Arnol'd's analytic perturbation for the polar equation

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Abstract

Based on the paper of Arnol'd [1] on the problems of Mathieu type, we suggest a modified Feynmann diagram technique to assist in solving problems of the form $A^\varepsilon y = (I + \varepsilon V)^{-1} A^0 y = \lambda y$. We then use this method to calculate eigenvalues and eigenfunctions to the polar problem,

$$-\frac{d^2 y}{dx^2} = \lambda(1 + \varepsilon \cos x)y,$$

with the quasiperiodic boundary condition on $[0, 2\pi]$.

1 Introduction

The finding of approximate solutions to additive perturbation problems, where the perturbed operator A^ε has the form $A^\varepsilon = A^0 + \varepsilon V$, is a process used widely in modern physics, see for instance [6]. In [1], Arnol'd examined problems of Mathieu type, $y'' + (\lambda - \varepsilon \cos x)y = 0$, using analytic perturbation technique in the modified form of Feynmann diagrams. Arnol'd calculated the eigenvalues and eigenfunctions by applying the Feynmann diagram technique to the analytic perturbation procedure, with eigenvalues and eigenfunctions of the perturbed operator determined from those of the unperturbed operator by the power expansion on ε^l, given by

$$\lambda_s^\varepsilon = \lambda_s^0 + \varepsilon \alpha_s^1 + \varepsilon^2 \alpha_s^2 + \cdots \quad \text{and} \quad |s^\varepsilon\rangle = |s^0\rangle + \varepsilon |s_1^\varepsilon\rangle + \varepsilon^2 |s_2^\varepsilon\rangle + \cdots \tag{1}$$

respectively. The diagram technique is useful in revealing the combinatorial nature of the terms in the expansion.

The standard diagram technique is not applicable to problems with multiplicative perturbations. For instance, in the polar problem with positive periodic density,

$$-y'' = \lambda(1 + \varepsilon \cos x)y \quad \text{where} \quad 0 < \varepsilon < 1, \tag{2}$$

the perturbed operators have the form $A^\varepsilon = (I+\varepsilon V)^{-1}A^0$ with $(I+\varepsilon V)$, depending on the application, being the density or refraction coefficient. The standard diagram technique is not applicable to this problem, because the terms in the expansion involve not powers of ε alone, but the product of it with the spectral parameter, namely $\varepsilon\lambda$.We suggest three different modification to Arnol'd's technique to treat (2) with the analytic perturbation approach.

2 Perturbation of metric, Riesz integral, and the plan of the proof

Consider the abstract Hilbert space H_0 with the inner product $\langle *|*\rangle_0$, and the associated Hilbert space H_ε with the inner product defined as

$$\langle *|*\rangle_\varepsilon = \langle (I+\varepsilon V)*|*\rangle_0.$$

Here $\rho_\varepsilon = I+\varepsilon V$ is a bounded, self-adjoint, strictly positive operator. Suppose that, in H_0, we have the positive self-adjoint operator A^0 with discrete spectrum $\{\lambda_s^0\}$ (where $(A^0)^{-1}$ is compact in H_0). Then in H_ε we have the associated operator $A^\varepsilon = (I+\varepsilon V)^{-1}A^0$, so that $\langle x, A^\varepsilon y\rangle_\varepsilon = \langle x, A^0 y\rangle_0$.

Assuming that the eigenvalue-eigenfunction pairs are$(\lambda_s^0, |s^0\rangle)$, we represent the bounded, self-adjoint, positive "density" $\rho_\varepsilon = [I+\varepsilon V]$ as

$$I+\varepsilon V = I + \varepsilon \sum_r \sum_t v_{rt}|r^0\rangle\langle t^0|$$

with the perturbation parameter ε. Hereafter we define the perturbed operator A^ε as above, and denote by $(\lambda_s^\varepsilon, |s^\varepsilon\rangle)$ its eigenvalue-eigenfunction pairs.

The spectrum of A^ε is discrete; to find the values of $(\lambda_s^\varepsilon, |s^\varepsilon\rangle)$ we write down the resolvent operator, $R_\lambda^\varepsilon = [A^\varepsilon - \lambda I]^{-1}$, as a sort of Neumann series of terms composed of products of the unperturbed resolvent R_λ^0, the perturbation operator V, and powers of λ and ε. Using the Riesz integral,

$$\frac{-1}{2\pi i}\oint_{\gamma_s} R_\lambda^\varepsilon d\lambda = P_s^\varepsilon. \tag{3}$$

see [4], we can find the spectral projection of A^ε in the form of the corresponding Neumann-type series.

For simple eigenvalues of A^0 and small ε the corresponding eigenvalues λ_s^ε are simple as well, and λ_s^ε is the only eigenvalue of A^ε in a sufficiently small disc D_s^ε centred at λ_s^0. The corresponding spectral projection P_s^ε can be calculated as the residue of R_λ^ε or, otherwise, as a sum of residues of the Neumann series for the perturbed resolvent.

In each term of the Neumann series we decompose $R_\lambda^0 = (A^0 - \lambda I)^{-1}$ as a sum of two terms, one analytic and the other non-analytic, within D_s^ε: the decomposition is

$$R_\lambda^0 = \frac{P_s^0}{(\lambda_s^0 - \lambda)} + Q_s^0, \quad \text{where} \quad Q_s^0 = \sum_{s'\neq s} \frac{P_{s'}^0}{(\lambda_{s'}^0 - \lambda)}.$$

We make sure that the perturbed eigenvalue, for ε sufficiently small, lies in D_s^ε. The residues of successive terms of the Neumann series are represented as products of the form

$$T^0_{s,1}VT^0_{s,2}V...T^0_{s,n}, \quad \text{with} \quad T^0_{s,n} \in \left\{P^0_s, Q^0_s, \frac{d}{d\lambda}Q^0_s, ...\right\}$$

evaluated at $\lambda = \lambda^0_s$. We shall write $(\overset{(k)}{\Sigma}{}^n_m)$ as the sum of all such sequences with $n - m$ P^0_s terms and with a total derivative order of k so that, in particular, $\dot{\Sigma}^n_m$ when $k = 1$ refers to the sum of sequences with one $\frac{d}{d\lambda}Q^0_s$ term. We shall also define $\frac{\lambda^0_s}{\lambda^0_{s'} - \lambda^0_s} := \mu^s_{s'}$ where $s' \neq s$. The terms, which arise in this series, can be represented by diagrams modified from those of Feynmann; we can then use other similar diagrams to simplify calculation of the terms involved in the trace. We can also calculate the perturbed eigenfunctions.

In the appendix, an alternative proof of the main result is given, based on comparison of terms with equal powers of ε.

3 Preliminary Results

Before we approach the main result, we need to verify for the polar problem two important details. The first is that, under certain conditions on V, we can represent the resolvent of the perturbed operator in terms of the resolvent of the unperturbed operator within the disc D^ε_s with $\delta D^\varepsilon_s = \gamma_s$ while excluding the eigenvalue λ^0_s. The second is to show that the eigenvalue λ^ε_s lies within D^ε_s for all sufficiently small ε. If we assume that the eigenvalue λ^0_s of the unperturbed operator is simple, we will have this property preserved by the perturbation.

In this paper we will simply perform the calculations with terms up to ε^2. The structure of the following modified Feynmann diagram will be clear from the analysis of the first two terms of the expansion.

Lemma 1 *Let R^ε_λ be the resolvent of the operator A^ε so that, in particular, R^0_λ is the resolvent of the operator A^0. If the disc D^ε_s centred at λ^0_s is chosen such that $\frac{\varepsilon|\lambda|}{|\lambda^0_s - \lambda|}\|V\|\Big|_{\delta D^\varepsilon_s} < 1$ then the resolvent of the perturbed operator A^ε is represented in the boundary δD^ε_s by the uniformly convergent series*

$$R^\varepsilon_\lambda = \sum_{r=0}^{\infty} \varepsilon^r \lambda^r (R^0_\lambda V)^r R^0_\lambda (I + \varepsilon V). \tag{4}$$

Proof By definition $R^\varepsilon_\lambda = (A^\varepsilon - \lambda I)^{-1}$. Note that $A^\varepsilon = (I + \varepsilon V)^{-1}A^0$; by simple substitution and rearranging of

$$R^\varepsilon_\lambda = (A^0 - \lambda I - \varepsilon\lambda V)^{-1}(I + \varepsilon V),$$

using $(A^0 - \lambda I - \varepsilon\lambda V)^{-1} = (I - \varepsilon R^0_\lambda V)^{-1}(A^0 - \lambda I)^{-1}$, we obtain

$$R^\varepsilon_\lambda = (I - \varepsilon\lambda R^0_\lambda V)^{-1}(A^0 - \lambda I)^{-1}(I + \varepsilon V) = (I - \varepsilon\lambda R^0_\lambda V)^{-1} R^0_\lambda (I + \varepsilon V).$$

On the boundary δD_s^ε the nearest eigenvalue of A^0 is λ_s^0, hence

$$|\varepsilon\lambda R_\lambda^0 V| \leq |\varepsilon|\frac{|\lambda|\|V\|}{|\lambda_s^0 - \lambda|} < 1,$$

and so $(I - \varepsilon\lambda R_\lambda^0 V)^{-1}$ is invertible. From the condition $\frac{\varepsilon|\lambda|}{|\lambda_s^0-\lambda|}\|V\|\Big|_{\delta D_s^\varepsilon} < 1$ we see that the series

$$(I - \varepsilon\lambda R_\lambda^0 V)^{-1} = \sum_{r=0}^{\infty} \varepsilon^r \lambda^r (R_\lambda^0 V)^r.$$

is convergent on δD_s^ε. $\square$

Remark: It is sufficient to take $D_s^\varepsilon = \left\{\lambda : |\lambda - \lambda_s^0| < \frac{1}{3}\min_{s'\neq s}|\lambda_{s'}^0 - \lambda_s^0|\right\}$.

For our purposes we take terms up to $O(\varepsilon^3)$, which gives us

$$R_\lambda^\varepsilon = R_\lambda^0(I + \varepsilon V) + \varepsilon\lambda R_\lambda^0 V R_\lambda^0(I + \varepsilon V) + \varepsilon^2\lambda^2 R_\lambda^0 V R_\lambda^0 V R_\lambda^0 + O(\varepsilon^3). \tag{5}$$

Lemma 2 *If λ_s^0 is a simple eigenvalue of the unperturbed operator A^0 then λ_s^ε is a simple eigenvalue of the perturbed operator A^ε.*

Proof For the proof of this we may refer to Gohberg [5]; however the result will be obtained independently within the proof of the next theorem. $\square$

We will also make use of calculating by residues of operator functions on a simple circular contour $D = \{|z - z_0| < \delta\}$ with f analytic in D with the kernel $(z - z_0)^{-n-1}$. See, for instance, the corresponding calculation for scalar functions in [2].

4 General Solution

We now use the previous results to prove the central statement of this paper. Analogously to the additive case, the multiplicative perturbation gives rise to a series of ε-powers whose coefficients are determined by the unperturbed systems eigenvalues and eigenfunctions. Again, we assume that the studied eigenvalue is simple and restrict our calculations to terms with ε power no greater than ε^2.

Theorem 1 *Assume that λ_s^0 is a simple eigenvalue of A^0 with eigenfunction $|s^0\rangle$. It follows that the eigenvalue λ_s^ε and the eigenfunction $|s^\varepsilon\rangle$ of A^ε arising from the pair $\lambda_s^0, |s^0\rangle$ are analytic functions of ε, and are represented by Taylor expansions in powers of ε ($|\varepsilon| < 1$). In particular*

$$\lambda_s^\varepsilon = \lambda_s^0 + \varepsilon\tilde{\lambda}_s^1 + \varepsilon^2\tilde{\lambda}_s^2 + O(\varepsilon^3) \qquad |s^\varepsilon\rangle = |s^0\rangle + \varepsilon|s_1^\varepsilon\rangle + \varepsilon^2|s_2^\varepsilon\rangle + O(\varepsilon^3), \tag{6}$$

where $\langle s^0, s^\varepsilon\rangle = 1$ *is assumed. The* ε*-power coefficients are given by*

$$\tilde{\lambda}_s^1 = -\lambda_s^0 v_{ss}, \quad \tilde{\lambda}_s^2 = \lambda_s^0\Big(v_{ss}^2 - \sum_{s'\neq s}\Big(\mu_{s'}^s v_{ss'} v_{s's}\Big)\Big), \quad |s_1^\varepsilon\rangle = \sum_{s'\neq s}\mu_{s'}^s v_{s's}|s'^0\rangle,$$

$$\textit{and} \quad |s_2^\varepsilon\rangle = \sum_{s'\neq s}\Big(\Big(\sum_{s''\neq s}\mu_{s'}^s\mu_{s''}^s v_{s's''}v_{s''s}\Big) - \mu_{s'}^s(\mu_{s'}^s+1)v_{s's}v_{ss}\Big)|s'^0\rangle.$$

Proof From Lemma 1 and (3) one can see that R_λ^ε is an analytic function of ε, for small ε. Then (3) gives the projection in the form of a Taylor series; in particular we have

$$P_s^\varepsilon = \frac{-1}{2\pi i}\oint_{\gamma_s} R_\lambda^0\rho_\varepsilon + \varepsilon\lambda R_\lambda^0 V R_\lambda^0\rho_\varepsilon + \varepsilon^2\lambda^2 R_\lambda^0 V R_\lambda^0 V R_\lambda^0 d\lambda + O(\varepsilon^3). \qquad (7)$$

We now consider the three integrals formed by integrating each term separately. Because of the independence of ρ^ε from λ the first integral gives

$$\frac{-1}{2\pi i}\oint_{\gamma_s} R_\lambda^0(I+\varepsilon V)d\lambda = \Big(\frac{-1}{2\pi i}\oint_{\gamma_s} R_\lambda^0 d\lambda\Big)(I+\varepsilon V) = P_s^0(I+\varepsilon V).$$

To calculate the second integral, note that

$$R_\lambda^0 = \sum_r \frac{P_r^0}{\lambda_r^0 - \lambda} = \frac{P_s^0}{\lambda_s^0-\lambda} + Q_s^0, \qquad (8)$$

so that we have

$$\frac{-\varepsilon}{2\pi i}\oint_{\gamma_s}\lambda R_\lambda^0 V R_\lambda^0 \rho_\varepsilon d\lambda = \Big(\frac{-\varepsilon}{2\pi i}\oint_{\gamma_s}\lambda\Big(\frac{P_s^0}{\lambda_s^0-\lambda}+Q_s^0\Big)V\Big(\frac{P_s^0}{\lambda_s^0-\lambda}+Q_s^0\Big)d\lambda\Big)\rho_\varepsilon.$$

On expanding these brackets we collect the terms which have the same power of $\lambda_s^0 - \lambda$ in the denominator and, noting that $\lambda Q_s^0 V Q_s^0$ is analytic over γ_s, we get

$$\frac{-\varepsilon}{2\pi i}\oint_{\gamma_s}\frac{\lambda P_s^0 V P_s^0}{(\lambda_s^0-\lambda)^2}d\lambda = -\varepsilon P_s^0 V P_s^0, \quad \frac{-\varepsilon}{2\pi i}\oint_{\gamma_s}\lambda Q_s^0 V Q_s^0 d\lambda = 0, \quad \text{and}$$

$$\frac{-\varepsilon}{2\pi i}\oint_{\gamma_s}\frac{\lambda(P_s^0 V Q_s^0 + Q_s^0 V P_s^0)}{\lambda_s^0-\lambda}d\lambda = \varepsilon\lambda_s^0(P_s^0 V Q_s^0 + Q_s^0 V P_s^0)\Big|_{\lambda=\lambda_s^0}.$$

Hence the second term is

$$\frac{-\varepsilon}{2\pi i}\oint_{\gamma_s}\lambda R_\lambda^0 V R_\lambda^0(I+\varepsilon V)d\lambda = \varepsilon(\lambda_s^0\Sigma_1^2 - \Sigma_0^2)(I+\varepsilon V),$$

where $(\overset{(k)}{\Sigma_m^n})$ is the sum of all terms arising from the expansion of the integral with ε-power $n-1$, with $n-m$ P_s^0 terms and with a total derivative order of k.

The last integral is the most complicated but the same arguments generally apply; the only additional factor to take into account appears from the derivative of Q_s^0 with respect to λ. For the final integral we have

$$\frac{-\varepsilon^2}{2\pi i}\oint_{\gamma_s}\lambda^2 R_\lambda^0 V R_\lambda^0 V R_\lambda^0 d\lambda = \varepsilon^2(\Sigma_0^3 - 2\lambda_s^0\Sigma_1^3 + (\lambda_s^0)^2(\Sigma_2^3 - \dot{\Sigma}_1^3)).$$

We now have the value for the integral on the contour $\gamma_s = \delta D_s^\varepsilon$. This gives an expression of P_s^ε in terms of the perturbation and the unperturbed spectral data, which is given by

$$P_s^\varepsilon = (P_s^0 + \varepsilon(\lambda_s^0\Sigma_1^2 - \Sigma_0^2) + \varepsilon^2(\Sigma_0^3 - 2\lambda_s^0\Sigma_1^3 + (\lambda_s^0)^2(\Sigma_2^3 - \dot{\Sigma}_1^3)))(I + \varepsilon V) + O(\varepsilon^3). \quad (9)$$

Remark: For a proof, independent from [5], of Lemma 2, note that we have P_s^ε represented in the form $P_s^0 + \varepsilon\widehat{P}$ with a bounded $\widehat{P}$. So we can say $\dim(P_s^\varepsilon) = \dim(P_s^0 + \varepsilon\widehat{P}) \underset{\varepsilon\to 0}{\longrightarrow} \dim(P_s^0)$. Hence $\dim(P_s^\varepsilon) = 1$ for small ε.

We can now find the eigenvalues of the perturbed operator: it is easy to see that $Tr(P_s^\varepsilon A^\varepsilon) = \lambda_s^\varepsilon$. This gives (up to $O(\varepsilon^3)$ terms)

$$\lambda_s^\varepsilon = Tr[(P_s^0 + \varepsilon(\lambda_s^0\Sigma_1^2 - \Sigma_0^2) + \varepsilon^2(\Sigma_0^3 - 2\lambda_s^0\Sigma_1^3 + (\lambda_s^0)^2(\Sigma_2^3 - \dot{\Sigma}_1^3)))\rho_\varepsilon\rho_\varepsilon^{-1}A^0].$$

To decide which elements contribute to the trace first note that, as A^0 is a diagonal operator then diagonal terms will only appear in terms in the sum involving P_s^0, $P_s^0VP_s^0$, $P_s^0VP_s^0VP_s^0$, $P_s^0VQ_s^0VP_s^0$, $P_s^0V\dot{Q}_s^0VP_s^0$, and $Q_s^0VP_s^0VQ_s^0$. Expanding out and taking the trace of each term gives us

$$\lambda_s^\varepsilon = \lambda_s^0 - \varepsilon\lambda_s^0 v_{ss} + \varepsilon^2\lambda_s^0\left(v_{ss}^2 - \sum_{s'\neq s}(\mu_{s'}^s v_{ss'}v_{s's})\right).$$

If we set $\tilde{\lambda}_s^1 = -\lambda_s^0 v_{ss}$ and $\tilde{\lambda}_s^2 = \lambda_s^0(v_{ss}^2 - \sum_{s'\neq s}(\mu_{s'}^s v_{ss'}v_{s's}))$, then we have the desired formula for the eigenvalues λ_s^ε.

Because we can scale eigenfunctions, we assume that $\langle s^0, s^\varepsilon\rangle = 1$ even though this means that we drop normality of the calculated perturbed eigenfunctions; normalization can be done after the calculations if it is desired. We calculate the perturbed eigenfunctions we wish to solve up to ε^3:

$$|s^\varepsilon\rangle = |s^0\rangle + \varepsilon|s_1^\varepsilon\rangle + \varepsilon^2|s_2^\varepsilon\rangle + O(\varepsilon^3).$$

From (9) and the definition of P_s^ε ($P_s^\varepsilon|s^\varepsilon\rangle = |s^\varepsilon\rangle$) we compare the coefficients for each power of ε on both sides of

$$(P_s^0 + \varepsilon\tilde{P}_s^{(1)} + \varepsilon^2\tilde{P}_s^{(2)})(I + \varepsilon V)(|s^0\rangle + \varepsilon|s_1^\varepsilon\rangle + \varepsilon^2|s_2^\varepsilon\rangle) = |s^0\rangle + \varepsilon|s_1^\varepsilon\rangle + \varepsilon^2|s_2^\varepsilon\rangle$$

and find the desired expressions for $|s_1^\varepsilon\rangle$ and $|s_2^\varepsilon\rangle$.The coefficient in front of $\varepsilon^0 = 1$ is already known from our assumptions, and is confirmed here. The coefficient in front of ε^1 gives us

$$P_s^0V|s^0\rangle + \tilde{P}_s^{(1)}|s^0\rangle + P_s^0|s_1^\varepsilon\rangle = P_s^0V|s^0\rangle + \lambda_s^0\Sigma_1^2|s^0\rangle - \Sigma_0^2|s^0\rangle + P_s^0|s_1^\varepsilon\rangle = |s_1^\varepsilon\rangle,$$

which implies

$$\sum_{s'\neq s} \mu^s_{s'} v_{s's} |s'^0\rangle = |s^\varepsilon_1\rangle.$$

Comparing the coefficients in front of ε^2 gives us, with $H^m_s = (I - mP^0_s)$,

$$((\lambda^0_s)^2(\Sigma^3_2 - \dot{\Sigma}^3_1) + \lambda^0_s \Sigma^2_1 V H^2_s - \Sigma^2_0 V H^1_s)|s^0\rangle + (P^0_s V H^1_s + \lambda^0_s \Sigma^2_1)|s^\varepsilon_1\rangle = H^1_s |s^\varepsilon_2\rangle,$$

which implies

$$\sum_{s',s''\neq s} \mu^s_{s'} \mu^s_{s''} v_{s's''} v_{s''s} |s'^0\rangle - \sum_{s'\neq s} (\mu^s_{s'} + 1) \mu^s_{s'} v_{s's} v_{ss} |s'^0\rangle = |s^\varepsilon_2\rangle.$$

So we have calculated the associated eigenfunctions up to $O(\varepsilon^3)$. □

Note that this proof holds only for simple eigenvalues, because we use the values of the eigenvalues during the evaluation of the eigenfunctions. In the case of multiple eigenvalues, the spectral projections P^ε_s onto the spectral subspaces corresponding to the eigenvalues of the perturbed operator A^ε inside D^ε_s can be obtained using the method listed previously. The eigenvalues are then found from solutions of the corresponding secular equation for $P^\varepsilon_s A^\varepsilon$.

5 Modified Diagram Techniques

The coefficients in front of higher powers of ε are more complicated. To obtain them we need to develop a corresponding modified diagram technique. We describe below three modifications of the Feynmann-Arnol'd diagram technique, which helps to label and calculate all terms appearing from the residue of the Riesz integral, which have the form

$$-\frac{\varepsilon^{l-1}}{2\pi i} \oint_{\gamma_s} \lambda^{l-1} \left(\left(\frac{P^0_s}{\lambda^0_s - \lambda} + Q^0_s \right) V \right)^{l-1} \left(\frac{P^0_s}{\lambda^0_s - \lambda} + Q^0_s \right) d\lambda. \tag{10}$$

We base the first diagram technique on two observations. The first comes from the decomposition of R^0_s into two terms: one, $\frac{P^0_s}{\lambda^0_s - \lambda}$, corresponding to the eigenvalue in the disc D^ε_s; the other, Q^0_s corresponds to all of the eigenvalues outside of this disc. The second is that the products alternate between R^0_λ terms and V terms. Consider the integral of an arbitrary term from (7), namely (10).

Enumerate the brackets in the product by the index n. For fixed s we let $\frac{P^0_s}{\lambda^0_s - \lambda} := P_n$ and $Q^0_s := Q_n$ for $n = 1, \dots, l$, so each of the products on expanding this term is of the form $T_1 V T_2 V \cdots V T_l$ where $T_n \in \{P_n, Q_n\}$. For each of these P_n and Q_n, let us create a vertex and label it P_n or Q_n as appropriate. Between each ordered pair of vertices of the form $(T_n, T_n + 1)$, with $1 \leq n < l$ and $T_n \in \{P_n, Q_n\}$, we make a directed edge from T_n to T_{n+1}. These edges represent the V terms between T_n and T_{n+1} in the product. It is easy to see that there will be one line for every product on expansion of the integral of the arbitrary term. For example, suppose $l = 4$. We are interested in all products from the expansion of

$$(P_1 + Q_1)V(P_2 + Q_2)V(P_3 + Q_3)V(P_4 + Q_4).$$

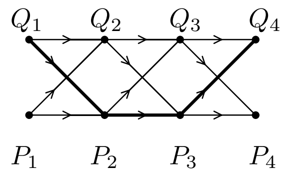

Figure 1: Modified Feynmann-Arnol'd graph, showing all terms that can be formed by expansion of the term $(R_\lambda^0 V)^{l-1} R_\lambda^0$ for the case $l = 4$. Each term corresponds to one of the directed paths in this graph.

Consider the bold line in the diagram: this corresponds to the product $Q_1 V P_2 V P_3 V Q_4$, and to the term

$$-\varepsilon^3 \frac{1}{1!} (\lambda^3 Q_s^0 V P_s^0 V P_s^0 V Q_s^0)' \Big|_{\lambda=\lambda_s^0}$$

from the integral given earlier. Note that this term differs from the corresponding term in Arnol'd's case by the factor λ^3.

Note that we will end up with the derivative of a finite product of functions of λ. From the generalization of Leibniz rule to several functions, see [9] for example, we know that

$$(\lambda^{l-1} T_1(\lambda) V \cdots V T_l(\lambda))^{(k)} = \sum_{\sum_r k_r = k} \frac{k!(l-1)!\lambda^{l-1-k_0} T_1^{(k_1)} V \cdots V T_l^{(k_l)}}{(l-1-k_0)! k_1! \cdots k_l!}. \quad (11)$$

Setting $k = l - q - 1$, because there are $l - q$ P_n-terms in the product and the integral from (10) is around a pole of order $l - q$, will give us the correct derivative order corresponding to the integral.

Further modification of Arnol'd's diagram technique comes from the sum of the terms due to the generalized Leibniz rule. Again the V terms correspond to the diagram's edges and the vertices will represent the terms in the decomposition of R_λ^0; however, instead of one vertex for each Q_s^0 term we will have a set of $l - q$ vertices. The P_s^0 terms will still be represented by a set of one vertex. Note that there are as many vertices in each set as there are possible derivatives of that term. Let us write U_n for the set of vertices corresponding to the nth P_s^0 or Q_s^0 term in the product. We label the vertices by the integers $0 \le k < |U_n|$, and say that the label of a vertex v is $k(v)$.

To construct the corresponding diagram, we take the sequence $T_1 V \cdots V T_l$ as before, the appropriate sets of vertices U_n for each of the T_n as detailed above. We add all possible edges between U_n and U_{n+1}. It is trivial to show that a path $v_1, v_2, \ldots, v_l$ with $v_n \in U_n$ for which

$$\sum_{n=1}^{l} k(v_n) \leq l - q - 1 \tag{12}$$

corresponds to a term arising from (11). Once we know the value of the sum from (12), we determine the appropriate derivative of λ^{l-1} which gives the complete expression of the term. Note that, in the Arnol'd case, only a path $v_1, v_2, \ldots, v_l$ with $v_n \in U_n$ for which

$$\sum_{n=1}^{l} k(v_n) = l - q - 1 \tag{13}$$

will correspond to a term arising from the Arnol'd analogue of (11).

For example, for the same product from the first diagram, $Q_1 V P_2 V P_3 V Q_4$, we have $l = 4$ and $q = 2$, so the order of the derivative of this term is $l - q - 1 = 1$. Hence we are interested in the expansion of $(\lambda^3 Q_s^0 V P_s^0 V P_s^0 V Q_s^0)'$. From the diagram, we see that the possible terms are

$$3\lambda^2 Q_s^0 V P_s^0 V P_s^0 V Q_s^0, \quad \lambda^3 \dot{Q}_s^0 V P_s^0 V P_s^0 V Q_s^0, \quad \text{and} \quad \lambda^3 Q_s^0 V P_s^0 V P_s^0 V \dot{Q}_s^0.$$

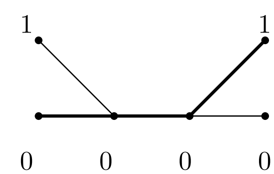

Figure 2: Modification of the Feynmann-Arnol'd diagram to label all terms that appear in the expansion of $(\lambda^3 Q_s^0 V P_s^0 V P_s^0 V Q_s^0)'$. The bold line represents the product $\lambda^3 Q_s^0 V P_s^0 V P_s^0 V \dot{Q}_s^0$.

The third diagram technique is designed to aid in the calculation of the eigenvalues of A^ε. To calculate the eigenvalues we have made use of the trace, so we must make sure that we do not include terms not appearing on the diagonal. To do so, we modify the first diagram. For all $1 \leq n < l$ we create a set of $\min(n+1, l-1)$ vertices U_n, and we create a set U_l of two vertices. Each vertex in U_n is labelled $s_n^{(k)}$ for $0 \leq k < |U_n|$. Each $v \in U_n$ represents the space generated by the eigenvalue $|s^{(k),0}\rangle$ with $s^{(k)} \neq s^{(j)}$ when $k \neq j$.

The edges once again correspond to the V operators appearing in the product. For $1 \leq n < l$, we include all directed edges from $s_n^{(k)}$ to $s_{n+1}^{(j)}$.

This diagram includes paths which do not contribute to the trace; however, the following three conditions collect all those paths which contribute to the trace:

- every path must contain at least one vertex $s_m^{(k_m)}$ with $1 \leq m \leq l$ and $k_m = 0$,
- $k_1 = k_l$, and

- for all $1 < m \leq l$, $k_m \leq 1 + \max\{k_{m'} : m' < m\}$.

To calculate the trace from this diagram, we take one of the paths with the above conditions. The term represented by this path is obtained by the following three rules: a vertex labelled s_n contributes a factor of P^0_s; a vertex labelled $s^{(k)}_n$ with $k \neq 0$ contributes a factor of

$$\sum_{s^{(k)} \neq s, \ldots, s^{(k-1)}} \frac{(m_n)! P^0_{s^{(k)}}}{(\lambda^0_{s^{(k)}} - \lambda^0_s)^{m_n+1}}$$

if Q_n corresponds to $(Q^0_s)^{(m_n)}$; and an edge labelled $v_{s^{(k)},s^{(j)}}$ contributes a factor of $v_{s^{(k)},s^{(j)}}|s^{(k)0}\rangle\langle s^{(j)0}|$.By proceeding along the path from $s^{(k_1)}_1$ to $s^{(k_l)}_l$ use these rules upon every vertex and edge included in the path to obtain their contributing factor. The product of all these factors, along with the appropriate coefficient, gives the trace element represented by this path.

For instance, consider the product $\varepsilon^3(\lambda^0_s)^3 P^0_s V Q^0_s V \dot{Q}^0_s V P^0_s$, which we obtain from using the appropriate first and second diagrams. One of the paths arising from this term's contribution to the trace has vertices labelled s_1, s'_2, s''_3 and s_4, where $s' \neq s$ and $s'' \neq s, s'$. On this path, the first and last vertex contribute P^0_s, the second contributes $\sum_{s' \neq s} \frac{P^0_{s'}}{\lambda^0_{s'} - \lambda^0_s}$ and the third $\sum_{s'' \neq s,s'} \frac{P^0_{s''}}{(\lambda^0_{s''} - \lambda^0_s)^2}$. The edges contribute $v_{ss'}|s^0\rangle\langle s'^0|$, $v_{s's''}|s'^0\rangle\langle s''^0|$, and $v_{s''s}|s''^0\rangle\langle s^0|$ respectively. The coefficient for this term is $\varepsilon^3(\lambda^0_s)^3$. Also, because we multiply the operator product at the end by A^0 to get the eigenvalue, we need to multiply by λ^0_s because the last vertex in this path corresponds to P^0_s. Upon taking the trace of the result, we get

$$\varepsilon^3 \sum_{s' \neq s} \sum_{s'' \neq s,s'} \frac{(\lambda^0_s)^4 v_{ss'} v_{s's''} v_{s''s}}{(\lambda^0_{s'} - \lambda^0_s)(\lambda^0_{s''} - \lambda^0_s)^2} = \varepsilon^3 \lambda^0_s \sum_{s' \neq s} \sum_{s'' \neq s,s'} v_{ss'} v_{s's''} v_{s''s} \mu^s_{s'} (\mu^s_{s''})^2,$$

which denotes this term's contribution to the trace. Note that this diagram functions independently of λ, and so can be applied the Arnol'd case as well (although with different coefficients).

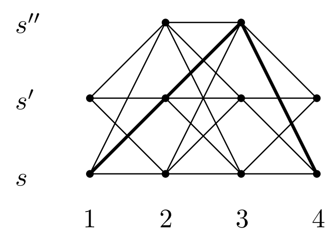

Figure 3: Example of the modified Feynmann-Arnol'd diagram for calculation of the eigenvalue λ^ε_s via the trace of $P^\varepsilon_s A^\varepsilon$.

6 Application to the Polar Problem

The spectral problem for the polar equation with periodic density,

$$-y'' = \lambda(1+\varepsilon\cos x)y, \quad 0<\varepsilon<1, \quad \text{and} \quad -\infty<x<\infty, \tag{14}$$

is reduced in the standard way, see [1], to the regular quasiperiodic Sturm-Liouville problem

$$-y'' = \lambda(1+\varepsilon\cos x)y, \quad y\Big|_{x=2\pi} = \Theta y\Big|_{x=0}, \quad y'\Big|_{x=2\pi} = \Theta y'\Big|_{x=0}, \tag{15}$$

with $0<\varepsilon<1$, $\Theta = e^{i\kappa}$, and $0 \le \kappa \le 2\pi$. The spectral bands of the periodic problem (14) are obtained as trajectories $\lambda_s(\kappa)$ for $0 \le \kappa \le 2\pi$. We obtain λ_s^ε based on the perturbation technique developed in the previous section. The value of the unperturbed eigenvalues and eigenfunctions fill the role of the case $\varepsilon = 0$, namely

$$\lambda_s^0 = \left(s - \frac{\kappa}{2\pi}\right)^2 \quad \text{and} \quad y_s^0 = \frac{1}{\sqrt{2\pi}} e^{-i(s-\frac{\kappa}{2\pi})x}, \tag{16}$$

where $s \in \mathbb{Z}$.

Theorem 2 *The system (15) has the eigenvalues λ_s^ε and eigenfunctions y_s^ε with*

$$\lambda_s^\varepsilon = \left(s-\frac{\kappa}{2\pi}\right)^2 + \varepsilon^2 \frac{(s-\frac{\kappa}{2\pi})^4}{8(s-\frac{\kappa}{2\pi})^2 - 2} + O(\varepsilon^3) \quad \textit{and}$$

$$y_s^\varepsilon = \frac{e^{-i(s-\frac{\kappa}{2\pi})x}}{\sqrt{2\pi}}\left(1 + \frac{\varepsilon}{2}(\beta_1+\beta_{-1}) + \frac{\varepsilon^2}{4}(\beta_2+\beta_{-2})\right) + O(\varepsilon^3)$$

when the unperturbed eigenvalues are simple and where

$$\beta_t = e^{-itx}\prod_{w=sgn(t)}^{t}\alpha_w \quad \textit{and} \quad \alpha_w = \frac{(s-\frac{\kappa}{2\pi})^2}{w(2s+w-\frac{\kappa}{\pi})}.$$

Proof Consider $V = \cos x$. If $|r-t| = 1$, then $v_{rt} = \frac{1}{2}$; otherwise $v_{rt} = 0$. Therefore, from simple substitution into the results from the general theorem we have

$$\tilde{\lambda}_s^1 = 0 \quad \text{and} \quad \tilde{\lambda}_s^2 = \frac{(s-\frac{\kappa}{2\pi})^4}{8(s-\frac{\kappa}{2\pi})^2 - 2}$$

for the respective coefficients of the ε and ε^2 correction terms for the eigenvalue λ_s^ε, so that $\lambda_s^\varepsilon = \lambda_s^0 + \varepsilon^2\tilde{\lambda}_s^2$. The corresponding coefficients of the ε and ε^2 terms of the eigenfunctions are

$$y_s^{(1)} = \frac{e^{-i(s-\frac{\kappa}{2\pi})x}}{2\sqrt{2\pi}}\left(\frac{(s-\frac{\kappa}{2\pi})^2 e^{-ix}}{2s+1-\frac{\kappa}{2\pi}} - \frac{(s-\frac{\kappa}{2\pi})^2 e^{-ix}}{2s-1-\frac{\kappa}{2\pi}}\right) \text{ and}$$

$$y_s^{(2)} = \frac{e^{-i(s-\frac{\kappa}{2\pi})x}}{8\sqrt{2\pi}}\left(\frac{(s-\frac{\kappa}{2\pi})^4 e^{-ix}}{(2s+1-\frac{\kappa}{2\pi})(2s+2-\frac{\kappa}{2\pi})} + \frac{(s-\frac{\kappa}{2\pi})^4 e^{-ix}}{(2s-1-\frac{\kappa}{2\pi})(2s-2-\frac{\kappa}{2\pi})}\right).$$

The perturbed eigenvalues are then of the form $y_s^\varepsilon = y_s^0 + \varepsilon y_s^{(1)} + \varepsilon^2 y_s^{(2)}$. Upon rearranging these values we can get them to be equivalent to the forms given in the statement of the theorem. $\square$

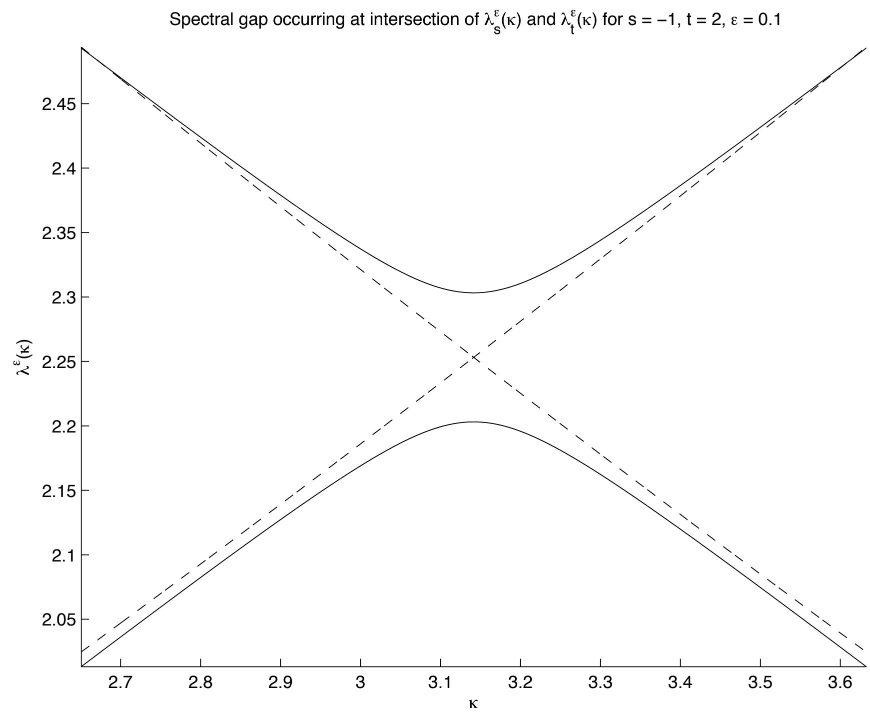

Figure 4: The ε-perturbation creates a quasi-intersection from the intersection of two terms. The spectral bands and spectral gaps are obtained as projections of the perturbed terms $\lambda_s^\varepsilon(\kappa)$ onto the vertical λ-axis.

Our technique allows us to calculate the spectral band gaps separating the spectral bands. For this problem, intersections of $\lambda_s^0(\kappa)$ and $\lambda_t^0(\kappa)$ will only occur when $\kappa = 0, \pi, 2\pi$, with $t = \frac{\kappa}{\pi} - s$. The degenerate eigenvalues will then be associated to the subspace of dimension 2 generated by $|s^0\rangle$ and $|t^0\rangle$. If we define the operator

$$A_s^\varepsilon = \begin{pmatrix} \langle s^0|A^\varepsilon|s^0\rangle & \langle s^0|A^\varepsilon|t^0\rangle \\ \langle t^0|A^\varepsilon|s^0\rangle & \langle t^0|A^\varepsilon|t^0\rangle \end{pmatrix},$$

then we see that we obtain the perturbed eigenvalues λ_s^ε and λ_t^ε by solving $\det(A_s^\varepsilon - \lambda^\varepsilon) = 0$ for λ^ε. Note that the entries off of the main diagonal of

$A_s^\varepsilon - \lambda^\varepsilon$ are of the order $O(\varepsilon^{|s-t|})$. This suggests that the spectral gaps decrease with increasing s. From [3], [7], [8], and [10], we obtain the creation of a quasi-intersection instead of an intersection, explaining the observation of a spectral gap.

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8 Appendix: Alternate Proof

Proof For an alternative proof to the main theorem, consider the expansion (up to $O(\varepsilon^3)$) of $A^\varepsilon|s^\varepsilon\rangle = \lambda_s^\varepsilon|s^\varepsilon\rangle$, with the same assumptions as before. Then

$$(I-\varepsilon V+\varepsilon^2 V^2)A^0(|s^0\rangle+\varepsilon|s_1^\varepsilon\rangle+\varepsilon^2|s_2^\varepsilon\rangle) = (\lambda_s^0+\varepsilon\tilde{\lambda}_s^1+\varepsilon^2\tilde{\lambda}_s^2)(|s^0\rangle+\varepsilon|s_1^\varepsilon\rangle+\varepsilon^2|s_2^\varepsilon\rangle),$$

so, for the terms with ε-powers 0, 1, and 2, we have

$$A^0|s^0\rangle = \lambda_s^0|s^0\rangle, \qquad A^0|s_1^\varepsilon\rangle - VA^0|s^0\rangle = \tilde{\lambda}_s^1|s^0\rangle + \lambda_s^0|s_1^\varepsilon\rangle, \quad \text{and}$$

$$A^0|s_2^\varepsilon\rangle - VA^0|s_1^\varepsilon\rangle + V^2A^0|s^0\rangle = \tilde{\lambda}_s^2|s^0\rangle + \tilde{\lambda}_s^1|s_1^\varepsilon\rangle + \lambda_s^0|s_2^\varepsilon\rangle$$

respectively for the coefficients. The first of these equations gives us what we already know from our assumptions. The second, when multiplied on the left by $\langle s^0|$, gives $\tilde{\lambda}_s^1 = -v_{ss}\lambda_s^0$ and, when multiplied on the left by $\langle s'^0|$, gives

$$-v_{s's}\lambda_s^0 + \lambda_{s'}^0\langle s'^0, s_1^\varepsilon\rangle = \lambda_s^0\langle s'^0, s_1^\varepsilon\rangle,$$

which after summation over all $s' \neq s$ gives

$$\sum_{s'\neq s}\mu_{s'}^s v_{s's}|s'^0\rangle = |s_1^\varepsilon\rangle.$$

By a similar argument, we see that we can obtain

$$\tilde{\lambda}_s^2 = \lambda_s^0\left(v_{ss}^2 - \sum_{s'\neq s}(\mu_{s'}^s v_{ss'}v_{s's})\right) \qquad \text{and}$$

$$\sum_{s',s''\neq s}\mu_{s'}^s\mu_{s''}^s v_{s's''}v_{s''s}|s'^0\rangle - \sum_{s'\neq s}(\mu_{s'}^s+1)\mu_{s'}^s v_{s's}v_{ss}|s'^0\rangle = |s_2^\varepsilon\rangle$$

from the third equation from multiplication on the left by $\langle s^0|$ and $\sum_{s'\neq s}\langle s'^0|$ respectively. □

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