RICH FAMILIES, W-SPACES AND THE PRODUCT OF BAIRE SPACES

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Abstract. In this paper we prove a theorem more general than the following. Suppose that X is a Baire space and Y is the product of hereditarily Baire metric spaces then $X \times Y$ is a Baire space.

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1. Introduction

A topological space X is said to be a *Baire* space if for each sequence $(O_n : n \in \mathbb{N})$ of dense open subsets of X, $\bigcap_{n\in\mathbb{N}} O_n$ is dense in X and a Baire space Y is called *barely Baire* if there exists a Baire space Z such that $Y \times Z$ is not Baire. It is well known that there exist metrizable barely Baire spaces, (see [5]). On the other hand it has recently been shown that the product of a Baire space X with a hereditarily Baire metric space Y is Baire, [7]. In that same paper the author claims in a "Remark" that the hypothesis on Y can be reduced to: "Y is the product of hereditarily Baire metric spaces". In this paper we substantiate this claim.

The main result of this paper relies upon two notions. The first, which is that of a W-space [6], is recalled in Section 2. The second, which is that of a "rich family" is considered in Section 3. In Section 4, we shall prove our main theorem which states that the product of a Baire space with a W-space that possesses a rich family of Baire subspaces is Baire.

2. W-spaces

In this paper all topological spaces are assumed to be regular, Hausdorff and nonempty. Furthermore, if X is a topological space and $a \in X$ then we shall always denote by $\mathcal{N}(a)$ the set of all neighbourhoods of a.

For any point a in a topological space X we can consider the following two player topological game, called the $G(a)$-*game*. This game is played between the players α and β and although it may seem unfair, β will always

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be granted the priviledge of the first move. To define this game we must first specify the rules and then also specify the definition of a win.

The moves of the player α are simple. He/she must always select a neighbourhood of the point a. However, the moves of the player β depend upon the previous move of α. Specifically, for his/her first move β may select any point $x_1 \in X$. For α's first move, as mentioned earlier, α must select a neighbourhood O_1 of a. Now, for β's second move he/she must select a point $x_2 \in O_1$. For α's second move he/she is entitled to select any neighbourhood O_2 of a. In general, if α has chosen $O_n \in \mathcal{N}(a)$ as his/her n^{th} move of the $G(a)$-game then β is obliged to choose a point $x_{n+1} \in O_n$. The response of α is then simply to choose any neighbourhood O_{n+1} of a. Continuing in this fashion indefinitely, the players α and β produce a sequence $((x_n, O_n) : n \in \mathbb{N})$ of ordered pairs with $x_{n+1} \in O_n \in \mathcal{N}(a)$ for all $n \in \mathbb{N}$, called a *play* of the $G(a)$-game. A *partial play* $((x_k, O_k) : 1 \leq k \leq n)$ of the $G(a)$-game consists of the first n moves of a play of the $G(a)$-game. We shall declare α the *winner* of a play $((x_n, O_n) : n \in \mathbb{N})$ of the $G(a)$-game if $a \in \overline{\{x_n : n \in \mathbb{N}\}}$, otherwise, β is the winner. That is, β is declared the winner of the play $((x_n, O_n) : n \in \mathbb{N})$ if, and only if, $a \notin \overline{\{x_n : n \in \mathbb{N}\}}$.

A *strategy* for the player α is a rule that specifies his/her moves in every possible situation that can occur. More precisely, a strategy for α is an inductively defined sequence of functions $t := (t_n : n \in \mathbb{N})$. The domain of t_1 is X^1 and for each $(x_1) \in X^1$, $t_1(x_1) \in \mathcal{N}(a)$, i.e., $((x_1, t_1(x_1)))$ is a partial play. Inductively, if $t_1, t_2, \ldots, t_n$ have been defined then the domain of t_{n+1} is defined to be,

$$\{(x_1, x_2, \ldots, x_{n+1}) \in X^{n+1} : (x_1, x_2, \ldots, x_n) \in \mathrm{Dom}(t_n) \\ \text{and } x_{n+1} \in t_n(x_1, x_2, \ldots, x_n)\}.$$

For each $(x_1, x_2, \ldots, x_{n+1}) \in \mathrm{Dom}(t_{n+1})$, $t_{n+1}(x_1, x_2, \ldots, x_{n+1}) \in \mathcal{N}(a)$. Equivalently, for each $(x_1, x_2, \ldots, x_{n+1}) \in \mathrm{Dom}(t_{n+1})$, $((x_k, t_k(x_1, \ldots, x_k)) : 1 \leq k \leq n+1)$ is a partial play.

A *partial t-play* is a finite sequence $(x_1, x_2, \ldots, x_n) \in X^n$ such that $(x_1, x_2, \ldots, x_n) \in \mathrm{Dom}(t_n)$ or, eqivalently, if $x_{k+1} \in t_k(x_1, x_2, \ldots, x_k)$ for all $1 \leq k < n$. A *t-play* is an infinite sequence $(x_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $(x_1, x_2, \ldots x_n)$ is a partial t-play.

A strategy $t := (t_n : n \in \mathbb{N})$ for the player α is said to be a *winning strategy* if each play of the form $((x_n, t_n(x_1, x_2, \ldots, x_n)) : n \in \mathbb{N})$ is won by α, or equivalently, if $a \in \overline{\{x_n : n \in \mathbb{N}\}}$ for each t-play $(x_n : n \in \mathbb{N})$.

A topological space X is called a *W-space* if α has a winning strategy in the $G(a)$-game for each $a \in X$, [6].

In the remainder of this section we shall recall some relevant facts concerning W-spaces.

Theorem 2.1. [6, Theorem 3.3] *Every first countable space is a W-space.*

There are of course many W-spaces that are not first countable, see Example 2.7.

A topological space X is said to have *countable tightness* if for each nonempty subset A of X and each $p \in \overline{A}$, there exists a countable subset $C \subseteq A$ such that $p \in \overline{C}$.

Proposition 2.2. [6, Corollary 3.4] *Every W-space has countable tightness.*

Proposition 2.3. [6, Theorem 3.1] *If X is a W-space and $\varnothing \neq A \subseteq X$ then A is a W-space.*

Lemma 2.4. [6, Theorem 3.9] *Suppose that X is a W-space and $a \in X$, then the player α possesses a strategy $s := (s_n : n \in \mathbb{N})$ in the $G(a)$-game such that every s-play converges to a.*

For the remainder of this paper whenever we shall consider a W-space X with $a \in X$ we shall assume that the player α is employing a strategy t, in the $G(a)$-game, in which every t-play converges to a.

Let $\{X_s : s \in S\}$ be a nonempty family of topological spaces and let $a \in \Pi_{s\in S} X_s$. The Σ-*product* of this family with *base point* a, denoted by $\Sigma_{s\in S} X_s(a)$, is the set of all $x \in \Pi_{s\in S} X_s$ such that $x(s) \neq a(s)$ for at most countably many $s \in S$. For each $x \in \Sigma_{s\in S} X_s(a)$, the *support of* x is defined by $\text{supp}(x) := \{s \in S : x(s) \neq a(s)\}$.

Theorem 2.5. [6, Theorem 4.6] *Suppose that $\{X_s : s \in S\}$ is a nonempty family of W-spaces. If $a \in \Pi_{s\in S} X_s$ then $\Sigma_{s\in S} X_s(a)$ is a W-space.*

Corollary 2.6. [6, Theorem 4.1] *If $\{X_n : n \in \mathbb{N}\}$ are W-spaces, then so is $\Pi_{n\in\mathbb{N}} X_n$.*

Example 2.7. Suppose that S is a nonempty set. For each $s \in S$, let $X_s := [0, 1]$ and define $a : S \to [0, 1]$ by, $a(s) := 0$ for all $s \in S$. Then by Theorem 2.5, $X := \Sigma_{s\in S} X_s(a)$ is a W-space. However, X is not first countable whenever S is uncountable.

3. Rich familes

Let X be a topological space, and let $\mathcal{F}$ be a family of nonempty, closed and separable subsets of X. Then $\mathcal{F}$ is *rich* if the following two conditions are satisfied:

(i) for every separable subspace Y of X, there exists an $F \in \mathcal{F}$ such that $Y \subseteq F$;
(ii) for every increasing sequence $(F_n : n \in \mathbb{N})$ in $\mathcal{F}$, $\overline{\bigcup_{n\in\mathbb{N}} F_n} \in \mathcal{F}$.

For any topological space X, the collection of all rich families of subsets forms a partially ordered set, under the binary relation of set inclusion. This partially ordered set has a greatest element, $\mathcal{S}_X := \{S \in 2^X : S$ is a nonempty, closed and separable subset of $X\}$. On the other hand, if X is a separable space, then the partially ordered set has a least element, namely $\{X\}$.

Next we present an important property of rich families. For a proof of this see [2, Proposition 1.1].

Proposition 3.1. *Suppose that X is a topological space. If $\{\mathcal{F}_n : n \in \mathbb{N}\}$ are rich families then so is $\bigcap_{n\in\mathbb{N}} \mathcal{F}_n$.*

Suppose that X is a topological space and S is a separable subset, it can be easily verified that the family $\mathcal{F}_S := \{F \in \mathcal{S}_X : S \subseteq F\}$ is rich. Hence, whenever X is an infinite set and $\mathcal{F}$ is a rich family of subsets of X, then we can always assume, by possibly passing to a sub-family, that all the members of $\mathcal{F}$ are infinite. Indeed, if X has a countably infinite subset A, then by Proposition 3.1, $\mathcal{F} \cap \mathcal{F}_A \subseteq \mathcal{F}$ is a rich family whose members are all infinite.

Proposition 3.2. *If X is a topological space with countable tightness (e.g. if X is a W-space) and E is a dense subset of X then*

$$\mathcal{F} := \{F \in \mathcal{S}_X :\ E \cap F \textit{ is dense in } F\}$$

is a rich family.

Proof: Let Y be a separable subspace of X, then Y has a countable dense subset $D := \{d_n : n \in \mathbb{N}\}$. Since X has countable tightness, for each $n \in \mathbb{N}$, there is a countable subset $C_n \subseteq E$ such that $d_n \in \overline{C_n}$. Let $F := \overline{\bigcup_{n\in\mathbb{N}} C_n}$, then $Y = \overline{D} \subseteq F \in \mathcal{S}_X$ and

$$F = \overline{\bigcup_{n\in\mathbb{N}} C_n} \subseteq \overline{E \cap F} \subseteq F.$$

Therefore, $F \in \mathcal{F}$. Now suppose that $(F_n : n \in \mathbb{N})$ is an increasing sequence in $\mathcal{F}$. Then $F' := \overline{\bigcup_{n\in\mathbb{N}} F_n} \in \mathcal{S}_X$ and $F' \cap E$ is dense in F'. Therefore, $F' \in \mathcal{F}$. □

Theorem 3.3. *Suppose that X is a topological space with countable tightness (in particular if X is a W-space) that possesses a rich family $\mathcal{F}$ of Baire subspaces then X is also a Baire space.*

Proof: Let $\{O_n : n \in \mathbb{N}\}$ be dense open subsets of X. For each $n \in \mathbb{N}$, let $\mathcal{F}_n := \{F \in \mathcal{S}_X : O_n \cap F \text{ is dense in } F\}$, then $\mathcal{F}_n$ is a rich family by Proposition 3.2. Let $\mathcal{F}^* = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n \cap \mathcal{F}$, then $\mathcal{F}^*$ is also a rich family by Proposition 3.1. For each $F \in \mathcal{F}^*$, $\bigcap_{n\in\mathbb{N}}(O_n \cap F)$ is dense in F since F is a Baire space. Let $x \in X$, then there is $F \in \mathcal{F}^*$ such that $x \in F$. Then $x \in \overline{\bigcap_{n\in\mathbb{N}}(O_n \cap F)} \subseteq \overline{\bigcap_{n\in\mathbb{N}} O_n}$. Therefore, $\overline{\bigcap_{n\in\mathbb{N}} O_n} = X$. □

Suppose that $\{X_s : s \in S\}$ is a nonempty family of topological spaces and $a \in \Pi_{s\in S} X_s$. A *cube E in $\Sigma_{s\in S} X_s(a)$* is any nonempty product $\Pi_{s\in S} E_s \subseteq \Sigma_{s\in S} X_s(a)$. The set $C_E := \{s \in S : E_s \neq \{a(s)\}\}$ is at most countable and E is homeomorphic to $\Pi_{s\in C_E} E_s$. If for each $s \in S$, $\mathcal{F}_s$ is a rich family of subsets of X_s then the Σ*-product of the rich families, with the base point*

$a \in \Pi_{s\in S}X_s$, denoted by $\Sigma_{s\in S}\mathcal{F}_s(a)$, is the set of all cubes $E := \Pi_{s\in S}E_s$ in $\Sigma_{s\in S}X_s(a)$ such that $E_s \in \mathcal{F}_s$ for each $s \in C_E$.

Lemma 3.4. *Let $\{X_s : s \in S\}$ be a nonempty family of topological spaces. For each $s \in S$, let $(E_n^s : n \in \mathbb{N})$ be an increasing sequence of nonempty subsets of X_s. Then $\overline{\bigcup_{n\in\mathbb{N}}(\Pi_{s\in S}E_n^s)} = \overline{\Pi_{s\in S}(\bigcup_{n\in\mathbb{N}} E_n^s)}$.*

Proof: It is easy to see that $\overline{\bigcup_{n\in\mathbb{N}}(\Pi_{s\in S}E_n^s)} \subseteq \overline{\Pi_{s\in S}(\bigcup_{n\in\mathbb{N}} E_n^s)}$ since for all $n \in \mathbb{N}$, $\Pi_{s\in S}E_n^s \subseteq \overline{\Pi_{s\in S}(\bigcup_{n\in\mathbb{N}} E_n^s)}$.

Let $x \in \overline{\Pi_{s\in S}(\bigcup_{n\in\mathbb{N}} E_n^s)}$ and let $U := \Pi_{s\in S}U_s$ be a basic neighbourhood of x. Then there exists $y \in U \cap \Pi_{s\in S}(\bigcup_{n\in\mathbb{N}} E_n^s)$. Let M be the finite set $\{s \in S : U_s \neq X_s\}$, and let $N_s := \min\{n \in \mathbb{N} : y(s) \in E_n^s\}$ for all $s \in M$. Let $N := \max\{N_s : s \in M\}$, then $y(s) \in E_N^s$ for all $s \in M$. Let $a \in \Pi_{s\in S}E_N^s$ and let $y' \in U$ be defined by $y'(s) := y(s)$ for all $s \in M$ and $y'(s) := a(s)$ for all $s \in S \setminus M$. Since $y' \in \Pi_{s\in S}E_N^s$, $U \cap \bigcup_{n\in\mathbb{N}}(\Pi_{s\in S}E_n^s) \neq \varnothing$. Therefore, $x \in \overline{\bigcup_{n\in\mathbb{N}}(\Pi_{s\in S}E_n^s)}$. $\square$

Theorem 3.5. *Suppose that $\{X_s : s \in S\}$ is a nonempty family of topological spaces and $a \in \Pi_{s\in S}X_s$. If for each $s \in S$, $\mathcal{F}_s$ is a rich family of subsets of X_s, then $\Sigma_{s\in S}\mathcal{F}_s(a)$ is a rich family of subsets of $\Sigma_{s\in S}X_s(a)$.*

Proof: Let Y be a separable subspace of $\Sigma_{s\in S}X_s(a)$, then it has a countable dense subset D. Let $C := \bigcup_{d\in D}\operatorname{supp}(d)$, then C is a countable set. For each $s \in C$, let P_s be the projection of D onto X_s, then P_s is countable and hence there is some $E_s \in \mathcal{F}_s$ such that $\overline{P_s} \subseteq E_s$. For each $s \in S \setminus C$, let $E_s := \{a(s)\}$. Let $F := \Pi_{s\in S}E_s$, then $F \in \Sigma_{s\in S}\mathcal{F}_s(a)$ and $Y \subseteq F$.

Let $(E_n : n \in \mathbb{N})$ be an increasing sequence in $\Sigma_{s\in S}\mathcal{F}_s(a)$. For each cube $E_n \in \Sigma_{s\in S}\mathcal{F}_s(a)$, let $E_n := \Pi_{s\in S}E_n^s$. Then by Lemma 3.4

$$\overline{\bigcup_{n\in\mathbb{N}} E_n} = \overline{\bigcup_{n\in\mathbb{N}}(\Pi_{s\in S}E_n^s)} = \overline{\Pi_{s\in S}(\bigcup_{n\in\mathbb{N}} E_n^s)} = \Pi_{s\in S}(\overline{\bigcup_{n\in\mathbb{N}} E_n^s}).$$

It now follows that $\overline{\bigcup_{n\in\mathbb{N}} E_n} \in \Sigma_{s\in S}\mathcal{F}_s(a)$. $\square$

4. Baire spaces and Σ-products

A subset R of a topological space X is *residual* in X if there exist dense open subsets $\{O_n : n \in \mathbb{N}\}$ of X such that $\bigcap_{n\in\mathbb{N}} O_n \subseteq R$.

For any subset R of a topological space X we can consider the following two player topological game, called the *$BM(R)$-game*. This game is played between two players α and β and, as with the $G(a)$-game, the player β is always granted the priviledge of the first move. To define this game we must first specify the rules and then specify the definition of a win.

The player β's first move is to select a nonempty open subset B_1 of X. For α's first move he/she must also select a nonempty open subset A_1 of B_1. Now, for β's second move he/she must select a nonempty open subset B_2 of A_1. For α's second move he/she must select a nonempty open subset A_2

of B_2. In general, if α has chosen A_n as his/her n^{th} move of the $BM(R)$-game then β is obliged to select a nonempty open subset B_{n+1} of A_n. The response of α is then simply to select any nonempty open subset A_{n+1} of B_{n+1}. Continuing in this fashion indefinitely the players α and β produce a sequence $((B_n, A_n) : n \in \mathbb{N})$ of ordered pairs of nonempty open subsets of X such that $B_{n+1} \subseteq A_n \subseteq B_n$ for all $n \in \mathbb{N}$, called a *play* of the $BM(R)$-game. A *partial play* $((B_k, A_k) : 1 \leq k \leq n)$ of the $BM(R)$-game consists of the first n moves of a play of the $BM(R)$-game. We shall declare α the *winner* of a play $((B_n, A_n) : n \in \mathbb{N})$ of the $BM(R)$-game if $\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} B_n \subseteq R$, otherwise, β is declared the winner. That is, β is the winner if, and only if, $\bigcap_{n\in\mathbb{N}} B_n \not\subseteq R$.

A *strategy* for the player α is an inductively defined sequence of functions $t := (t_n : n \in \mathbb{N})$. The domain of t_1 is the family of all nonempty open subsets of X and for each $B_1 \in \mathrm{Dom}(t_1)$, $t_1(B_1)$ must be a nonempty open subset of B_1 or, equivalently, for each $B_1 \in \mathrm{Dom}(t_1)$, $t_1(B_1)$ is defined so that $((B_1, t_1(B_1)))$ is a partial play of the $BM(R)$-game. Inductively, if $t_1, t_2, \ldots, t_n$ have been defined then the domain of t_{n+1} is defined to be:

$$\{(B_1, B_2, \ldots, B_{n+1}) : (B_1, B_2, \ldots, B_n) \in \mathrm{Dom}(t_n) \text{ and } B_{n+1} \text{ is a nonempty open subset of } t_n(B_1, B_2, \ldots, B_n)\}.$$

For each $(B_1, B_2, \ldots, B_{n+1}) \in \mathrm{Dom}(t_{n+1})$, $t_{n+1}(B_1, B_2, \ldots, B_{n+1})$ must be a nonempty open subset of B_{n+1}. Alternatively, but equivalently, for each $(B_1, B_2, \ldots, B_{n+1}) \in \mathrm{Dom}(t_{n+1})$, $t_{n+1}(B_1, B_2, \ldots, B_{n+1})$ is defined so that $((B_k, t_k(B_1, B_2, \ldots, B_k)) : 1 \leq k \leq n+1)$ is a partial play. A *partial t-play* is a finite sequence $(B_1, B_2, \ldots, B_n)$ such that $(B_1, B_2, \ldots, B_n) \in \mathrm{Dom}(t_n)$ or, equivalently, B_{k+1} is a nonempty open subset of $t_k(B_1, B_2, \ldots, B_k)$ for all $1 \leq k < n$. A *t-play* is an infinite sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $(B_1, B_2, \ldots, B_n)$ is a partial t-play.

A strategy $t := (t_n : n \in \mathbb{N})$ for the player α is said to be a *winning strategy* if each play of the form $((B_n, t_n(B_1, B_2, \ldots, B_n)) : n \in \mathbb{N})$ is won by α, or equivalently, if $\bigcap_{n\in\mathbb{N}} B_n \subseteq R$ for each t-play $(B_n : n \in \mathbb{N})$. For more information on the $BM(R)$-game see [3].

Our interest in the $BM(R)$-game is revealed in the next lemma.

Lemma 4.1 ([9]). *Let R be a subset of a topological space X. Then R is residual in X if, and only if, the player α has a winning strategy in the $BM(R)$-game played on X.*

The next simple result plays a key role in the proof of our main theorem (Theorem 4.3).

Lemma 4.2. *Let X and Y be topological spaces and let O be a dense open subset of $X \times Y$. Given nonempty open subsets V_1, V_2,...,V_m of Y and a nonempty open subset U of X, there exists a nonempty open subset $W \subseteq U$ and elements $y_i \in V_i$, $1 \leq i \leq m$, such that $W \times \{y_1, ..., y_m\} \subseteq O$.*

Proof: The result will be shown inductively on m.

Base Step: $m = 1$. Since $U \times V_1$ is nonempty and open in $X \times Y$ and O is dense and open in $X \times Y$, $(U \times V_1) \cap O$ is a nonempty open subset of $X \times Y$. Therefore, there is a nonempty open subset $W \subseteq U$ and an element $y_1 \in V_1$ such that $W \times \{y_1\} \subseteq (U \times V_1) \cap O \subseteq O$.

Inductive Step: Suppose that the result holds for $m = k$ and consider the case when $m = k + 1$. According to the inductive hypothesis, there exists a nonempty open subset $W' \subseteq U$ and elements $y_i \in V_i$, $1 \leq i \leq k$, such that $W' \times \{y_1, ..., y_k\} \subseteq O$. By repeating the base step, there is a nonempty open subset $W \subseteq W'$ and an element $y_{k+1} \in V_{k+1}$ such that $W \times \{y_{k+1}\} \subseteq O$. Clearly, $W \times \{y_1, ..., y_{k+1}\} \subseteq O$. $\square$

Theorem 4.3. *Suppose that Y is a W-space and X is a topological space. If Z is a separable subset of Y and $\{O_n : n \in \mathbb{N}\}$ are dense open subsets of $X \times Y$ then for each rich family $\mathcal{F}$ of Y the subset*

$$R := \{x \in X : \text{ there exists a } F_x \in \mathcal{F} \text{ containing } Z \text{ such that } \{y \in F_x : (x, y) \in O_n\} \text{ is dense in } F_x \text{ for all } n \in \mathbb{N}\}$$

is residual in X.

Proof: We are going to apply the $BM(R)$-game and Lemma 4.1 to show that R is residual in X. We shall only consider the case when Y is infinite as the case when Y is finite (and hence has the discrete topology) follows from Lemma 4.2. Thus we can assume that all the members of $\mathcal{F}$ are infinite. Moreover, without loss of generality, we can also assume that all the sets $\{O_n : n \in \mathbb{N}\}$ are decreasing. For each $a \in Y$, let $t^a := (t^a_n : n \in \mathbb{N})$ be a winning strategy for the player α in the $G(a)$-game.

We shall inductively define a strategy $s := (s_n : n \in \mathbb{N})$ for the player α in the $BM(R)$-game played on X, but first let us choose $y \in Y$, set $z_{(i,j,0)} := y$ for all $(i, j) \in \mathbb{N}^2$, set $Z_0 := \{z_{(1,1,0)}\}$ and let $\mathscr{F}_0$ be any countable subset of Y such that $Z \subseteq \overline{\mathscr{F}_0} \in \mathcal{F}$.

Base Step: Suppose that (B_1) is a partial s-play. We shall define the following:

(i) a countable set $\mathscr{F}_1 := \{f_{(1,n)} : n \in \mathbb{N}\}$ such that $Z_0 \cup \mathscr{F}_0 \subseteq \overline{\mathscr{F}_1} \in \mathcal{F}$;
(ii) $s_1(B_1)$ and $z_{(1,1,1)}$ so that:
   (a) $s_1(B_1)$ is a nonempty open subset of B_1;
   (b) $z_{(1,1,1)} \in t_1^{f_{(1,1)}}(z_{(1,1,0)})$, i.e., $(z_{(1,1,0)}, z_{(1,1,1)}) \in \mathrm{Dom}(t_2^{f_{(1,1)}})$;
   (c) $s_1(B_1) \times \{z_{(1,1,1)}\} \subseteq O_1$.
   Note that this is possible by Lemma 4.2.

Finally, define $Z_1 := \{z_{(1,1,1)}\}$.

Inductive Hypothesis: Suppose that $(B_1, ..., B_k)$ is a partial s-play, and for each $1 \leq n \leq k$, the following terms have been defined, $\mathscr{F}_n = \{f_{(n,j)} : j \in \mathbb{N}\}$, $Z_n = \{z_{(i,j,l)} : (i, j, l) \in \mathbb{N}^3$ and $i + j + l \leq n + 2\}$ and s_n so that:

(i) $(\mathscr{F}_{n-1} \cup Z_{n-1}) \subseteq \overline{\mathscr{F}_n} \in \mathcal{F}$;

(ii) $(z_{(i,j,0)}, ..., z_{(i,j,l)}) \in \mathrm{Dom}(t_{l+1}^{f_{(i,j)}})$ for all $i + j + l = n + 2$ and

$$s_n(B_1, ..., B_n) \times \{z_{(i,j,l)} : i + j + l = n + 2\} \subseteq O_n.$$

Inductive Step: Suppose that $(B_1, ..., B_{k+1})$ is a partial s-play, that is, $(B_1, ..., B_k) \in \mathrm{Dom}(s_k)$ and B_{k+1} is a nonempty open subset of $s_k(B_1, ..., B_k)$. Then:

(i) $Z_k \cup \mathscr{F}_k$ is countable, hence it is contained in some $F \in \mathcal{F}$. Define $\mathscr{F}_{k+1} := \{f_{(k+1,n)} : n \in \mathbb{N}\}$ to be a countable dense subset of F;

(ii) by the inductive hypothesis, $(z_{(i,j,0)}, ..., z_{(i,j,l)}) \in \mathrm{Dom}(t_{l+1}^{f_{(i,j)}})$ for all $i+j+l = k+2$. By re-indexing and noting $(z_{(i,j,0)}) \in \mathrm{Dom}(t_1^{f_{(i,j)}})$ for all $i + j = (k + 1) + 2$, we get that $(z_{(i,j,0)}, ..., z_{(i,j,l-1)}) \in \mathrm{Dom}(t_l^{f_{(i,j)}})$ for all $i + j + l = (k + 1) + 2$.

Next, we define $s_{k+1}(B_1, ..., B_{k+1})$ and $z_{(i,j,l)}$ for all $i + j + l = (k + 1) + 2$ so that:

(a) $s_{k+1}(B_1, ..., B_{k+1})$ is a nonempty open subset of B_{k+1};

(b) $z_{(i,j,l)} \in t_l^{f_{(i,j)}}(z_{(i,j,0)}, ..., z_{(i,j,l-1)})$ for all $i + j + l = (k + 1) + 2$, i.e., $(z_{(i,j,0)}, ..., z_{(i,j,l)}) \in \mathrm{Dom}(t_{l+1}^{f_{(i,j)}})$ for all $i+j+l = (k+1)+2$;

(c) $s_{k+1}(B_1, ..., B_{k+1}) \times \{z_{(i,j,l)} : i + j + l = (k + 1) + 2\} \subseteq O_{k+1}$.

Note that this is possible by Lemma 4.2.

Finally, define $Z_{k+1} := \{z_{(i,j,l)} : i + j + l \leq (k + 1) + 2\}$. This completes the inductive definition of s.

Consider an s-play $(B_n : n \in \mathbb{N})$ of the $BM(R)$-game played on X. For any $x \in \bigcap_{n\in\mathbb{N}} B_n$, let $F_x := \overline{\bigcup_{n\in\mathbb{N}} \mathscr{F}_n} \in \mathcal{F}$. Clearly, $Z \subseteq F_x$. Let $N \in \mathbb{N}$, we will show that the set $\{y \in F_x : (x, y) \in O_N\}$ is dense in F_x. For any open subset U of Y that intersects F_x, there is $f_{(i,j)} \in U \cap (\bigcup_{n\in\mathbb{N}} \mathscr{F}_n)$. Since $t^{f_{(i,j)}}$ is a winning strategy for the player α in the $G(f_{(i,j)})$-game, there is $m > N$ such that $z_{(i,j,m)} \in U \cap F_x$. Moreover, according to the definition of the the strategy s, $(x, z_{(i,j,m)}) \in O_{i+j+m-2} \subseteq O_m \subseteq O_N$. Therefore, $\{y \in F_x : (x, y) \in O_N\}$ is dense in F_x. Hence $\bigcap_{n\in\mathbb{N}} B_n \subseteq R$, which means s is a winning strategy for the player α is the $BM(R)$-game. Hence, by Lemma 4.1, R is residual in X. □

Theorem 4.4. *Suppose that Y is a W-space and X is a Baire space. If Y possesses a rich family $\mathcal{F}$ of Baire subspaces then $X \times Y$ is a Baire space. In fact, if Z is any topological space that contains Y as a dense subspace then $X \times Z$ is also a Baire space.*

Proof: Suppose that $\{O_n : n \in \mathbb{N}\}$ are dense open subsets of $X \times Y$ and $U \times V$ is the product of a nonempty open subset U of X with a nonempty open subset V of Y; we will show that $(U \times V) \cap \bigcap_{n\in\mathbb{N}} O_n \neq \varnothing$. To this end, choose $y \in V$ and set $Z := \{y\}$. By the previous theorem there exists a residual subset R of X such that for each $x \in R$ there exists an $F_x \in \mathcal{F}$ such that (i) $y \in F_x$ and (ii) $\{y' \in F_x : (x, y') \in \bigcap_{n\in\mathbb{N}} O_n\}$ is dense in F_x. Choose $x_0 \in U \cap R \neq \varnothing$ and $F_{x_0} \in \mathcal{F}$ such that $y \in F_{x_0}$ and $\{y' \in F_{x_0} : (x_0, y') \in \bigcap_{n\in\mathbb{N}} O_n\}$ is dense in F_{x_0}. In particular, $\{y' \in F_{x_0} : (x_0, y') \in \bigcap_{n\in\mathbb{N}} O_n\} \cap V \neq \varnothing$. Hence, if we choose $y_0 \in \{y' \in F_{x_0} : (x_0, y') \in \bigcap_{n\in\mathbb{N}} O_n\} \cap V$ then $(x_0, y_0) \in (U \times V) \cap \bigcap_{n\in\mathbb{N}} O_n$. This completes the first part of the proof. To see that $X \times Z$ is a Baire space it is sufficient to realise that $X \times Y$ is a dense Baire subspace of $X \times Z$. □

There are many examples of spaces that admit a rich family of Baire spaces that are not hereditarily Baire. For example, if (i) X is a separable Baire space that is not hereditarily Baire; in which case $\mathcal{F} := \{X\}$ is a rich family of Baire spaces, [1] or (ii) Y is a hereditarily Baire W-space such that $Y \times Y$ is not hereditarily Baire, [1], then the family of all nonempty closed separable rectangles gives a rich family of Baire subspaces of $Y \times Y$.

Corollary 4.5. *Suppose that $\{X_s : s \in S\}$ is a nonempty family of W-spaces. If each X_s, $s \in S$, possesses a rich family of Baire subspaces $\mathcal{F}_s$ then for each $a \in \Pi_{s\in S} X_s$, $\Sigma_{s\in S} X_s(a)$ is a W-space with a rich family of Baire subspaces. In particular, $\Sigma_{s\in S} X_s(a)$ is a Baire space.*

Proof: The fact that $\Sigma_{s\in S} X_s(a)$ is a W-space follows directly from Theorem 2.5. Moreover, from Theorem 3.5 we know that $\Sigma_{s\in S} \mathcal{F}_s(a)$ is a rich family, so it remains to show that all the members of $\Sigma_{s\in S} \mathcal{F}_s(a)$ are Baire spaces. To this end, suppose that $E := \Pi_{s\in S} E_s \in \Sigma_{s\in S} \mathcal{F}_s(a)$. Then E is homeomorphic to $\Pi_{s\in C_E} E_s$. However, by [6, Theorem 3.6] E is a separable first countable space. Therefore, by [8, Theorem 3], $\Pi_{s\in C_E} E_s$ is a Baire space. Finally, the fact that $\Sigma_{s\in S} X_s(a)$ is a Baire space now follows from Theorem 3.3. □

Corollary 4.6. *Suppose that $\{X_s : s \in S\}$ is a nonempty family of W-spaces. If each X_s, $s \in S$, possesses a rich family of Baire subspaces $\mathcal{F}_s$ then $\Pi_{s\in S} X_s$ is a Baire space.*

Proof: This follows directly from Corollary 4.5 since for any $a \in \Pi_{s\in S} X_s$, $\Sigma_{s\in S} X_s(a)$ is a dense Baire subspace. □

As a tribute to Professor I. Namioka, let us end this paper with what is essentially a folklore result, apart from the phrasing in terms of rich families, concerning the Namioka property.

Recall that a Baire space X has the *Namioka property* if for each compact Hausdorff space K and continuous mapping $f : X \to C_p(K)$ there exists a dense subset D of X such that f is continuous with respect to the $\|\cdot\|_\infty$-topology on $C(K)$ at each point of D.

Theorem 4.7. *Suppose that X is a topological space with countable tightness (in particular if X is a W-space) that possesses a rich family $\mathcal{F}$ of Baire subspaces then X has the Namioka property.*

Proof: In order to obtain a contradiction let us suppose that X does not have the Namioka property. Then there exists a compact Hausdorff space K and a continuous mapping $f : X \to C_p(K)$ that does not have a dense set of points of continuity with respect to the $\|\cdot\|_\infty$-topology. In particular, since X is a Baire space (by Theorem 3.3), this implies that for some $\varepsilon > 0$ the open set:

$$O_\varepsilon := \bigcup\{U \in 2^X : \ U \text{ is open and } \|\cdot\|_\infty\text{-diam}[f(U)] \leq 2\varepsilon\}$$

is not dense in X. That is, there exists a nonempty open subset W of X such that $W \cap O_\varepsilon = \varnothing$. For each $x \in X$, let $F_x := \{y \in X : \|f(y) - f(x)\|_\infty > \varepsilon\}$. Then $x \in \overline{F_x}$ for each $x \in W$. Moreover, since X has countable tightness, for each $x \in W$, there exists a countable subset C_x of F_x such that $x \in \overline{C_x}$.

Next, we inductively define an increasing sequence of separable subspaces $(F_n : n \in \mathbb{N})$ of X such that:

(i) $W \cap F_1 \neq \varnothing$;
(ii) $\bigcup\{C_x : x \in D_n \cap W\} \cup F_n \subseteq F_{n+1} \in \mathcal{F}$ for all $n \in \mathbb{N}$, where D_n is any countable dense subset of F_n.

Note that since the family $\mathcal{F}$ is rich this construction is possible.

Let $F := \overline{\bigcup_{n\in\mathbb{N}} F_n}$ and $D := \bigcup_{n\in\mathbb{N}} D_n$. Then $\overline{D} = F \in \mathcal{F}$ and $\|\cdot\|_\infty$-diam$[f(U)] \geq \varepsilon$ every nonempty open subset U of $F \cap W$. Therefore, $f|_F$ has no points of continuity in $F \cap W$ with respect to the $\|\cdot\|_\infty$-topology. This however, contradicts [10, Theorem 6] which states the every separable Baire space has the Namioka property. Therefore, the space X must have the Naimoka property. □

This theorem improves upon some results from [4].

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