Some results on linearly Lindelöf spaces *

Hongfeng Guo, Shouli Jiang

(School of Mathematics and System Sciences, Shandong University, Jinan, 250100, P. R. China)

Abstract: Some new results about linearly Lindelöf spaces are given here. It is proved that if X is a space of countable spread and $X = Y \cup Z$, where Y and Z are meta-Lindelöf spaces, then X is linearly Lindelöf. Moreover, we give a positive answer to a problem raised by A.V. Arhangel'skii and R.Z. Buzyakova.

Key words: extent; spread; regular cardinality; linearly Lindelöf space

Classification (MSC 2000): 54A25; 54D20

1. Introduction

In the study of Lindelöf spaces, it is interesting to notice the following condition (introduced in [8]):

(**CAP**) *every uncountable subset A of X of countable regular cardinality has a point of complete accumulation in X*,

which dose not characterize a Lindelöf space. Probably, the first example of a space of this kind was constructed by A.S.Mischenko (see [1]). The spaces satisfying (**CAP**) were later renamed *linearly Lindelöf* or *Chain-Lindelöf* spaces, since the condition (**CAP**) turned out to be equivalent to the following requirement: every open covering γ of X which is a chain (that is, for any two elements of γ, one is a subset of the other) contains a countable subcovering of X.

In [4], Arhangel'skii and Buzyakova obtained the following result about linearly Lindelöf spaces.

Theorem. Suppose that X is a space of countable extent such that $X = Y \cup Z$, where Y

* This work is supported by the NSF of Shandong Province of China (No. Y2004A03)

and Z are paracompact spaces. Then X is linearly Lindelöf.

They also raised some problems and the following is one of them.

Problem. Can the above theorem be extended to finite unions of spaces?

In section 3, we can strengthen the above theorem by Theorem 3.1, and answer the above problem positively by the corollary of Theorem 3.8.

2. Definitions

In this section, we shall give the main definitions used in this paper.

Definition 2.1. The *extent* $e(X)$ of a topological space X is the smallest infinite cardinal number τ such that for any locally finite subset A of X, $|A| \leq \tau$, that is , $e(X)$=sup$\{|A|$: A is a locally finite subset of $X\}$.

Note that this definition obviously coincides with the usual definition of the extent of X for all T_1-spaces.

In the above definition, if we substitute a discrete set for the locally finite set, we have the following (see [2]).

Definition 2.2. The *spread* $s(X)$ of a topological space X is the smallest infinite cardinal number τ such that for any discrete subset A of X, $|A| \leq \tau$, that is, $s(X)$=sup$\{|A|$: A is a discrete subset of $X\}$.

Definition 2.3. $x \in X$ is called a *point of complete accumulation* for A if for any neighborhood U of x, $|U \cap A| = |A|$.

Now we can recall the definition of linearly Lindelöf spaces.

Definition 2.4. A topological space X is called a *linearly Lindelöf* space, if for any uncountable subset A of X of regular cardinality, there exists a point of complete accumulation for A in X. This is well-known to be equivalent to the statement that every increasing open cover of X has a countable subcover.

The class of D-spaces, introduced by E. van Douwen in [6], is a very natural and interesting one. In the following, we shall give two kinds of definitions of D-spaces. Definition 2.5 (ii) is introduced in [6]. Arhangel'skii and Buzyakova have done a lot of interesting recent work under Definition 2.5 (i) (see [3], [4] and [10]).

Definition 2.5. A neighborhood assignment on a topological space is a mapping ϕ of X into the topology $\mathcal{T}$ of X such that $x \in \phi(x)$, for each $x \in X$.

(i) A space X is called a D-space if, for every neighborhood assignment ϕ on X, there exists a locally finite in X subset A of X such that the family $\phi(A) = \{\phi(x) : x \in A\}$ covers X.

(ii) A space X is called a D-space if, for every neighborhood assignment ϕ on X, there exists a closed discrete subset A of X such that the family $\phi(A)$ covers X.

It is easy to see that the above two definitions are the same in T_1-spaces, since in such spaces a set is locally finite if and only if it is closed discrete.

Other terminologies and notations follow [3], [4] and [9].

3. Main results and their proofs

Now we will give the main results and their proofs. Note that all spaces in this section are general topological spaces satisfying no axiom of separation, unless explicitly stated.

Theorem 3.1. Suppose that X is a space of countable spread such that $X = Y \cup Z$, where Y and Z are meta-Lindelöf spaces. Then X is linearly Lindelöf.

Before giving the proof of Theorem 1, we provide two Lemmas and their proofs.

Lemma 3.2. Suppose that X is a meta-Lindelöf space and A is an uncountable subset of X of regular cardinality. Then either there exists a point of complete accumulation for A in X, or there exists a subset B of A such that B is discrete in X and $|B| = |A|$.

Proof. Assume $|A| = \kappa$, where κ is an uncountable regular cardinal number. If none of the points of X is a point of complete accumulation for A in X, we show that there exists a subset B of A such that B is discrete in X and $|B| = |A|$.

For any $x \in \overline{A}$, since x is not a point of complete accumulation for A, then there exists an open neighborhood U_x of x, such that $|U_x \cap A| < |A|$. Thus, we get a family $\mathcal{U}$ of open sets, such that $\bigcup \mathcal{U} \supset A$ and for any $U \in \mathcal{U}$, $|U \cap A| < |A|$. For any $U \in \mathcal{U}$, let $U_{\overline{A}} = U \cap \overline{A}$. Then we get a family $\mathcal{U}_{\overline{A}}$ such that $\bigcup \mathcal{U}_{\overline{A}} = \overline{A}$ and for any $U_{\overline{A}} \in \mathcal{U}_{\overline{A}}$, there exists $U \in \mathcal{U}$ satisfying $U_{\overline{A}} = U \cap \overline{A}$.

Since X is a meta-Lindelöf space and $\overline{A}$ is a closed subspace of X, then $\overline{A}$ is also a meta-Lindelöf space. Therefore, $\mathcal{U}_{\overline{A}}$ has a point-countable open refinement $\mathcal{V}$ in the subspace $\overline{A}$.

For any $V \in \mathcal{V}$, $V \cap A \neq \emptyset$. Or else, there exists $x \in V$ and an open set U of X such that $V = U \cap \overline{A}$. Then $U \cap A = \emptyset$ since $V \cap A = \emptyset$, which is contradicted by the fact that $x \in \overline{A}$.

Now we show that $|\mathcal{V}| = |A| = \kappa$. We know that $|\mathcal{V}| \leq \kappa$, since $\mathcal{V}$ is a point-finite open refinement. Besides, $|\mathcal{V}| \geq |A|$, since for any $V \in \mathcal{V}$, $|V \cap A| < \kappa$, and κ is a regular cardinal number.

Take $x_0 \in A$. Since $\mathcal{V}_0 = \{V \in \mathcal{V} : x_0 \in V\}$ is a countable family, then $A \setminus \bigcup \mathcal{V}_0 \neq \emptyset$. Therefore, we can take $x_1 \in A \setminus \bigcup \mathcal{V}_0$ and let $\mathcal{V}_1 = \{V \in \mathcal{V} : x_1 \in V\}$. Similarly, for any ordinal number α, if $A \setminus \{\bigcup \mathcal{V}_\beta : \beta < \alpha\} \neq \emptyset$, then we can take $x_\alpha \in A \setminus \bigcup\{\mathcal{V}_\beta : \beta < \alpha\}$ and let $\mathcal{V}_\alpha = \{V \in \mathcal{V} : x_\alpha \in V\}$. Thus, there must exist an ordinal number τ such that $\bigcup\{\mathcal{V}_\alpha : \alpha < \tau\}$ covers A and we get a set $B = \{x_\alpha \in A : \alpha < \tau\}$. We assume that $|B| = \lambda$.

Since κ is a regular cardinal number and for any $V \in \bigcup\{\mathcal{V}_\alpha : \alpha < \tau\}$, $|V \cap A| < \kappa$, then we know that $\lambda \geq \kappa$. Besides, since $B \subset A$, then $\lambda \leq \kappa$. Therefore, we know that $\lambda = \kappa$, that is, $|B| = |A|$.

The following claim will complete the proof.

Claim. B is discrete in X.

Proof. For any $x \in X$, $x \notin \overline{A}$ or $x \in \overline{A}$.

(i) If $x \notin \overline{A}$, then $X \setminus \overline{A}$ is an open neighborhood of x and $(X \setminus \overline{A}) \cap B = \emptyset$. Thus, B is discrete at such a point.

(ii) If $x \in \overline{A}$, then there exists $V \in \mathcal{V}$ such that $x \in V$. If $V \cap B \neq \emptyset$, then there exists $x_\beta \in V \cap B$. For any $\alpha > \beta$, according to the definition of x_α, we have that $x_\alpha \notin V$. So $V \cap \{x_\alpha : \beta < \alpha < \tau\} = \emptyset$. On the other hand, if there exists $\alpha_0 < \beta$ such that $x_{\alpha_0} \in V$, then we have $x_\beta \notin V$. Thus, we get a contradiction. Therefore, $V \cap B = \{x_\beta\}$. Since there exists an open set U of X satisfying $V = U \cap \overline{A}$, then $U \cap B = \{x_\beta\}$. Thus, B is also discrete at such a point.

According to *(i)* and *(ii)*, we know that B is discrete at any point of X. So the set B is discrete in X.

The following lemmas are well-known, and are required for us to prove Theorem 3.1. In fact, in [5], it has been showed that every $\delta\theta$-refinable space of countable extent is Lindelöf. But in order to understand more easily and clearly, here we'll give the proof of Lemma 3.3 again.

Lemma 3.3. Every meta-Lindelöf space of countable spread is a Lindelöf space.

Proof. Assume that X is a meta-Lindelöf space of countable spread and that it is not Lindelöf. Therefore, there exists an open cover $\mathcal{U}$ of X which dose not contain a countable subcover. Since X is meta-Lindelöf, then $\mathcal{U}$ has a point-countable refinement $\mathcal{V}$. $\mathcal{V}$ is not countable and dose not contain a countable subcover, or else $\mathcal{U}$ has a countable subcover. Assume $\mathcal{V}= \{V_\alpha : \alpha < \kappa\}$. Take $x_0 \in V_0$ and let $\mathcal{V}_0 = \{V \in \mathcal{V}\colon x_0 \in V\}$. We know that $\mathcal{V}_0$ is countable and $\bigcup \mathcal{V}_0 \neq X$. Then we can take $x_1 \in X \setminus \bigcup \mathcal{V}_0$ and let $\mathcal{V}_1 = \{V \in \mathcal{V}\colon x_1 \in V\}$. Obviously, $\mathcal{V}_1$ is countable and $\bigcup(\mathcal{V}_0 \cup \mathcal{V}_1) \neq X$. Similarly, we take $x_2 \in X \setminus \bigcup(\mathcal{V}_0 \cup \mathcal{V}_1)$ and let $\mathcal{V}_2 = \{V \in \mathcal{V}\colon x_2 \in V\}$. Thus, we can get a set $A = \{x_\alpha : \alpha < \tau\}$ and the family $\bigcup\{\mathcal{V}_\alpha : \alpha < \tau\}$ which covers X. Since $\mathcal{V}$ has no countable subcover , it is easy to see that $\{\mathcal{V}_\alpha : \alpha < \tau\}$ is not countable. So the set A is uncountable.

Since X is a space of countable spread, then A is not discrete. So there exists a point y such that for any open neighborhood U of y, $|U \cap A| \geq 2$. Assume that $y \in V_y$ where $V_y \in \mathcal{V}$. Then V_y contains at least two elements of A. We assume that $x_m, x_n \in V_y \cap A$ and $m < n$. Then we have that $V_y \in \mathcal{V}_m$ since $x_m \in V_y$. On the other hand, $x_n \notin \bigcup \mathcal{V}_m$, so $x_n \notin V_y$. Thus, we get a contradiction.

Hence X is Lindelöf.

Lemma 3.4. Every closed subspace of a space having countable spread is also a space of countable spread.

On the basis of these three lemmas we can now prove Theorem 3.1.

Proof of Theorem 3.1. Suppose that the space X is not a linearly Lindelöf space. Then there exists an uncountable subset A of X such that $|A| = \kappa$ where κ is a regular cardinal number and there is no point of complete accumulation for A in X.

It is clear that at least one of $A \cap Y$ and $A \cap Z$ has the same cardinality as A. We can assume that $|A \cap Y| = \kappa$. Thus, we can also assume that $A \subset Y$. According to Lemma 3.2, in the space Y there exists a discrete set $B \subset A$ satisfying $|B| = |A|$.

Let $C = \{x \in X : B$ is not discrete at x in $X\}$. It is obvious that $C \subset Z$. Then we prove that C is closed in X. Since for any $x \in \overline{C}$ and any open neighborhood U of x,

$U \cap C \neq \emptyset$, then there exists $y \in U \cap C$. Therefore, U is also an open neighborhood of y, so $|U \cap B| > 1$. Then we know that B is not discrete at x, so $x \in C$. Thus, we have shown that C is closed in X. According to Lemma 3.4, the subspace C is a space of countable spread. Then according to Lemma 3.3, the subspace C is Lindelöf.

For any $z \in C$, there exists an open neighborhood U_z of z such that $|U_z \cap B| < |B|$. Thus, we get an open family $\mathcal{U}$ satisfying $\bigcup \mathcal{U} \supset C$ and for any $U \in \mathcal{U}$, $|U \cap B| < |B|$. Since the subspace C is a Lindelöf space, there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcup \mathcal{V} \supset C$. Let $W = \bigcup \mathcal{V}$. Obviously, W is an open subset of X and $C \subset W$. Since $|B| = |A| = \kappa$ where κ is a regular cardinal number, we have $|W \cap B| < |B|$. Then $B \setminus (W \cap B)$ is discrete in X and $|B \setminus (W \cap B)| = \kappa > \omega$. This contradicts the fact that $s(X) = \omega$.

Hence X is a linearly Lindelöf space.

As is well-known, all countably compact linearly Lindelöf spaces are compact and every countably compact space has countable extent. Hence we have the following corollary of Theorem 3.1.

Corollary 3.5. Suppose that X is a countably compact space such that $X = Y \cup Z$, where Y and Z are meta-Lindelöf spaces. Then X is compact.

Theorem 3.6. Suppose that X is a space of countable spread such that $X = \bigcup_{i=1}^{n} X_i$ $(n \geq 3)$, where each space X_i is a meta-Lindelöf space and $n-2$ spaces of them are closed subspaces, then X is linearly Lindelöf.

Proof. We firstly give the proof of the case $n = 3$. We assume that X_1 is a closed subspace.

Suppose that X is not a linearly Lindelöf space. Then there exists an uncountable subset A such that $|A| = \kappa$ and there is no point of complete accumulation for A in X, where κ is a regular cardinal number.

(i) Suppose that $|A \cap X_1| = \kappa$. Since X_1 is closed, then by Lemma 3.4, the subspace X_1 is a space of countable spread. By Lemma 3.2, there is a point of complete accumulation for A in X_1, or there exists a subset B of A such that B is discrete in X_1 and $|B| = |A|$. If x is a point of complete accumulation for A in X_1, then x is also a point of complete accumulation for A in X, which is a contradiction to the fact that there is no point of complete accumulation for A in X. Otherwise, if there exists a subset B of A such that B is discrete in the space X_1, then we know $|B| \leq \omega$ since X_1 is a space of countable spread. Then $|B| < |A|$. Thus, we have that $|A \cap X_1| < \kappa$.

(ii) Suppose that $|A \cap X_2| = \kappa$. We can assume that $A \subset X_2$. It is easy to see that there is no point of complete accumulation for A in X_2. Then by Lemma 3.2, there exists a subset B of A such that B is discrete in X_2 and $|B| = |A|$.

Now consider the space $X' = X_1 \cup X_2$. Let $C' = \{x \in X' : B \text{ is not discrete at } x \text{ in } X'\}$. Then C is closed in the space X' and $C' \subset X_1$. Since X_1 is closed in X, C' is closed in X. Therefore, the subspace C' is a space of countable spread. By Lemma 3.3, the subspace C' is a Lindelöf space. Consequently, in view of the proof of Theorem 3.1, we can get $B' \subset B$ such that B' is discrete in X' and $|B'| = |B|$.

Then we let $C = \{x \in X : B' \text{ is not discrete at } x \text{ in } X\}$. Obviously, $C \subset X_3$ and C is closed in X. By Lemma 3.3, the subspace C is Lindelöf. Since there is no point of complete accumulation for B' in X, then we can get a subset B'' of B' such that B'' is discrete in X and $|B''| = |B'|$. Then $|B''| = |A| = \kappa > \omega$, which is a contradiction to $s(X) = \omega$. Thus, we have $|A \cap X_2| < \kappa$.

Similarly, we can prove $|A \cap X_3| < \kappa$.

Since at least one of $A \cap X_i (i = 1, 2, 3)$ has the same cardinality as A, then we get a contradiction by the above proof. Thus we know that the space X is a linearly Lindelöf space when $n = 3$.

For the case $n > 3$, where $n \in N$, by the same way as in the above proof, we can also prove that X is a linearly Lindelöf space.

In [7], under Definition 2.5 (ii), Gruenhage has proved that if X has countable extent and can be written as the union of finitely many D-spaces, then X is linearly Lindelöf.

Here we can obtain the same result under Definition 2.5.(i) by modifying the proof of Theorem 4.2 [7]. So the method of the proof is due to Gruenhage. Before that, we need the following lemma.

Lemma 3.7 (see [7]). A space X is linearly Lindelöf iff whenever $\mathcal{O}$ is an open cover of X of cardinality κ and $\mathcal{O}$ has no subcover of cardinality $< \kappa$, then $cf(\kappa) \leq \omega$.

Theorem 3.8. If X has countable extent and can be written as the union of finitely many D-spaces, then X is linearly Lindelöf.

Proof. Suppose that X is a space of countable extent such that $X = \bigcup_{i \leq k} X_i$ where $k \in N$ and each subspace X_i is a D-space, but X is not linearly Lindelöf. We can also assume that k is the least possible value for any counterexample to the theorem. It is easy to see that $k > 1$ since every D-space of countable extent is Lindelöf.

By Lemma 3.7, there is an open cover $\mathcal{O} = \{O_\alpha : \alpha < \kappa\}$ of some cardinality κ with $cf(\kappa) > \omega$, and $\mathcal{O}$ has no subcover of cardinality $< \kappa$. For each $x \in X$, let α_x be the least such that $x \in O_{\alpha_x}$. Define a neighborhood assignment ϕ on X as follows. For any $x \in X$,

$\phi(x) = O_{\alpha_x}$.

For any $i \leq k$, there exists a relative locally finite subset D_i of X_i such that the family $\phi(D_i) = \{\phi(x) : x \in D_i\}$ covers X_i. It is easy to see that there must be some $i_0 < k$ such that $|\{\alpha_d : d \in D_{i_0}\}| = \kappa$ since $\mathcal{O}$ has no subcover of cardinality ¡κ. Let $C = \{x \in X : D_{i_0}$ is not locally finite at $x\}$. Clearly, C is closed in X and $C \subset \bigcup_{i \neq i_0} X_i$. By the minimality of k, C is a linearly Lindelöf subspace. Thus, for the increasing open cover $\{\bigcup_{\beta<\alpha} \mathcal{O}_\beta : \alpha < \kappa\}$, there are $\alpha_n < \kappa$, $n \in \omega$, such that $\mathcal{U} = \{\bigcup_{\beta<\alpha_n} \mathcal{O}_\beta : n \in \omega\}$ covers C. Then $C \setminus \bigcup\mathcal{U}$ is locally finite in X. Since X has countable extent, then $C \setminus \bigcup\mathcal{U}$ is a countable subset. By $cf(\kappa) > \omega$, $\lambda = sup\{\alpha_n : n \in \omega\} < \kappa$. Hence there exists some $d \in D_{i_0}$ such that $d \in \bigcup\mathcal{U}$ and $\alpha_d > \lambda$. Since $d \in \bigcup\mathcal{U}$ implies that for some $\beta < \lambda$, $d \in O_\beta$, then $\alpha_d < \beta$. Thus we get a contradiction, and the proof is complete.

Since under Definition 2.5 (i), every paracompact space is a D-space, then we have the following corollary which is a positive answer to the problem we have stated in the introduction.

Corollary 3.9. If X has countable extent and can be written as the union of finitely many paracompact spaces, then X is linearly Lindelöf.

REFERENCES

1. A.S. Mischenko, Finally compact spaces, Soviet Math. Dokl. 145 (1962) 1199-1202.
2. A.V. Arhangel'skii, On Lindelöf property and spread in C_p-theory, Topology Appl. 74 (1996) 83-90.
3. A.V. Arhangel'skii, D-spaces and covering properties, Topology Appl. 146-147 (2005) 437-449.
4. A.V. Arhangel'skii, R.Z. Buzyakova, Addition theorems and D-spaces, Comment. Math. Univ. Carolin. 43 (4) (2002) 653-663.
5. C.E. Aull, A generalization of a theorem of Aquaro, Bull. Austral. Math. Soc. 9 (1973) 105-108.
6. E.K. van Douwen, W. Pfeffer, Some properties of the Sorgenfrey line and related space, Pacific J. Math. 81 (1979), 371-377. .

7. G. Gruenhage, A note on D-spaces, Topology Appl., in press.

8. P.S. Alexandroff and P.S. Urysohn, Memoire sur les espaces topologiques compacts, Nederl. Akad. Wetensch. Proc. Ser. A, 14 (1929) 1-96.

9. R. Engelking, General Topology, Polish Scientific Publishers, Warszawa, (1977).

10. R.Z. Buzyakova, On D-property of strong Σ-spaces, Comment. Math. Univ. Carolin. 43 (3) (2002) 493-495.

School of Mathematics and System Science, Shandong University, Jinan, 250100, P. R. China
E-mail: ghf@mail.sdu.edu.cn
School of Mathematics and System Science, Shandong University, Jinan, 250100, P. R. China
E-mail: jiang@sdu.edu.cn