least desirable alternative and (iii) $F(R) = A$. F is not constant, so there exists a profile $S = (S_1, S_2, S_3, ...)$ such that

$F(S) = B \neq A$. Consider the finite sequence of profiles

$$\begin{aligned}
&R^0 = R, \\
&R^1 = R_{-\{1\}}(S_1), \\
&R^2 = (R_{-\{1\}}(S_1))_{-\{2\}}(S_2), \\
&R^3 = ((R_{-\{1\}}(S_1))_{-\{2\}}(S_2))_{-\{3\}}(S_3), \\
&\ldots \\
&S
\end{aligned}$$

The value of F at R is A. The value of F at S is B. Let m be the largest integer such that $F(R^m) = A$. Then if R^m was a profile of sincere preferences it would be safely manipulable by voter $m+1$. In fact, $m+1$ could escape at R^m. □

5 Social choice rules: three alternatives

This section proves that all onto, non-dictatorial social choice rules used to choose one of precisely three alternatives are safely manipulable. We will make use of the theorem that we are extending:

Theorem 1. *(Gibbard-Satterthwaite). An onto and non-dictatorial social choice rule used to choose one of at least three alternatives is manipulable.*

Proofs can be found in the original papers Gibbard (1973) and Satterthwaite (1975). Easily read proofs may also be found in Schmeidler and Sonnenschein (1978) and Barbera and Peleg (1990).

Lemma 1. *Suppose the number of voters is two, the number of alternatives is three, and F is an onto and non-dictatorial social choice rule. Then there exists a voter i, a profile of sincere preferences R, and a linear order $L \neq R_i$ such that if V is the set of all voters with preferences identical to i at the profile R, then*

$$F(R_{-U}(L)) \succeq_V F(R)$$

for all subsets $U \subseteq V$, and $F(R_{-W}(L)) \succ_V F(R)$ for at least one subset $W \subseteq V$.

Proof. F is onto and non-dictatorial, so the Gibbard-Satterthwaite theorem implies F is manipulable. Suppose that voter $i \in \{1, 2\}$ can manipulate the profile R with a vote of $L \neq R_i$. If $R_1 \neq R_2$ then clearly we are done. So suppose $R_1 = R_2$. Then the set of voters with preferences identical to i at R is $V = \{1, 2\}$. For some $V_1 \subset V$ we have $F(R_{-V_1}(L)) \succ_V F(R)$. We further may assume that this manipulation is unsafe. Then $F(R) \succ_V F(R_{-V}(L))$. Without loss of any generality let i be an ABC type, and $V_1 = \{1\}$. Then

$$A = F(L, ABC) \succ_V B = F(ABC, ABC) \succ_V C = F(L, L).$$

Note that finding either a profile at which a voter can escape or a manipulable non-unanimous profile is sufficient to complete the proof.

If $L = BAC$ then any voter can escape from (BAC, BAC). For $L = BCA$ and $L = CBA$ escapes are also easily found. These will be escapes from (BCA, ABC) and (CBA, ABC), respectively. For the remaining two cases $L = ACB$ and $L = CAB$ see Appendix. □

An immediate consequence of Lemma 1 is the following corollary.

Corollary 1. *If the number of voters is two, the number of alternatives is three, and F is an onto and non-dictatorial social choice rule, then F is safely manipulable.*

We now move on to the case that $n = 3$ or more voters are present, and the number of alternatives is three. A very useful construction will be the following one which reduces an arbitrary rule to a two-voter rule.

Suppose F is a social choice rule, and there is more than one voter. Let V_1 and V_2 be two non-empty sets that partition the set of voters. The *two voter social choice rule, F_{V_1,V_2}, generated by V_1 and V_2* is constructed as follows. The value of F_{V_1,V_2} at the two voter profile (R_1, R_2) shall be the value of F when all voters in V_1 report their preferences to be R_1, while all voters in V_2 report their preferences to be R_2.

Proposition 5. *Suppose F is a social choice rule, there are three alternatives, and $[n] = V_1 \cup V_2$ is a non-trivial partition of the set of voters. If F_{V_1,V_2} is onto and non-dictatorial then F is safely manipulable.*

Proof. By Lemma 1 F_{V_1,V_2} is safely manipulable. We thus may assume that there exist voter $i \in \{1, 2\}$, and we assume that $i = 1$, the linear order L, and the profile $R = (R_1, R_2)$ such that voter 1 can manipulate at R safely. Let R' be the n tuple of preferences such that $R'_v = R_1$ when $v \in V_1$ and $R'_v = R_2$ when $v \in V_2$. Let W be the set of voters with, at R', preferences identical to those in V_1. The preferences at R' of the voters in W are the same as those of voter 1 at R. Either $W = V_1$ or $W = V_1 \cup V_2$ (the entire set of voters). Lemma 1 implies that for either $W_1 = V_1$ or $W_1 = V_1 \cup V_2$ one has

$$F(R'_{-W_1}(L)) \succ_W F(R'),$$

which shaows that R' is manipulable. If this manipulation is safe we are done. If this manipulation is unsafe then by Lemma 1 we have

$$F(R'_{-W}(L)) \succeq_W F(R').$$

We may now apply Proposition 3 to deduce that F is safely manipulable. □

Lemma 2. *Suppose the number of alternatives is three. If F is a non-dictatorial and weakly unanimous social choice rule then it is safely manipulable.*

Proof. See Appendix. □

Lemma 3. *Suppose the number of alternatives is three. If F is an onto and non-dictatorial social choice rule then it is safely manipulable.*

Proof. If the number of voters is either one or two, or if F is weakly unanimous or antagonistic, then the proposition is already proven. So assume that the number of voters, n, is three or more, and that F is neither weakly unanimous nor antagonistic.

The proof consists of several simple steps. The steps will either show directly that F is safely manipulable, or they will show that a particular partition of the set of voters allows us to appeal to Proposition 5 and also conclude that F is safely manipulable.

If F is not weakly unanimous then there exists a profile in which every voter has identical preferences, but the value of F at that profile is not equal to every voter's first choice. Let $R = (R_1, ..., R_n)$ be the n tuple of preferences in which every voter is of type ABC. Without loss of generality, let the value of F at R be B (were it to be C then F would be antagonistic).

Consider the profile in which every voter is of type ACB. The value of F at this profile must be C. Were it B, F would be antagonistic. Suppose it were A. Then consider the manipulation of R by ABC types voting ACB. Either this move is safe, or we may refer to Proposition 3 to show that F is safely manipulable.

Let $R' = (R'_1, ..., R'_n)$ be a profile such that $F(R') = A$. Such a profile exists because F is onto. Consider the finite sequence of profiles

$$
\begin{aligned}
&R, \\
&R_{-\{1\}}(R'_1), \\
&(R_{-\{1\}}(R'_1))_{-\{2\}}(R'_2), \\
&\cdots \\
&R'
\end{aligned}
$$

Suppose the mth profile is the first such that the value of F at that profile is A. Denote the mth profile by R^m. Let R^{m-1} be the profile in the sequence immediate preceeding R^m.

Consider the set of voters that, at the profile R^{m-1}, have preferences ABC. If they can safely manipulate with a vote of R'_m then we are done. If not, then this manipulation is unsafe. If they can stragically undershoot but not overshoot then by Proposition 2 we are done. Suppose that they can strategically overshoot. We may suppose (see the proof of Lemma 1) that the preference order represented by R'_m is either ACB or CAB. This allows us to deduce two facts about the profile R^m. Firstly, at this profile the set of voters that have preferences ABC is not empty (otherwise the voters that, at R^{m-1}, have preferences ABC could not strategically overshoot). Secondly, at the profile R^m the set of voters with preferences ACB and the set of voters with preferences CAB cannot both be empty.

Let the profile R^m in set form be $(1, 2, 3, 4, 5, 6)$. We have established that $1 \neq \emptyset$ and $2 \cup 5 \neq \emptyset$. Recall $F(R^m) = A$. If, at the profile R^m, any members of the sets 4 and 6 (supposing they are non-empty) voted differently, the outcome would still have been A. Otherwise the voters in these sets would find some profile to escape. Thus at the profile $R^* = (1, 2 \cup 4 \cup 6, 3, \emptyset, 5, \emptyset)$, F must take the value A.

Consider again the profile R^*. If the entire set 5 voted differently, the value of F could not be C. Otherwise either the voters in 5 would find R^* to be safely manipulable, or we could apply Proposition 3 to deduce that F is safely manipulable. Let $R^{**} = (1, 2 \cup 4 \cup 5 \cup 6, 3, \emptyset, \emptyset, \emptyset)$. The value of F at R^{**} cannot be B, otherwise voters with preferences ACB could safely manipulate at R^{**} with a vote of CAB. Therefore $F(R^{**}) = A$.

Now reconsider profile R^{**}. If the entire set 3 voted differently, the value of F could not be B. Otherwise either the voters in 3 would find R^{**} to be safely manipulable, or we could apply Proposition 3 to deduce that F is safely manipulable. Let $R^{***} = (1 \cup 3, 2 \cup 4 \cup 5 \cup 6, \emptyset, \emptyset, \emptyset, \emptyset)$. The value of F at R^{***} cannot be C, otherwise voters with preferences ABC could safely manipulate at R^{***} with a vote of BAC. So $F(R^{***}) = A$.

We can now finish the proof. If all voters state their preferences as ABC, F takes value B. If all voters state their preferences are ACB, F takes value C. Neither $1 \cup 3$ nor $2 \cup 4 \cup 5 \cup 6$ are empty. The social choice rule $F_{1\cup 3, 2\cup 4\cup 5\cup 6}$ takes value A when the first voter reports their preferences as ABC while the second reports their preferences as ACB. Hence $F_{1\cup 3, 2\cup 4\cup 5\cup 6}$ is onto. $F_{1\cup 3, 2\cup 4\cup 5\cup 6}$ is non-dictatorial: when both voters state they have preferences ABC, it does not take value A. So by Proposition 5 F is safely manipulable. □

6 Four or more alternatives

This section deals with situations in which there are an unspecified number (greater than two) of voters, and four or more alternatives.

Definition 6. *The social choice rule F_{-A} generated by a non-antagonistic social choice rule F and an alternative A.*

Suppose F is not antagonistic. Fix the set of voters and an alternative A. We construct a new social choice rule, denoted F_{-A}, designed to choose one of the alternatives $\mathcal{A} - \{A\}$. Let R be an arbitrary profile of preferences over the set $\mathcal{A} - \{A\}$. Let R' be the profile of preferences over the original alternative set $\mathcal{A}$ formed by appending an A to the bottom of every preference order in R. Then the value of F_{-A} at R shall be the value of F at R'. Note that F_{-A} cannot select A, so it is satisfactorily defined.

Proposition 6. *Suppose there are at least four alternatives. Suppose an onto, non-dictatorial social choice rule F is not safely manipulable. Then there cannot exist two alternatives A and B such that both F_{-A} and F_{-B} are dictatorial.*

Proof. If F is antagonistic then it is safely manipulable, so suppose it is not antagonistic. Suppose voter one is a dictator for F_{-A}. If her strongest preference is B, C, or D, then she can achieve that outcome by placing A at the bottom of her ballot paper. If F is not safely manipulable then F takes value B (C, D, ...) whenever voter one ranks B (C, D, ...) first. F is not dictatorial. So there exists a profile in which voter one reports A first (and, without loss of generality, B last), but F does not take value A at that profile. Then voter one is not a dictator for F_{-B}. Suppose voter two is a dictator for F_{-B}. By similar reasoning to that above, either F is safely manipulable or F takes value A (C, D) whenever voter two ranks A (C, D) first. If the latter then we have reached an absurdity. If voter one ranks C first and A last, and voter two ranks D first and B last, F cannot take value C and D simultaneously. □

Proposition 7. *If a social choice rule F is non-dictatorial and weakly unanimous then it is safely manipulable.*

Proof. The three alternative case is proven. We shall proceed by induction. Suppose the proposition is true when the number of alternatives is $N-1$. Now let the number of alternatives present be $N \geq 4$. Let A be an alternative such that F_{-A} is not dictatorial. The social choice rule F_{-A} inherits weak unanimity from F. Hence F_{-A} is onto and non-dictatorial which exists by Proposition 6. By the induction hypothesis it is safely manipulable at the profile R (of preferences over $\mathcal{A} \setminus \{A\}$) . Let R' be the profile of preferences over $\mathcal{A}$ formed by appending A's to the bottom of every preference order in R. Then clearly R' will be safely manipulable under F. □

We turn our attention to social choice rules that are not weakly unanimous. Lemma 2 and Proposition 6 together imply Proposition 7, which in turn implies our main result, Proposition 8.

Let $\mathcal{R}$ denote the set of completely agreed profiles.

Proposition 8. *Suppose F is a social choice rule and there are at least four alternatives. Suppose A and B are two alternatives and B is not in the range of F_{-A}. If $B \in F(\mathcal{R})$ then F is safely manipulable.*

Proof. Let R be the completely agreed profile in which every voter ranks B first and A last. Suppose $R' \in \mathcal{R}$ and $F(R') = B$. At R, voters can manipulate by voting R'_1. If this manipulation is safe then we are done. If it is unsafe then we may apply Proposition 3 to deduce F is safely manipulable. □

Proposition 9. *Suppose F is an onto, non-dictatorial social choice rule and there are at least four alternatives. If alternative $B \notin F(\mathcal{R})$ and voter i is a dictator for F_{-B} then F is safely manipulable.*

Proof. Without loss of generality we assume $i = 1$. Suppose F takes the value $A \in \mathcal{A} \setminus \{B\}$ whenever voter 1 ranks A first (if not, there will be a profile at which voter 1 can safely manipulate simply by moving B to the last place on her ballot paper). If voter 1 ranks B highest then the outcome of F must either be B or voter 1's second choice (if not, there will be a profile at which voter 1 can safely manipulate by placing her second choice first, and B last). If the value of F at the profile R is B then R_1 must be a linear order with B at the top (if not, voter 1 can safely manipulate at R by shifting B to the bottom of her ranking).

We now find a manipulation that allows us to apply Proposition 3. Let R be a profile such that $F(R) = B$. Suppose the linear order R_1 has A second. Let L be a linear order over the alternatives such that B is first, A is second, but $L \neq R_1$ (such a linear order exists because there are at least four alternatives present). Let $R^L \in \mathcal{R}$ be the completely agreed profile in which every voter ranks alternatives according to the preference order L. Given that $B \notin F(\mathcal{R})$, $F(R^L) \neq B$. Given that voter 1 ranks A second at R^L, $F(R^L) = A$. Consider the finite sequence of profiles

$$
\begin{aligned}
R^0 &= R^L, \\
R^1 &= R^L_{-\{1\}}(R_1), \\
R^2 &= (R^L_{-\{1\}}(R_1))_{-\{2\}}(R_2), \\
R^3 &= ((R^L_{-\{1\}}(R_1))_{-\{2\}}(R_2))_{-\{3\}}(R_3), \\
&\dots \\
&R.
\end{aligned}
$$

Notice $F(R^0) = A$ while $F(R) = B$. Let R^m be such that $F(R^m) = A$ while $F(R^{m+1}) = B$. Now suppose R^m were a profile of sincere preferences, and let V be the set of all voters with preference order L at R^m. Then $m+1 \in V$ and $B = F(R^{m+1}) = F(R^m{}_{-\{m+1\}}(R_{m+1})) \succ_V F(R^m) = A$, so voter $m+1$ may manipulate at R^m. Moreover, this manipulation is safe: if $V_1 \subseteq V$ then at the profile $R^m{}_{-V_1}(R_{m+1})$ voter 1 ranks B first and A second, and hence $F(R^m{}_{-V_1}(R_{m+1})) = A$ or B. □

Proposition 10. *Suppose F is an onto, non-dictatorial social choice rule and the number of alternatives is four or more. Then either F is safely manipulable or at least two alternatives are absent from the set of values F can take at completely agreed profiles.*

Proof. If F is antagonistic then we are done, so suppose it is not. Let F_{-A} be non-dictatorial. If F_{-A} is onto then we are done by induction. So suppose F_{-A} is not onto, and can never take the value B. If $B \in F(\mathcal{R})$ then by Proposition 8 we are done, so suppose the contrary. If F_{-B} is dictatorial then we may apply Proposition 9 to deduce that F is safely manipulable. Suppose F_{-B} is non-dictatorial. If F_{-B} is onto then again by a similar proof to that of Proposition 7 we are done. If F_{-B} is not onto then F_{-B} can never take the value $C \in \mathcal{A} - \{B\}$. If $C \in F(\mathcal{R})$ then by Proposition 8 we are done, so suppose the contrary. Then neither B nor C is in $F(\mathcal{R})$. □

Theorem 2. *Suppose that the number of alternatives is at least three. Then any onto and non-dictatorial social choice rule is safely manipulable by a single voter.*

Proof. Suppose F is a social choice rule that is neither antagonistic nor weakly unanimous, and that the number of alternatives is four or more. Further suppose neither A nor B is an element of the set $F(\mathcal{R})$. Let L^{AB} be a fixed but otherwise arbitrary linear order of the elements of $\mathcal{A}$ that has A first and B second. Let L^{BA} be the linear order of the elements of $\mathcal{A}$ formed by taking L^{AB} and reversing the spots of A and B while leaving all other spots unchanged. A voter whose preferences can be represented by L^{AB} (L^{BA}) shall be referred to as an L^{AB} (L^{BA}) type. Let $m \geq 2$ be the minimum possible number of voter types present when F takes the values from the set $\{A, B\}$. Let $\mathcal{S}$ denote the set of profiles that have exactly m voter types present and are mapped to A or B by F.

Suppose that no profile in $\mathcal{S}$ has an L^{AB} type present. Pick $R \in \mathcal{S}$. Let V be the entire set of voters having the preference order R_1 at R. Consider the profile $R_{-V}(L^{AB})$. F does not map this profile to A or B because it has $m(A)$ voter types present, and one of those types is L^{AB}. Then

$$A = F(R) = F((R_{-V}(L^{AB}))_{-V}(R_1)) \neq F(R_{-V}(L^{AB})).$$

Thus an L^{AB} type can manipulate at $R_{-V}(L^{AB})$. If this manipulation is unsafe then we may apply Proposition 3 to deduce F is safely manipulable. In the event that no $R \in \mathcal{S}$ has an L^{BA} type present, the analysis proceeds similarly to that immediately above.

Suppose that some profile in $\mathcal{S}$ has an L^{AB} type present, another has an L^{BA} type present, but no profile in $\mathcal{S}$ has both present. Let $R \in \mathcal{S}$ have L^{AB} types present. Let V be the entire set of voters having the preference order R_1 (without loss of generality $R_1 \neq L^{AB}$) at R. Clearly, either $F(R_{-V}(L^{BA})) \in \{A, B\}$ or the profile $R_{-V}(L^{BA})$ is manipulable by an L^{BA} type. If the latter manipulation is unsafe, it is nonetheless such that we may apply Proposition 3. In the former case we obtain a profile $R_{-V}(L^{BA})$ in $\mathcal{S}$ where both types L^{AB} and L^{BA} present. This was assumed not to be the case.

The remaining case is that some profile in $\mathcal{S}$ has both L^{AB} and L^{BA} types present. Let $R \in \mathcal{S}$ be one such profile. Let U be the set of voters with preference L^{AB} at R, and W be the set of voters with preference L^{BA} at R. Define a new partial preference relation on $\mathcal{A}$ as follows: $\succ_{U\&W}$ if and only if $\succ_U$ and $\succ_W$. Note that the restriction of $\succ_{U\&W}$ on $\mathcal{A} \setminus \{A, B\}$ is a linear order. Then

$$A, B \neq F(R_{-U}(L^{BA})) \succeq_{U\&W} F(R_{-W}(L^{AB})) \neq A, B.$$

The inequalities follow from the fact that the profiles in question have only $m-1$ types present. The direction of the preference relation is arbitrary and without loss of generality. Now

$$A = F(R) = F((R_{-W}(L^{AB}))_{-W}(L^{BA})) \succ_{U\&W} F(R_{-W}(L^{AB})) \neq A, B.$$

So voters in $W \subset U \cup W$ can manipulate at $R_{-W}(L^{AB})$ (by insincerely voting L^{BA} rather than sincerely voting L^{AB}). If this manipulation is safe we are done; if not then notice that the manipulation is such that we may apply Proposition 3. □

7 Conclusion

This paper has formally distinguished between safe and unsafe manipulation of voting rules. Examples of unsafe manipulations were presented. The Gibbard-Satterthwaite theorem was extended to show that all onto, non-dictatorial social choice rules are safely manipulable.

Favardin and Lepelley (2006) write that 'There has been a significant and increasing interest during recent years in research trying to evaluate the degree of manipulability of various social choice rules'. An investigation into the the ratio of possible unsafe manipulations to possible manipulations under different social choice rules would be a contribution to this research.

We focused on social choice rules for two reasons. Firstly, it allowed us to formally introduce and illustrate the concepts of over and undershooting relatively simply. Secondly, our main result applied only to social choice rules. But the difference between safe and unsafe strategic votes is applicable to a much wider class of choice rules. For example, Slinko and White (2006) identified that strategic over and undershooting can occur under systems of proportional representation. Future research might consider the over and undershooting phenomena under different classes of choice rules. We would inparticular like to know if all systems of proportional representation are safely manipulable by a voter more realistic than a 'daft' one.

Informal comments: My barber volunteered the following story. He always votes because he thinks doing so permits him to criticise the government. Several elections ago, before the introduction of MMP, he voted for what he called one of the 'crazy' (fringe) parties, even though he did not particularly support them. He said he was frustrated with all the main parties, and wanted to signal that. I asked him how he would have felt if the party he voted for had ended up forming the government. He said he would have been horrified.

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Appendix

7.1 Proof of the remaining two cases in Lemma 1

Case 1: $L = ACB$. It may be convenient to refer to the table below while reading the proof. The rows represent the vote of voter one. The columns represent the vote of voter two. Table entries represent the value of F at that particular voter-one-vote, voter-two-vote combination.

	ABC	ACB	BAC
ABC	B		
ACB	A	C	
CAB			

If $F(CAB, ACB) \neq C$ then voter one can manipulate from (CAB, ACB) to (ACB, ACB). So suppose $F(CAB, ACB) = C$. If $F(CAB, ABC) = B$ then voter one can escape from (CAB, ABC) to (ACB, ABC); if $F(CAB, ABC) = A$ then voter two can manipulate from (CAB, ACB) to (CAB, ABC). So suppose $F(CAB, ABC) = C$. Then $F(CAB, BAC) = C$ (if not, voter two can escape from (CAB, ABC) to (CAB, BAC)). Next consider (ACB, BAC). If $F(ACB, BAC) = B$ then voter one can escape from (ACB, BAC) to (CAB, BAC). If $F(ACB, BAC) = C$ then voter two can escape from (ACB, BAC) to (ACB, ABC). So let $F(ACB, BAC) = A$. Then $F(ABC, BAC) = A$, otherwise voter one can manipulate from (ABC, BAC) to (ACB, BAC). But now voter two can manipulate from (ABC, BAC) to (ABC, ABC).

Case 2: $L = CAB$. It may be convenient to refer to the table below while reading the proof.

	ABC	ACB
ABC	B	
ACB		
CAB	A	

If $F(ACB, ABC) \neq A$ then voter one can manipulate (ACB, ABC) by voting CAB. So let $F(ACB, ABC) = A$. If $F(CAB, ACB) \neq A$ then voter two can manipulate (CAB, ACB) by voting ABC. So let $F(CAB, ACB) = A$. If $F(ABC, ACB) \neq A$ then voter one can manipulate (ABC, ACB) by voting CAB. So let $F(ABC, ACB) = A$. If $F(ACB, ACB) = A$ or B then a safe manipulation is possible at (ABC, ABC). If $F(ACB, ACB) = C$ then voter one can manipulate from (CAB, ACB) to (ACB, ACB).

7.2 Proof of Lemma 2

Throughout this proof we shall depict profiles in set form. We may assume that three or more voters are present.

It suffices to show that F being unsafely manipulable implies F is safely manipulable. By Proposition 2, without any loss of generality, suppose that ABC types may strategically overshoot at the profile R^0. If, at R^0, these ABC types can strategically overshoot by voting BAC, BCA, or CBA then an escape can easily be found, and F is safely manipulable (see the proof of Lemma 1 for a similar argument). So we suppose that, at R^0, the ABC types are strategically overshooting by voting ACB or CAB.

Case 1: overshooting by voting ACB. Suppose that, at R^0, some ABC types can strategically overshoot by voting ACB. Let V be the set of ABC types at R^0. Then $F(R^0) = B$ and there must exist some $V_1 \subset V$ such that $F(R^0{}_{-V_1}(ACB)) = A$. Let

$$R^2 = R^0_{-V_1}(ACB) = (1, 2, 3, 4, 5, 6).$$

The intent now is to either directly show that F is safely manipulable or to demonstrate that the two voter social choice rule generated by $1 \cup 2 \cup 3$ and $4 \cup 5 \cup 6$ is well defined, onto, and non-dictatorial. This will imply safe manipulability by Proposition 5. We know that $F(R^2) = A$, and that both 1 and 2 are not empty. Let

$$R^3 = R^0_{-V}(ACB) = (\emptyset, 1 \cup 2, 3, 4, 5, 6).$$

As $F(R^0) = B$, if $F(R^3) = A$ or B then we may apply Proposition 3 to deduce F is safely manipulable. So suppose $F(R^3) = C$. Next let

$$R^1 = (1 \cup 2, \emptyset, 3, 4, 5, 6).$$

If $F(R^1) = A$ or C then consider the manipulation of R^3 by some ACB types voting ABC; if this is unsafe we may then use Proposition 3 to deduce F is safely manipulable. So suppose $F(R^1) = B$. Let

$$\begin{aligned}
R^4 &= (1, 2, 3, \emptyset, 4 \cup 5, 6),\\
R^5 &= (1, 2, 3, \emptyset, 4 \cup 5 \cup 6, \emptyset),\\
R^6 &= (1 \cup 3, 2, \emptyset, \emptyset, 4 \cup 5 \cup 6, \emptyset),\\
R^7 &= (1 \cup 2 \cup 3, \emptyset, \emptyset, \emptyset, 4 \cup 5 \cup 6, \emptyset).
\end{aligned}$$

If $F(R^4) \neq A$ and $4 \neq \emptyset$, then BCA types can escape (to R^4) at R^2. So suppose $F(R^4) = A$ (if $4 = \emptyset$ this is immediate as $R^4 = R^2$). If $F(R^5) \neq A$ and $6 \neq \emptyset$, then CBA types can escape (to R^5) at R^4. So suppose $F(R^5) = A$ (if $6 = \emptyset$ this is immediate as $R^5 = R^2$). Suppose, for now, $3 \neq \emptyset$. If $F(R^6) = B$ then consider the manipulation of R^5 by BAC types voting ABC; if this is unsafe we may then use Proposition 3 to deduce F is safely manipulable. If $F(R^6) = C$ then some ABC types can escape from R^6 (to R^5). So suppose $F(R^6) = A$ (if $3 = \emptyset$ this is immediate). If $2 = \emptyset$, then $F(R^7) = F(R^6) = A \neq C$. If $2 \neq \emptyset$ and $F(R^7) = C$, some ABC types can escape R^7 (to R^6). So suppose $F(R^7) \neq C$. This implies that in the event $4 \cup 5 \cup 6 \neq \emptyset$, the second voter is not a dictator for $F_{1\cup 2\cup 3, 4\cup 5\cup 6}$.

We now show (assuming F is not safely manipulable) that $4 \cup 5 \cup 6 \neq \emptyset$. We will then show (again assuming F is not safely manipulable) the first voter is not a dictator for $F_{1\cup 2\cup 3, 4\cup 5\cup 6}$. Given that F is weakly unanimous, hence onto, we will then have enough to use Proposition 5. We need just three more profiles:

$$R^8 = (\emptyset, 1 \cup 2 \cup 3, \emptyset, 4, 5, 6),$$
$$R^9 = (\emptyset, 1 \cup 2 \cup 3, \emptyset, \emptyset, 5, 4 \cup 6),$$
$$R^{10} = (\emptyset, 1 \cup 2 \cup 3, \emptyset, \emptyset, \emptyset, 4 \cup 5 \cup 6).$$

If $3 = \emptyset$, then $F(R^8) = F(R^3) = C$. Suppose $3 \neq \emptyset$. If $F(R^8) \neq C$, then BAC types can escape from R^3 (to R^8). So set $F(R^8) = C$. Since $F(R^7) \neq C$, by the weak unanimity of F, this implies $4 \cup 5 \cup 6 \neq \emptyset$. If $4 = \emptyset$ then $F(R^9) = F(R^8) = C$. Suppose $4 \neq \emptyset$. If $F(R^9) = A$, then some CBA types can escape from R^9 (to R^8). So let $F(R^9) \neq A$. If $5 = \emptyset$ then $F(R^{10}) \neq A$. Suppose $5 \neq \emptyset$. If $F(R^{10}) = A$, then some CBA types can escape from R^{10} (to R^9). So let $F(R^{10}) \neq A$. This implies that the first voter is not a dictator for $F_{1\cup 2\cup 3, 4\cup 5\cup 6}$.

$F_{1\cup 2\cup 3, 4\cup 5\cup 6}$ inherits weak unanimity from F. Hence $F_{1\cup 2\cup 3, 4\cup 5\cup 6}$ is onto. Then by Proposition 5, F is safely manipulable.

Case 2: overshooting by voting CAB. Suppose that, at R^0, some ABC types can strategically overshoot by voting CAB. This implies $F(R^0) = B$. Let V be the set of ABC types at R^0. There must exist some $V_1 \subset V$ such that $F({R^0}_{-V_1}(CAB)) = A$. Let

$$R^2 = R^0_{-V_1}(CAB) = (1, 2, 3, 4, 5, 6).$$

The intent now is to either directly show that F is safely manipulable or to demonstrate that the two voter social choice rule generated by $1 \cup 2 \cup 5$ and $3 \cup 4 \cup 6$ is onto, and non-dictatorial. We know $F(R^2) = A$, and $1, 5 \neq \emptyset$. Let

$$R^3 = R^0_{-V}(CAB) = (\emptyset, 2, 3, 4, 1 \cup 5, 6).$$

If $F(R^3) = A$ or B then we may apply Proposition 3 to deduce F is safely manipulable. So suppose $F(R^3) = C$. Next let

$$R^1 = (1 \cup 5, 2, 3, 4, \emptyset, 6).$$

If there are no CAB types present at R^0 then $R^0 = R^1$, and $F(R^1) = B$. Now suppose that there are CAB types present at R^0, and $F(R^1) \neq B$; given $F(R^0) = B$, CAB types are then capable of escaping from R^0 to R^1. So let us suppose $F(R^1) = B$. Let

$$\begin{aligned} R^4 &= (1 \cup 2 \cup 5, \emptyset, 3, 4, \emptyset, 6), \\ R^5 &= (1 \cup 2 \cup 5, \emptyset, \emptyset, 4, \emptyset, 3 \cup 6), \\ R^6 &= (1 \cup 2 \cup 5, \emptyset, \emptyset, \emptyset, \emptyset, 3 \cup 4 \cup 6). \end{aligned}$$

If $2 = \emptyset$, then $F(R^4) = F(R^1) = B$. Suppose $2 \neq \emptyset$. If $F(R^4) \neq B$ then ACB types can escape from R^1 to R^4. So let $F(R^4) = B$. We note that by weak unanimity this implies $3 \cup 4 \cup 6 \neq \emptyset$. If $3 = \emptyset$, then $F(R^5) = F(R^4) = B$. Suppose $3 \neq \emptyset$; if $F(R^5) = A$ then some CBA types can escape from R^5 to R^4. Let $F(R^5) \neq A$ then. If $4 = \emptyset$, $F(R^6) \neq A$. If $4 \neq \emptyset$ then in the event $F(R^6) = A$, some CBA types can escape from R^6 to R^5. So $F(R^6) \neq A$. This implies the first voter is not a dictator for $F_{1 \cup 2 \cup 5, 3 \cup 4 \cup 6}$.

It remains to show that the second voter is not a dictator for $F_{1 \cup 2 \cup 5, 3 \cup 4 \cup 6}$. For this purpose we will need four more profiles:

$$\begin{aligned} R^7 &= (1, 2, 3 \cup 4, \emptyset, 5, 6), \\ R^8 &= (1, 2, 3 \cup 4 \cup 6, \emptyset, 5, \emptyset), \\ R^9 &= (\emptyset, 2, 3 \cup 4 \cup 6, \emptyset, 1 \cup 5, \emptyset), \\ R^{10} &= (\emptyset, \emptyset, 3 \cup 4 \cup 6, \emptyset, 1 \cup 2 \cup 5, \emptyset). \end{aligned}$$

If $4 = \emptyset$, then $F(R^7) = F(R^2) = A$. Suppose $4 \neq \emptyset$. If $F(R^7) \neq A$ then BCA types can escape from R^2 to R^7. So let $F(R^7) = A$. If $6 = \emptyset$, then $F(R^8) = F(R^7) = A$. Suppose $6 \neq \emptyset$. If $F(R^8) \neq A$ then CBA types can escape from R^7 to R^8. So let $F(R^8) = A$. If $F(R^9) = B$ then some CAB types can escape from R^9 to R^8. So let $F(R^9) \neq B$. If $2 = \emptyset$, then $F(R^{10}) = F(R^9) \neq B$. Suppose $2 \neq \emptyset$. If $F(R^{10}) = B$ then some CAB types can escape from R^{10} to R^9. So let $F(R^{10}) \neq B$. But then the second voter is not a dictator for $F_{1 \cup 2 \cup 5, 3 \cup 4 \cup 6}$.

$F_{1 \cup 2 \cup 5, 3 \cup 4 \cup 6}$ inherits weak unanimity from F. Hence $F_{1 \cup 2 \cup 5, 3 \cup 4 \cup 6}$ is onto. Then by Proposition 5, F is safely manipulable.

7.3 Strategic undershooting, overshooting under plurality

Suppose 90 voters are using a plurality scoring rule to select one of the four alternatives A, B, C, D. Suppose the rule incorporates the following tie breaking procedure: in a two-way tie the highest scoring alternative whose name appears earliest in the alphabet wins; in a three-way tie the highest scoring alternative whose name appears latest in the alphabet wins. As examples, A beats B in a two-way tie, while D beats B and C in a three-way tie. Let the number of different voter types present at the profile of sincere preferences be

Preference order	Number of voters
$ABCD$	22
$BCDA$	24
$CDBA$	24
$DABC$	20

If all voters vote truthfully then B beats C in a two way tie. Consider the position of the $DABC$ types. If two of them claim to rank A highest (ceteris paribus) then C wins in a three way tie. This renders these types worse off. But if three or more of them claim to rank A first (ceteris paribus) then A wins. Thus $DABC$ types can strategically undershoot at the given profile of sincere preferences. Now suppose the profile of sincere preferences is

Preference order	Number of voters
$ABCD$	22
$BCDA$	24
$CDBA$	24
$DCBA$	20

If all voters vote truthfully B beats C. If two $DCBA$ types claim to rank A highest (ceteris paribus) then C wins in a three way tie, while if three or more $DCBA$ types claim to rank A highest (ceteris paribus) then A wins. Thus $DCBA$ types can strategically undershoot at the given profile of sincere preferences. Note that in the first case the $DCBA$ types have no alternative strategic possibilities, whilst in the second case they do, and would in fact do better to claim to rank C rather than A highest.

Consider a scoring social choice rule F. If R is a profile, then let $top_F(R)$ denote the set of alternatives with the highest score, according to F, at R. We classify F as *consistent* if it meets the following two conditions. Firstly, if R is any profile then $F(R) \in top_F(R)$. Secondly, if R and R' are any two profiles such that $top_F(R) = top_F(R')$ then $F(R) = F(R')$. Consistency is a property derived from the tie-breaking procedure. If a scoring social choice rule is consistent then, for example, whenever alternatives A and B tie for first place, the rule always picks A or always picks B.

Proposition 11. *Suppose F is a consistent plurality scoring social choice rule. Suppose that voter i can make an unsafe strategic vote at R. Then F needs to employ a tie-breaking procedure at R.*

Proof. Let us use the following notation: if $X \in \mathcal{A}$ is an alternative and R is a profile then we shall denote by $sc_X(R)$ the score of X at R (i.e. the number of voters ranking X first at

R). Suppose that when the profile of sincere preferences is R, voter i can strategically over or undershoot by voting $L \neq R_i$. Let V be the set of voters with preferences identical to i at R. If i can over or undershoot at R then the set $\mathcal{S} := \{F(R_{-U}(L)) \mid U \subseteq V\}$ must contain at least three elements.

Let us suppose $F(R) = A$. Then the linear order represented by L cannot place A first. Otherwise no subset of V could bring about a change in the value of F by voting L (cet. par.). So suppose the linear order represented by L places B first. Consequently, voter i's favorite alternative cannot be A or B; suppose it is C.

If either $sc_A(R) < sc_B(R)$ or $sc_A(R) < sc_C(R)$ then we contradict the supposition that $F(R) = A$. If either $sc_A(R) = sc_B(R)$ or $sc_A(R) = sc_C(R)$ then our desired result is immediate. So here-on suppose $sc_A(R) > sc_B(R)$ and $sc_A(R) > sc_C(R)$.

Suppose, for contradiction, that $sc_A(R_{-V}(L)) > sc_B(R_{-V}(L))$. Then $F(R_{-U}(L)) = F(R) = A$ for all $U \subseteq V$ and $\mathcal{S}$ has just one element. Suppose, again for contradiction, that $sc_A(R_{-V}(L)) = sc_B(R_{-V}(L))$. Let $U \subseteq V$. Then $F(R_{-U}(L)) = F(R) = A$ for all $U \subset V$ and $F(R_{-U}(L)) \neq F(R) = A$ if and only if $U = V$. In which case $\mathcal{S}$ has just two elements. So, in order for $\mathcal{S}$ to contain three or more elements, it must be that $sc_A(R_{-V}(L)) < sc_B(R_{-V}(L))$

We now have $sc_A(R) > sc_B(R)$ and $sc_A(R_{-V}(L)) < sc_B(R_{-V}(L))$. Thus there exists $W \subset V$ such that $W \neq \emptyset$ and $sc_A(R_{-W}(L)) = sc_B(R_{-W}(L))$.

Suppose there exists a set U such that $\emptyset \neq U \subset W$. Then $sc_A(R_{-U}(L)) > sc_B(R_{-U}(L))$. The set of alternatives scoring highest at R is the same as the set of alternatives scoring highest at $R_{-U}(L)$. Hence, by the consistency of F, $F(R_{-U}(L)) = A$. Next, let U be such that $W \subset U \subseteq V$. One then has $sc_A(R_{-U}(L)) < sc_B(R_{-U}(L))$, and $F(R_{-U}(L)) = B$.

Therefore if $\mathcal{S}$ contains an element other than A or B, it is realised at $R_{-W}(L)$, when $sc_A(R_{-W}(L)) = sc_B(R_{-W}(L))$, and when F must therefore be breaking a three-(or more)-way tie.

Suppose that the set of alternatives scoring highest at $R_{-W}(L)$ contains A, B, and D. It is not possible that $C = D$: consider the fact that $sc_C(R_{-W}(L)) < sc_C(R) < sc_A(R) = sc_A(R_{-W}(L))$. The only alternatives whose scores are altered by the manipulative moves of the members of W are B and C. Then $sc_D(R) = sc_D(R_{-W}(L)) = sc_A(R_{-W}(L)) = sc_A(R)$, and F must break a tie at R.

□