Approximability of Dodgson's Rule

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ABSTRACT. It is known that Dodgson's rule is computationally very demanding. Tideman (1987) suggested an approximation to it but did not investigate how often his approximation selects the Dodgson winner. We show that under the Impartial Culture assumption the probability that the Tideman winner is the Dodgson winner tend to 1. However we show that the convergence of this probability to 1 is slow. We suggest another approximation — we call it Dodgson Quick — for which this convergence is exponentially fast. Also we show that Simpson and Dodgson rules are asymptotically different. We formulate, and heavily use in construction of examples, the generalization of M[c]Garvey's theorem (1953) for weighted majority relations.

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1 Introduction

Condorcet proposed that a winner of an election is not legitimate unless a majority of the population prefer that alternative to all other alternatives. However such a winner does not always exist. A number of voting rules have been proposed which select the Condorcet winner if it exists, and otherwise selects an alternative that is in some sense closest to being a Condorcet Winner. A prime example of such as rule was the one proposed by Dodgson (1876).

Unfortunately, Bartholdi et al. (1989) proved that finding the Dodgson winner is an NP-hard problem. Hemaspaandra et al. (1997) refined this result by proving that it is Θ_2^p-complete and hence is not NP-complete unless the polynomial hierarchy collapses. For this reason, we investigate the asymptotic behaviour of simple approximations to the Dodgson rule as the number of agents gets large. Under the assumption that all votes are independent and each type of vote is equally likely, the probability that the Tideman (1987) approximation picks the Dodgson winner does asymptotically converge to 1, but not exponentially fast.

We propose a new social choice rule, which we call Dodgson Quick. The Dodgson Quick approximation does exhibit exponential convergence, and we can quickly verify that it has chosen the Dodgson winner. This, together with other nice properties, makes our new approximation useful in computing the Dodgson winner. This approximation could be used to develop an algorithm to choose the Dodgson winner with $\mathcal{O}(\ln n)$ expected running time for a fixed number of alternatives and n agents.

A similar result was independently obtained by Homan and Hemaspaandra (2005). They developed a "greedy" algorithm that finds the Dodgson winner with probability approaching 1 as we increase the number of voters. This convergence is also exponentially fast. However they do not suggest any rule and, unlike their algorithm, the Dodgson Quick rule requires only the information in the weighed majority relation.

Our experimental results (M^c^Cabe-Dansted and Slinko, 2006) showed that Simpson's and Dodgson's rules are very close. However, we discover that the frequency that the Simpson rule picks the Dodgson winner does not converge to one.

M^c^Garvey (1953) proved that for any tournament, we can find a profile such that the tournament is the majority relation on that profile. A tournament can be represented (and defined as) a complete and asymmetric graph. However the graphs of ordinary majority relations do not have advantages attached to their edges. For the purposes of constructing the examples in this paper, we also needed the advantages to be correct. Thus we cannot use the M^c^Garvey theorem here.

Fortunately for any weighted tournament, we can find a society with that weighted tournament as its weighted majority relation, if and only if all the weights are even or all the weights are odd. Thus we can check at a glance that all the weights on the weighted tournament are odd. We also check that there are no missing edges, because this is equivalent to an edge with weight of zero, which is even. This can be done easily without even needing to reach for a pencil.

The first paper that mentions this result was probably Debord's PhD thesis (1987) as quoted by Vidu (1999). However this source is inaccessible to the author, and we will give an independent proof in this paper.

2 Preliminaries

In the introduction we described voting procedures which took rankings from each voter and output either a ranking of the candidates or a single winner. Procedures which only output a winner are called social choice functions. These are the procedures we will study in this paper. Historically SCFs were often called **rules**. Below we will formally define SCFs and, discuss classifications for such rules and give definitions for the SCFs we will study in this paper.

We assume that agents' preferences are transitive, i.e. if they prefer a to b and prefer b to c they also prefer a to c. We also assume that agents preferences' are strict, if a and b are distinct they either prefer a to b or b to a. Thus we may consider each agent's preferences to be a ranking of each alternative from best to worst.

Let A and $\mathcal{N}$ be two finite sets of cardinality m and n respectively. The elements of A will be called alternatives, the elements of $\mathcal{N}$ agents. We assume that the agents have preferences over the set of alternatives. By $\mathcal{L}(A)$ we denote the set of all linear orders on A; they represent the preferences of agents over A. For example if A is $\{a, b, c\}$, then $\mathcal{L}(A)$ is the set of all permutations of abc, i.e. $\{abc, acb, bac, bca, cab, cba\}$. The elements of the Cartesian product

$$\mathcal{L}(A)^n = \mathcal{L}(A) \times \cdots \times \mathcal{L}(A) \quad (n \text{ times})$$

are called **profiles**, for example, where $A = \{a, b, c\}$ the ordered set (abc, bca) is a possible profile for a society with two voters. The profiles represent the collection of preferences of an n-element society of agents $\mathcal{N}$. Let $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ be our profile. If a linear order $P_i \in \mathcal{L}(A)$ represents the preferences of the i-th agent, then by aP_ib, where $a, b \in A$, we denote that this agent prefers a to b.

A family of mappings $F = \{F_n\}$, $n \in \mathbb{N}$,

$$F_n \colon \mathcal{L}(A)^n \to A,$$

is called a **social choice function** (SCF).

Let $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ be our profile. We define n_{xy} to be the number of linear orders in $\mathcal{P}$ that rank x above y, i.e. $n_{xy} \equiv \#\{i \mid xP_iy\}$. The approximations we consider depend upon the information contained in the matrix $N_{\mathcal{P}}$, where $(N_{\mathcal{P}})_{ab} = n_{ab}$. Many of them use the numbers

$$\text{adv}(a, b) = \max(0, n_{ab} - n_{ba}) = (n_{ab} - n_{ba})^+,$$

which will be called **advantages**. Note that $\text{adv}(a, b) = \max(0, W(a, b)) = W(a, b)^+$ where W is the weighted majority relation on $\mathcal{P}$.

A **Condorcet winner** is an alternative a for which $\text{adv}(b, a) = 0$ for all other alternatives b. A Condorcet winner does not always exist. The rules we consider below attempt to pick an alternative that is in some sense closest to being a Condorcet winner, and will always pick the Condorcet winner when it exists.

The social choice rules we consider are based on calculating the vector of **scores**. In the rules we describe below, the alternative with the lowest score wins. Let the lowest score be s. It is possible that more than one alternative has a score of s. In this case we may have a set of winners with cardinality greater than one. Strictly speaking, to be a social choice function, a rule has to output a single winner. Rules are commonly modified to achieve this by splitting ties according to the preference of the first voter. However we will usually study the set of tied winners rather than the single winner output from a tie-breaking procedure, as this will give us more information about the rules.

The **Dodgson score** (Dodgson 1876, see e.g. Black 1958; Tideman 1987), which we denote as $Sc_{\mathbf{d}}(a)$, of an alternative a is the minimum number of neighbouring alternatives that must be swapped to make a a Condorcet winner. For example, say our profile is the single vote cba. Then we swapping a and c would make a a Condorcet winner, however c and a are not neighbouring, so we would have to first swap ba and then swap ca. Hence the Dodgson score of a is 2. We call the alternative(s) with the lowest Dodgson score(s) the **Dodgson winner**(s).

The **Simpson score** (Simpson 1969, see e.g. Laslier 1997) $Sc_{\mathbf{s}}(a)$ of an alternative a is

$$Sc_{\mathbf{s}}(a) = \max_{b \neq a} \operatorname{adv}(b, a).$$

Although the Simpson rule was not designed to approximate the Dodgson rule, it is slightly simpler than the other approximations we study and so we are interested in whether it approximates the Dodgson rule well for a large number of voters. We call the alternative(s) with the lowest Simpson score(s) the **Simpson winner**(s). That is, the alternative with the smallest maximum defeat is the Simpson winner. This is why the rule is often known as the Maximin or Minimax rule.

The **Tideman score** (Tideman, 1987) $Sc_{\mathbf{t}}(a)$ of an alternative a is

$$Sc_{\mathbf{t}}(a) = \sum_{b \neq a} \operatorname{adv}(b, a).$$

We call the alternative(s) with the lowest Tideman score(s) the **Tideman winner**(s). Tideman suggested this approximation as it can be quite hard to compute the Dodgson winner.

The **Dodgson Quick (DQ) score** $Sc_{\mathbf{q}}(a)$, which is introduced in this paper for the first time, of an alternative a is

$$\begin{aligned} Sc_{\mathbf{q}}(a) &= \sum_{b \neq a} F(b, a), \\ \text{where } F(b, a) &= \left\lceil \frac{\operatorname{adv}(b, a)}{2} \right\rceil. \end{aligned}$$

We call the alternative(s) with the lowest Dodgson Quick score(s) the **Dodgson Quick winner**(s) or **DQ-winner**.

The **impartial culture** assumption is that all possible profiles $\mathcal{P}$ are equally likely, i.e. all agents are independent and all linear orders are equally likely. This assumption is of course unrealistic. Worse, we have found that the choice of probability model can affect the similarities between approximations to the Dodgson rule (M^cCabe-Dansted and Slinko, 2006). However it is impossible to select an assumption that accurately reflects the voting behaviour of all voting societies. Berg (1985) suggests studying voting properties under a variety of voting assumptions. We have conducted a broader survey of relationships between voting rules M^cCabe-Dansted and Slinko (2006), in this paper we instead seek to gain an in depth understanding of the Approximability of Dodgson's rule. This requires us to focus on a single assumption of voting behaviour. The impartial culture is the simplest assumption available. As noted by Berg (1985), many voting theorists have chosen to focus their research based upon the impartial culture assumption. Thus an in depth study of the Approximability of Dodgson's rule under the impartial culture is a natural first step.

We may derive a multinomial distribution from the impartial culture assumption as follows. Let $\mathcal{P}$ be a random profile defined on a set of m alternatives A and n agents. Let then X be a vector where each X_i represents the number of occurrences of a distinct linear order in the profile $\mathcal{P}$. Then, under the impartial culture assumption, the vector X is $(n, k, \mathbf{p})$-multinomially distributed with $k = m!$ and $\mathbf{p} = \mathbf{1}_k/k = (\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k})$.

An $(n, k, \mathbf{p})$-multinomial distribution is similar to a binomial distribution with n trials. However such a multinomial distribution has k elementary outcomes instead of just "success" and "failure". An (n, p)-binomial distribution is defined similarly.

3 A M^cGarvey Theorem for Weighted Tournaments

The M^cGarvey Theorem (1953) is a famous theorem that states that every tournament can be represented as a majority relation for a certain society of voters. We will prove a generalization of the M^cGarvey Theorem to weighted tournaments and weighted majority relations.

Laslier (1997), calls weighted tournaments "generalized tournaments". However the term "generalized tournament" seems to be less popular with other authors and the term "weighted tournament" gives an indication of how the tournament has been generalized.

Like most other authors, Laslier defines weighted tournaments as matrices and tournaments as (complete and asymmetric) binary relations. However Laslier notes that there are many different equivalent definitions of tournaments, of which Laslier gives four examples. In this paper we define both tournaments and weighted tournaments as functions, for consistency.

Definition 3.1 *Let a* ***weighted tournament*** *on A be a function $W\colon A \times A \to \mathbb{Z}$, such that $W(a,b) = -W(b,a)$ for all a, b. We call $W(a,b)$ a* ***weight*** *if $a \neq b$.*

We may equivalently draw a weighted tournament as a weighted graph, i.e. a directed graph with integers (weights) attached to edges. An edge is drawn from a to b, with an arrow pointing to b, if and only if $W(a,b) > 0$.

Note 3.2 *Tournaments are not indifferent between any pair of distinct alternatives. For this reason we cannot convert a weighted tournament which contains a 0 weight to an ordinary tournament simply by removing the weights form the edges of the directed graph.*

Definition 3.3 *We define the sum $W_1 + W_2$ of two weighted tournaments as a function f, where for all alternatives a and b we have:*

$$f(a,b) = W_1(a,b) + W_2(a,b).$$

Similarly, we define the difference between two weighted tournaments $W_1 - W_2$ as a function f where for all alternatives a and b we have:

$$f(a,b) = W_1(a,b) - W_2(a,b).$$

Definition 3.4 *We define the* ***weighted majority relation*** *$W^{\mathcal{P}}$ on a profile $\mathcal{P}$ as the weighted tournament where each weight $W^{\mathcal{P}}(a,b)$ of a pair of alternatives (a,b) equals $n_{ab} - n_{ba}$[1]. We say that a profile $\mathcal{P}$* ***generates*** *a weighted tournament $W^{\mathcal{P}}$ if $W^{\mathcal{P}}$ is the weighted majority relation on $\mathcal{P}$.*

For example, we say that the profile $\{abc, abc, cab, cab, bca\}$ generates the weighted tournament above.

Note 3.5 *$adv(a,b) = W^{\mathcal{P}}(a,b)^+$, where $x^+ = \max(0,x)$. Similarly $W^{\mathcal{P}}(a,b) = adv(a,b) - adv(b,a)$.*

Below we define the tournament and majority relation as function. The definition of tournament below is equivalent to the more traditional definition as a complete and asymmetric binary relation, used by Laslier (1997); we simply write $W(a,b) = 1$ where Laslier would write aWb.

Our definition of majority relation is also equivalent to Laslier's (1997, p. 34, definition 2.1.2). Some other authors include a λ -parameter or tie-breaking in their definition of a majority relation. We do not include λ or tie-breaking in our definition; these concepts are not needed to express the M^{c}Garvey theorem.

Definition 3.6 *A* ***tournament*** *W on A is a weighted tournament where all weights are 1 or -1.*

Definition 3.7 *We define the* ***majority relation*** *$W^{\mathcal{P}}$ on a profile $\mathcal{P}$ as the tournament where each weight $W^{\mathcal{P}}(a,b)$ of a pair of alternatives (a,b) equals 1 if and only if $n_{ab} - n_{ba}$ is greater than zero. We say that a profile $\mathcal{P}$* ***generates*** *a tournament $W^{\mathcal{P}}$ if $W^{\mathcal{P}}$ is the majority relation on $\mathcal{P}$.*

Definition 3.8 *For a weighted tournament W, for which all weights are non-zero, we define the* ***reduction*** *of W to be the tournament W_S where:*

$$W_S(a,b) = 1 \iff W(a,b) > 0.$$

Thus, from the definition of a weighted tournament:

$$W_S(a,b) = \begin{cases} 1 & \textit{if} \quad W(a,b) > 0 \\ -1 & \textit{if} \quad W(a,b) < 0 \\ 0 & \textit{if} \quad a = b \end{cases}.$$

[1]We define n_{dc}, for any pair of alternatives d and c, as the number of agents in our profile $\mathcal{P}$ who rank c above d, i.e. $\#\{d\mathcal{P}_i c\}$.

Note 3.9 *If $W^{\mathcal{P}}$ is the weighted majority relation on the profile $\mathcal{P}$, then the reduction $W^{\mathcal{P}}_S$ is the majority relation on the profile $\mathcal{P}$.*

Definition 3.10 *The **majority relation** on a profile $\mathcal{P}$ is the reduction $W^{\mathcal{P}}_S$ of $W^{\mathcal{P}}$, where $W^{\mathcal{P}}$ is the weighted majority relation on $\mathcal{P}$.*

We will now state the M^c^Garvey Theorem in terms of tournaments and majority relations:

Theorem 3.11 (McGarvey 1953) *For every tournament W there exists a profile $\mathcal{P}$ such that W is the majority relation generated by $\mathcal{P}$.*

Lemma 3.12 *Let $W^{\mathcal{P}}$ be a weighted majority relation on a profile $\mathcal{P}$ with n agents, then all weights in $W^{\mathcal{P}}$ have the same parity as n. That is, for each pair of distinct alternatives a and b, the weight $W^{\mathcal{P}}(a,b)$ is even if and only if n is even.*

Proof. We know that for all alternatives a and b we have $W^{\mathcal{P}}(a,b) = n_{ab} - n_{ba}$ and $n = n_{ba} + n_{ab}$. Hence $W^{\mathcal{P}}(a,b) + n = 2n_{ab}$ and so $W^{\mathcal{P}}(a,b)$ and n have the same parity. □

Lemma 3.13 *For a weighted tournament W with all weights being even, we may construct a profile $\mathcal{P}$ for which W is a weighted majority relation. This profile has exactly $\sum W(a,b)^+$ agents.*

Proof. We may construct such a profile as follows:

We start with an empty profile $\mathcal{P}$. For each pair of alternatives (a,b), for which the weight $W(a,b)$ is positive, we let $k = W(a,b)/2$. We take a linear order $\mathbf{v}$, on the set of alternatives A, such that $a\mathbf{v}b$ and $b\mathbf{v}x$ for all $x \neq a,b$. For example, $\mathbf{v} = abcde$. We then reverse the linear order, keeping a ranked above b, in this case producing $\mathbf{w} = edcab$. We add k instances of $\mathbf{v}$ and k instances of $\mathbf{w}$ to the profile $\mathcal{P}$. This ensures that the weight of (a,b) generated by profile $\mathcal{P}$ is equal to $W(a,b)$ without affecting the weight of (x,y) where $(b,a) \neq (x,y) \neq (a,b)$.

For each positive weight $W(a,b)$ we used exactly $W(a,b)$ agents. Thus there are exactly $\sum W(a,b)^+$ agents in the constructed profile. □

Note 3.14 *In the lemma above we have found an upper bound $\sum W(a,b)^+$ on the number of agents required. Where we have m alternatives and all positive weights are 2 (the smallest positive even number), we require $2\binom{m}{2}$ agents. This is the same upper bound that McGarvey found for ordinary tournaments. As McGarvey suspected, there was a tighter upper bound. Erdös and Moser (1964) has shown that for ordinary tournaments we will require no more than $c_1 m/\ln m$ agents, where c_1 is some fixed positive constant. From the previous work by Stearns (1959) we know that it is not possible to find tighter bound than this for ordinary tournaments, as there exists a positive constant c_2 such that for all m there exists a tournament, with only m alternatives, for which more than $c_2 m/\ln m$ agents are required.*

For a weighted tournament with a weight $W(a,b) = w$, we know that the profile will need to contain at least w agents that prefer a to b. Thus $\max_{a,b}(W(a,b))$ provides a lower bound on the number of agents *required for weighted tournaments, and so the number of required agents n is unbounded for a fixed number of alternatives m. It follows that Erdös and Moser's bound does not apply to weighted tournaments. Never-the-less, it may be possible to find an equivalent of this bound for weighted tournaments. We have not attempted to do so.*

Lemma 3.15 *For a weighted tournament W with all weights being odd, we may construct a profile which generates this weighted tournament. We will need no more than*

$$\frac{m(m-1)}{2} + \sum_{(a,b)} W(a,b)^+$$

agents to construct this profile.

Proof. Let W_1 be the weighted majority relation of a profile consisting of a single arbitrarily chosen linear order $\mathbf{v}$. Let $W_2 = W - W_1$.

Note that as W_1 is generated from a profile with an odd number (i.e. one) of linear orders, all the weights in W_1 must be odd. Thus all weights in W_2 are the difference between two odd numbers. Hence all weights in W_2 are even and we can construct a profile for which W_2 is the majority relation, as shown by Lemma 3.13. Since $W = W_1 + W_2$, joining the profiles that generate W_1 and W_2 constructs a profile that generates W.

Now we shall determine an upper bound on how many agents we will need. Say that $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}$ are the profiles we constructed that generate W_1, W_2, and W respectively. From the construction in Lemma 3.13, there will be exactly $\sum_{(a,b)} W_2(a,b)^+$ agents in $\mathcal{P}_2$, and thus exactly $1 + \sum_{(a,b)} W_2(a,b)^+$ agents in $\mathcal{P}$. We may pick $\mathbf{v}$ such that $c\mathbf{v}d$, where c, d are alternatives such that $W(c,d) \geq 1$. As $W_2 = W - W_1$, we have $W_2(c,d)^+ = W(c,d)^+ - 1$.

In a complete graph with m vertices, there are $m(m-1)/2$ edges. Thus, for $m(m-1)/2$ pairs of alternatives $W_1(a,b) = -1$, and hence $W_2(a,b) = W(a,b) + 1$. Thus for no more than $m(m-1)/2$ pairs (a,b) of alternatives $W_2(a,b) = W(a,b) + 1$. Thus,

$$\sum_{(a,b)} W_2(a,b)^+ \quad \leq \quad (-1) + \frac{m(m-1)}{2} + \sum_{(a,b)} W(a,b)^+.$$

Now we have an upper bound for the number of agents we will need to construct $\mathcal{P}_2$. As there is one more agent in $\mathcal{P}$ that is not in $\mathcal{P}_2$, we know that we need at most

$$\frac{m(m-1)}{2} + \sum_{(a,b)} W(a,b)^+$$

agents to construct $\mathcal{P}$. □

We may now prove our generalisation to the M^{c}Garvey theorem.

Theorem 3.16 *There exists a profile that generates a weighted tournament W if and only if all weights in W have the same parity.*

Proof. ($\Longleftarrow$) From the last two lemmas, we know that if all weights are even or if all weights are odd, we can construct a profile that generates W.

($\Longrightarrow$) We know that if n is odd our profile will generate a weighted tournament with all weights odd, if n is even our profile will generate a weighted tournament with all weights even. Thus every profile generates a weighted tournament for which either all weights are even or all weights are odd.

□

Note 3.17 *All weights in a tournament W_S are odd (1 or -1). Thus from Theorem 3.16, we may find a profile $\mathcal{P}$ such that the weighted majority relation $W^{\mathcal{P}}$ is equal to W_S. As $W^{\mathcal{P}}$ is already an ordinary tournament, its reduction $W_S^{\mathcal{P}}$ is equal to $W^{\mathcal{P}}$, and hence also equal to W_S.*

Hence the M^c Garvey theorem, that for all tournaments W_S there exists a profile $\mathcal{P}$ such that the majority relation W_S^P is equal to W_S, is a special case of Theorem 3.16.

4 Dodgson Quick, A New Approximation

Definition 4.1 *We say that b is **ranked directly above** a in a linear order $\mathbf{v}$ if and only if $a\mathbf{v}b$ and there does not exist c different from a, b such that $a\mathbf{v}c \wedge c\mathbf{v}b$.*

Definition 4.2 *Recall that given a profile $\mathcal{P}$, we define $D(b, a)$ as the number of agents who rank b directly above a in their preference list, and we define $F(b, a)$ and the Dodgson Quick (DQ) score $Sc_{\mathbf{q}}(a)$ of an alternative a as follows*[2]

$$\begin{aligned} F(b,a) &= \left\lceil \frac{adv(b,a)}{2} \right\rceil, \\ Sc_{\mathbf{q}}(a) &= \Sigma_{b \neq a} F(b,a). \end{aligned}$$

The Dodgson Quick score and the rule Dodgson Quick (DQ) based on that score is introduced in this paper. Recall also that we define the Dodgson score $Sc_{\mathbf{d}}(a)$ of an alternative a as the minimum number of neighbouring preferences that must be swapped to make a a Condorcet winner.

Lemma 4.3 *For distinct alternatives $a, b \in A$, under the impartial culture assumption n_{ba} and $D(b, a)$ are binomial random variables with means of $n/2$ and n/m respectively.*

Proof. For each linear order $\mathbf{v}$, we may reverse the order $\mathbf{v}$ to produce its opposite $\bar{\mathbf{v}}$, i.e. $c\mathbf{v}d \Leftrightarrow d\bar{\mathbf{v}}c$ for all $c, d \in A$. This operation $v \mapsto \bar{v}$ provides a bijection between linear orders where b is ranked above a and those where b is ranked below a. Hence these two sets of linear

[2] We define $\lceil x \rceil$ as the ceiling of x, the smallest integer that is greater than or equal to x. We define n_{ba} as the number of agents who ranked alternative b above alternative a in their preference list.

orders have the same cardinality. Under the impartial culture assumption, this implies that the probability that any agent ranks b above a is $1/2$.

The number of ways that b can be ranked *directly* above a is easily calculated if we consider the pair ba to be one object. Then we see that this number is equal to the number of permutations of $(m-1)$ objects, i.e. $(m-1)!$. The probability that b is ranked directly above a is $(m-1)!/m!$, which is equal to $1/m$.

Since votes are independent under the impartial culture assumption, n_{ba} and $D(b,a)$ are binomially distributed random variables. The mean of a binomially distributed random variable is np, so the means of n_{ba} and $D(b,a)$ are $n/2$ and n/m, respectively. □

Lemma 4.4 *Under the impartial culture assumption, the probability that $D(x,a) > F(x,a)$ for all x converges exponentially fast to 1 as the number of agents n tends to infinity.*

Proof. Under the impartial culture assumption, n_{ba} and $D(b,a)$ are binomially distributed with means of $n/2$ and n/m respectively. From Chomsky's (Dembo and Zeitouni, 1993) large deviation theorem, we know that for a fixed number of alternatives m there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$\begin{aligned} P\left(\frac{D(b,a)}{n} < \frac{1}{2m}\right) &\leq e^{-\beta_1 n}, \\ P\left(\frac{n_{ba}}{n} - \frac{1}{2} > \frac{1}{4m}\right) &\leq e^{-\beta_2 n}. \end{aligned}$$

We can rearrange the second equation to involve $F(b,a)$,

$$\begin{aligned} P\left(\frac{n_{ba}}{n} - \frac{1}{2} > \frac{1}{4m}\right) &= P\left(\frac{2n_{ba}}{n} - 1 > \frac{1}{2m}\right) \\ &= P\left(\frac{2n_{ba} - n}{n} > \frac{1}{2m}\right) \\ &= P\left(\frac{n_{ba} - (n - n_{ba})}{n} > \frac{1}{2m}\right) \\ &= P\left(\frac{n_{ba} - n_{ab}}{n} > \frac{1}{2m}\right) \\ &= P\left(\frac{\text{adv}(b,a)}{n} > \frac{1}{2m}\right). \end{aligned}$$

Since $\text{adv}(b,a) \geq F(b,a)$,

$$P\left(\frac{n_{ba}}{n} - \frac{1}{2} > \frac{1}{4m}\right) \geq P\left(\frac{F(b,a)}{n} > \frac{1}{2m}\right).$$

From this and the law of probability $P(A \vee B) \leq P(A) + P(B)$ it follows that

$$P\left(\frac{F(b,a)}{n} > \frac{1}{2m}\right) \leq e^{-\beta_2 n},$$
$$P\left(\frac{D(b,a)}{n} < \frac{1}{2m}\right) \leq e^{-\beta_1 n},$$

and so, where $\beta = \min(\beta_1, \beta_2)$,

$$\begin{aligned} P\left(\frac{F(b,a)}{n} > \frac{1}{2m} \vee \frac{D(b,a)}{n} < \frac{1}{2m}\right) &\leq e^{-\beta_1 n} + e^{-\beta_2 n} \\ &\leq 2e^{-\beta n}. \end{aligned}$$

Hence

$$P\left(\exists_x \frac{F(x,a)}{n} > \frac{1}{2m} \vee \frac{D(x,a)}{n} < \frac{1}{2m}\right) \leq 2me^{-\beta n}.$$

Using $P(\bar{E}) = 1 - P(E)$, we find that

$$P\left(\forall_x \frac{F(x,a)}{n} < \frac{1}{2m} < \frac{D(x,a)}{n}\right) \geq 1 - 2me^{-\beta n}.$$

□

Lemma 4.5 *The DQ-score $Sc_{\mathbf{q}}(a)$ is a lower bound for the Dodgson Score $Sc_{\mathbf{d}}(a)$ of a.*

Proof. Let $\mathcal{P}$ be a profile and $a \in A$. Suppose we are allowed to change linear orders in $\mathcal{P}$, by repeated swapping neighbouring alternatives. Then to make a a Condorcet winner we must reduce $\text{adv}(x,a)$ to 0 for all x and we know that $\text{adv}(x,a) = 0$ if and only if $F(x,a) = 0$. Swapping a over neighbouring alternative b will reduce $(n_{ba} - n_{ab})$ by two, but this will not affect $(n_{ca} - n_{ac})$ where $a \neq c$. Thus swapping a over neighbouring b will reduce $F(b,a)$ by one, but will not affect $F(c,a)$ where $b \neq c$. Therefore, making a a Condorcet winner will require at least $\Sigma_b F(b,a)$ swaps. This is the DQ-Score $Sc_{\mathbf{q}}(a)$ of a. □

Lemma 4.6 *If $D(x,a) \geq F(x,a)$ for every alternative x, then the DQ-Score $Sc_{\mathbf{q}}(a)$ of a is equal to the Dodgson Score $Sc_{\mathbf{d}}(a)$.*

Proof. If $F(b,a) \leq D(b,a)$, we can find at least $F(b,a)$ linear orders in the profile where b is ranked directly above a. Thus we can swap a directly over b, $F(b,a)$ times, reducing $F(b,a)$ to 0. Hence we can reduce $F(x,a)$ to 0 for all x, making a a Condorcet winner, using $\Sigma_x F(x,a)$ swaps of neighbouring preferences. In this case, $Sc_{\mathbf{q}}(a) = \Sigma_b F(b,a)$ is an upper bound for the Dodgson Score $Sc_{\mathbf{d}}(a)$ of a. From Lemma 4.5 above, $Sc_{\mathbf{q}}(a)$ is also a lower bound for $Sc_{\mathbf{d}}(a)$. Hence $Sc_{\mathbf{q}}(a) = Sc_{\mathbf{d}}(a)$. □

Corollary 4.7 *If $D(x, a) \geq F(x, a)$ for every pair of distinct alternatives (x, a), then the DQ-Winner is equal to the Dodgson Winner.*

Theorem 4.8 *Under the impartial culture assumption, the probability that the DQ-Score $Sc_{\mathbf{q}}(a)$ of an arbitrary alternative a is equal to the Dodgson Score $Sc_{\mathbf{d}}(a)$, converges to 1 exponentially fast.*

Proof. From Lemma 4.6, if $D(x, a) \geq F(x, a)$ for all alternatives x then $Sc_{\mathbf{q}}(a)$= $Sc_{\mathbf{d}}(a)$ however from Lemma 4.4, $P(\forall x \ D(x, a) \geq F(x, a))$ converges exponentially fast to 1 as $n \to \infty$. □

Corollary 4.9 *There exists an algorithm that computes the Dodgson score of an alternative a given the frequency of each linear order in the profile $\mathcal{P}$ as input, with expected running time that is logarithmic with respect the number of agents (i.e. is $\mathcal{O}(\ln n)$ for a fixed number of alternatives m).*

Proof. The are at most $m!$ distinct linear orders in the profile. Hence for a fixed number of alternatives the number of distinct linear orders is bounded. Hence we may find the DQ-score using and check whether $D(x, a) \geq F(x, a)$ for all alternatives x using a fixed number of additions. The largest number that needs to be added is proportional to the number of agents n. Additions can be performed in time linear with respect to the number of bits - logarithmic with respect to the size of the number. So we have only used an amount of time that is logarithmic with respect to the number of agents.

If $D(x, a) \geq F(x, a)$ for all alternatives x, we know that the DQ-score is the Dodgson score and we do not need to go further. From Lemma 4.4 we know that the probability that we need go further declines exponentially fast, and we can still find the Dodgson score in time polynomial with respect to the number of agents (Bartholdi et al., 1989). □

Corollary 4.10 *There exists an algorithm that computes the Dodgson winner given the frequency of each linear order in the profile $\mathcal{P}$, and has expected running time that is logarithmic with respect the number of agents.*

Corollary 4.11 *Under the impartial culture assumption, the probability that the DQ-Winner is the Dodgson Winner converges to 1 exponentially fast as we increase the number of agents.*

Obvious from Theorem 4.8 above.

5 Tideman's Rule

Recall that in Section 2 we defined the Tideman score $Sc_{\mathbf{t}}(a)$ of an alternative a as

$$Sc_{\mathbf{t}}(a) = \sum_{b \neq a} \mathrm{adv}(b, a),$$

and that the Tideman winner is the candidate with the lowest score.

Lemma 5.1 *Given an even number of agents, the Tideman winner and the DQ-winner will be the same.*

Proof. Since the n is even, we know from Lemma 3.12 that all weights in the majority relation W are even. Since the $\text{adv}(a, b) \equiv W(a, b)^+$ it is clear that all advantages will also be even. Since $\text{adv}(a, b)$ will always be even, $\lceil \text{adv}(a, b)/2 \rceil$ will be exactly half $\text{adv}(a, b)$ and so the DQ-score will be exactly half the Tideman score. Hence the DQ-winner and the Tideman winner will be the same. □

Corollary 5.2 *Under the impartial culture assumption, for $2n$ agents and a fixed number of alternatives m, the probability that the Tideman winner is the Dodgson winner converges to 1 exponentially fast as n approaches infinity.*

Proof. If there are an even number of agents the Tideman winner equals the DQ-winner (Lemma 5.1) and the probability that the DQ-winner is the Dodgson winner exponentially fast (Corollary 4.11). □

Corollary 5.3 *If the number of agents is even and $D(x, a) \geq F(x, a)$ for every pair of distinct alternatives (x, a), then the Tideman Winner is equal to the Dodgson Winner.*

Obvious from Lemma 5.1 above and Corollary 4.7.

Corollary 5.4 *If the Tideman winner is not the DQ-winner, all non-zero advantages are odd.*

Proof. As we must have an odd number of agents, from Lemma 3.12 all weights in the majority relation W must be odd. Since the $\text{adv}(a, b) \equiv W(a, b)^+$ the advantage $\text{adv}(a, b)$ must be zero or equal to the weight $W(a, b)$. □

Lemma 5.5 *There is no profile with three alternatives such that the Tideman winner is not the DQ-winner.*

Proof. The Tideman and Dodgson Quick rules both pick the Condorcet winner when it exists, so if a Condorcet winner exists the Tideman winner and DQ-winner will be the same. It is well known that the absence of a Condorcet winner on three alternatives means that we can rename these alternatives a, b and c so that $\text{adv}(a, b) > 0$, $\text{adv}(b, c) > 0$, and $\text{adv}(c, a) > 0$. These advantages must be odd from the previous corollary. Hence for some $i, j, k \in \mathbb{Z}$ such that $\text{adv}(a, b) = 2i - 1$, $\text{adv}(b, c) = 2j - 1$, and $\text{adv}(c, a) = 2k - 1$. The DQ-Scores and Tideman scores of a, b, c are i, j, k and $2i - 1,\ 2j - 1,\ 2k - 1$ respectively. From here the result is clear, since if $i > j > k$ then $2i - 1 > 2j - 1 > 2k - 1$. □

Lemma 5.6 *For a profile with four alternatives there does not exist a pair of alternatives such that a is a DQ-winner but not a Tideman winner, and b is a Tideman winner but not a DQ-winner.*

Proof. By way of contradiction assume that there exist such a, b. Thus there is no Condorcet winner, and so for each alternative c there are one to three alternatives d such that $\text{adv}(c, d) > 0$. Also, since the set of Tideman winners and DQ-winners differ, n must be odd and hence all non-zero advantages must be odd. The relationship between the Tideman score $Sc_{\mathbf{t}}(c)$ and the DQ-score $Sc_{\mathbf{q}}(c)$ is as follows:

$$\begin{aligned} Sc_{\mathbf{t}}(c) &= \sum_{d \in A} \text{adv}(c, d) \\ &= \sum_{d \in A} \left\lceil \frac{\text{adv}(c, d)}{2} \right\rceil - \#\{c : \text{adv}(c, d) \notin 2\mathbb{Z}\} \\ &= Sc_{\mathbf{q}}(c) - (1 \text{ or } 2 \text{ or } 3). \end{aligned}$$

Thus,

$$2Sc_{\mathbf{q}}(c) - 3 \quad \leq Sc_{\mathbf{t}}(c) \leq \quad 2Sc_{\mathbf{q}}(c) - 1,$$

and so,

$$Sc_{\mathbf{t}}(a) \quad \leq \quad 2Sc_{\mathbf{q}}(a) - 1.$$

Given that a is DQ-winner and b is not, we know

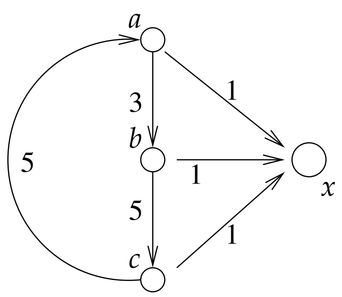
$$Sc_{\mathbf{q}}(a) \quad \leq \quad Sc_{\mathbf{q}}(b) - 1.$$

Thus by substitution,

$$\begin{aligned} Sc_{\mathbf{t}}(a) &\leq 2(Sc_{\mathbf{q}}(b) - 1) - 1 \\ &= 2Sc_{\mathbf{q}}(b) - 3 \\ &\leq Sc_{\mathbf{t}}(b). \end{aligned}$$

This shows that if b is a Tideman winner, so is a. By contradiction the result must be correct. □

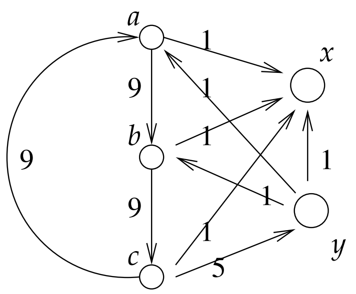
Example 5.7 *There do exist profiles with four alternatives where the set of tied Tideman winners differs from the set of tied DQ-winners. By Theorem 3.16, we know we may construct a profile with the following advantages:*



Scores	a	b	c	x
Tideman	5	3	5	3
DQ	3	2	3	3

Here x, b are tied Tideman winners, but b is the sole DQ-winner.

Example 5.8 *There do exist profiles with five alternatives where there is a unique Tideman winner that differs from the unique DQ-winner. By Theorem 3.16, we know we may construct a profile with the following advantages:*



Scores	a	b	c	x	y
Tideman	10	10	9	4	5
DQ	6	6	5	4	3

Here x is the sole Tideman winner, but y is the sole DQ-winner.

Theorem 5.9 *For any $m \geq 5$ there exists a profile with m alternatives and an odd number of agents, where the Tideman winner is not the DQ-winner.*

On page 14 there is an example of a weighted majority relation where the Tideman winner is not the Dodgson Quick winner. To extend this example for larger numbers of alternatives, we may add additional alternatives who lose to all of a, b, c, x, y. From Theorem 3.16 and Lemma 3.12, there exists a profile with an odd number of agents that generates that weighted majority relation.

Theorem 5.10 *Under the impartial culture assumption, if we have an even number of agents, the probability that all of the advantages are 0 does not converge to 0 faster than $\mathcal{O}(n^{-\frac{m!}{4}})$.*

Proof. Let $\mathcal{P}$ be a random profile, $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m!}\}$ be an ordered set containing all $m!$ possible linear orders on m alternatives, and X be a random vector, with elements X_i representing the number of occurrences of $\mathbf{v}_i$ in $\mathcal{P}$. Under the impartial culture assumption, X is distributed according to a multinomial distribution with n trials and $m!$ possible outcomes. Let us group the $m!$ outcomes into $m!/2$ pairs $S_i = \{\mathbf{v}_i, \bar{\mathbf{v}}_i\}$. Denote the number of occurrences of $\mathbf{v}$ as $n(\mathbf{v})$. Let the random variable Y_i^1 be $n(\mathbf{v}_i)$ and Y_i^2 be $n(\bar{\mathbf{v}}_i)$. Let $Y_i = Y_i^1 + Y_i^2$.

It is easy to show that, given $Y_i = y_i$ for all i, each Y_i^1 is independently binomially distributed with $p = 1/2$ and y_i trials. It is also easy to show that an arbitrary integer n, a $(2n, 0.5)$-binomial random variable X has a probability of at least $\frac{1}{\sqrt{2n}}$ of equaling n where $n > 0$; thus if y_i is even then the probability that $Y_i^1 = Y_i^2$ is at least $\frac{1}{2\sqrt{y_i}}$. Combining these results we get

$$\begin{aligned} P(\forall_i Y_i^1 = Y_i^2 | \forall_i Y_i = y_i \in 2\mathbb{Z}) &\geq \prod_i \frac{1}{2\sqrt{y_i}} \\ &\geq \prod_i \frac{1}{2\sqrt{n}} = 2^{-\frac{m!}{2}} n^{-\frac{m!}{4}}. \end{aligned}$$

It is easy to show that for any k-dimensional multinomially distributed random vector, the probability that all k elements are even is at least 2^{-k+1}; hence the probability that all X_i are even is at least 2^{-k+1} where $k = m!/2$. Hence

$$P(\forall_i X_{i,1} = X_{i,2}) \quad \geq \quad \left(2^{-\frac{m!}{2}+1}\right)\left(2^{-\frac{m!}{2}} n^{-\frac{m!}{4}}\right) = 2^{1-m!} n^{-\frac{m!}{4}}.$$

If for all i, $X_{i,1} = X_{i,2}$ then for all i, $n(\mathbf{v}_i) = n(\bar{\mathbf{v}}_i)$, i.e. the number of each type of vote is the same as its complement. Thus

$$n_{ba} \quad = \quad \sum_{\mathbf{v} \in \{\mathbf{v} : b\mathbf{v}a\}} n(v) = \sum_{\bar{\mathbf{v}} \in \{\bar{\mathbf{v}} : a\bar{\mathbf{v}}b\}} n(\bar{v}) = \sum_{\mathbf{v} \in \{\mathbf{v} : a\mathbf{v}b\}} n(v) = n_{ab},$$

so $\text{adv}(b, a) = 0$ for all alternatives b and a. □

Corollary 5.11 *Under the impartial culture assumption, if we have an even number of agents, the probability that all of the advantages are 0, does not converge to 0 at an exponentially fast rate.*

Lemma 5.12 *Under the impartial culture assumption, the probability that the Tideman winner is not the DQ-winner does not converge to 0 faster than $\mathcal{O}(n^{-\frac{m!}{4}})$ as the number of agents n tends to infinity.*

Let $\mathcal{P}$ be our random profile with n agents, for some odd number n. Let $|C|$ be the size of the profile from Theorem 5.9. Let us place the first $|C|$ agents from profile $\mathcal{P}$ into sub-profiles C and the remainder of the agents into sub-profile D. There is a small but constant probability that C forms the example from Theorem 5.9, resulting in the Tideman winner of C differing from its DQ-winner. As n, $|C|$ are odd, $|D|$ is even. Thus from Theorem 5.10 the probability that the advantages in D are zero does not converge to 0 faster than $\mathcal{O}(n^{-\frac{m!}{4}})$. If all the advantages in D are zero then adding D to C will not affect the Tideman or DQ-winners. Hence the probability that the Tideman winner is not the DQ-winner does not converge to 0 faster than $\mathcal{O}(n^{-\frac{m!}{4}})$.

Corollary 5.13 *Under the impartial culture assumption, the probability that the Tideman winner is not the DQ-winner does not converge to 0 exponentially fast as the number of agents n tends to infinity.*

Theorem 5.14 *Under the impartial culture assumption, the probability that the Tideman winner is not the Dodgson winner does not converge to 0 faster than $\mathcal{O}(n^{-\frac{m!}{4}})$ as the number of agents n tends to infinity.*

Proof. From Corollary 4.11 the DQ-winner converges to the Dodgson winner exponentially fast. However, the Tideman winner does not converge faster than $\mathcal{O}(n^{-\frac{m!}{4}})$ to the DQ-winner, and hence also does not converge faster than $\mathcal{O}(n^{-\frac{m!}{4}})$ to the Dodgson winner. □

Lemma 5.15 *Let S be a subset of A. Let $a_1 a_2 \ldots a_{|S|}$ and $b_1 b_2 \ldots b_{|S|}$ be two linear orderings of S. Then the number of linear orders $\mathbf{v}$ in $\mathcal{L}(A)$ where $a_1\mathbf{v}a_2, a_2\mathbf{v}a_3, \ldots, a_{|S|-1}\mathbf{v}a_{|S|}$ is equal to the number of linear orders $\mathbf{v}$ where $b_1\mathbf{v}b_2, b_2\mathbf{v}b_3, \ldots, b_{|S|-1}\mathbf{v}b_{|S|}$, i.e.*

$$\#\{\mathbf{v} : \forall_{i \in [2,|S|]} a_{i-1}\mathbf{v}a_i\} \quad = \quad \#\{\mathbf{v} : \forall_{i \in [2,|S|]} b_{i-1}\mathbf{v}b_i\}.$$

Proof. Let the function f be defined with domain and range $\mathcal{L}(A)$ as follows. If a_i is ranked in position j in $\mathbf{v}$ then b_i is ranked in position j in $f(\mathbf{v})$. If $x \notin S$ is ranked in some position j in v then x is still ranked in position j in $f(\mathbf{v})$.

Clearly, if b_i is ranked in position j in $\mathbf{v}$ then a_i is ranked in position j in all members of $f^{-1}(\mathbf{v})$. If $x \notin S$ is ranked in some position j in $\mathbf{v}$ then x is still ranked in position j in all members of $f^{-1}(\mathbf{v})$. Hence $f^{-1}(\mathbf{v})$ is a function.

The function f provides a bijection between the sets $\{\mathbf{v}\colon \forall_{i\in[2,|S|]} a_{i-1}\mathbf{v}a_i\}$, $\{\mathbf{v}\colon \forall_{i\in[2,|S|]} b_{i-1}\mathbf{v}b_i\}$. Hence the result. □

Corollary 5.16 *Let S be a subset of A. The number of linear orders $\mathbf{v}$ where $a_{i-1}\mathbf{v}a_i$ for all $i = 2, 3, \ldots, |S|$ is equal to $n!/|S|!$.*

Definition 5.17 *We define the adjacency matrix M, of a linear order $\mathbf{v}$, as follows:*

$$M_{ij} = \begin{cases} 1 & \textit{if} \quad i\mathbf{v}j \\ -1 & \textit{if} \quad j\mathbf{v}i \\ 0 & \textit{if} \quad i = j \end{cases} .$$

Lemma 5.18 *Suppose that each linear order is equally likely, then M is an m^2-dimensional random variable satisfying the following equations for all $i, j, r, s \in A$.*

$$\begin{aligned} E[M] &= 0 \\ = cov(M_{ij}, M_{rs}) &= E[M_{ij}M_{rs}] \\ &= \begin{cases} 1 & \textit{if} \quad i = r \neq j = s \\ 1/3 & \textit{if} \quad i = r, \textit{ but } i, j, s \textit{ distinct } \vee\ j = s, \textit{ others distinct} \\ -1/3 & \textit{if} \quad i = s, \textit{ others distinct } \vee\ j = r, \textit{ others distinct} \\ 0 & \textit{if} \quad i, j, r, s \textit{ distinct } \vee\ i = j = r = s \\ -1 & \textit{if} \quad i = s \neq j = r \end{cases} . \end{aligned}$$

Proof. From Lemma 5.15,

$$E[M_{ij}] = \frac{(1) + (-1)}{2} = 0.$$

It is well known that $\text{cov}(X, Y) = E[XY] - E[X]E[Y]$ (see e.g. Walpole and Myers 1993; p97). Thus $\text{cov}(M_{ij}, M_{rs}) = E[M_{ij}M_{rs}] - (0)(0) = E[M_{ij}M_{rs}]$. Note that for all $i \neq j$ we know that $M_{ii}M_{ii} = 0$, $M_{ij}M_{ij} = 1$, and $M_{ij}M_{ji} = -1$. If $i = r$ and i, j, s are all distinct then the sign of

$M_{ij}M_{is}$ for each permutation of i, j and s is as shown below.

$$\begin{array}{rcccccc} & i & i & j & j & s & s \\ & j & s & i & s & i & j \\ & s & j & s & i & j & i \\ M_{ij} & + & + & - & - & + & - \\ M_{is} & + & + & + & - & - & - \\ M_{ij}M_{is} & + & + & - & + & - & + \end{array}$$

Thus from Lemma 5.15,

$$E[M_{ij}M_{rs}] \quad = \quad \frac{+1+1-1+1-1+1}{6} = \frac{1}{3}.$$

If i, j, r, s are all distinct then there are six linear orders $\mathbf{v}$ where $i\mathbf{v}j \wedge r\mathbf{v}s$, six linear orders $\mathbf{v}$ where $i\mathbf{v}j \wedge s\mathbf{v}r$, six linear orders $\mathbf{v}$ where $j\mathbf{v}i \wedge r\mathbf{v}s$, and six linear orders $\mathbf{v}$ where $j\mathbf{v}i \wedge s\mathbf{v}r$. Hence from Lemma 5.15,

$$E[M_{ij}M_{rs}] \quad = \frac{6(1)(1)+6(1)(-1)+6(-1)(1)+6(-1)(-1)}{24} = 0 \quad .$$

We may prove the other cases for $\text{cov}(M_{ij}, M_{rs})$ in much the same way. □

Corollary 5.19 *As var$(X) =$ cov(X, X) we also have,*

$$var(M_{ij}) \quad = \quad \begin{cases} 1 & if\ i \neq j \\ 0 & if\ i = j \end{cases} \quad .$$

Example 1 *For example, for $m = 4$ the covariances with M_{12} are shown in the matrix*

$$\mathfrak{L} \quad = \quad \begin{bmatrix} 0 & 1 & 1/3 & 1/3 \\ -1 & 0 & -1/3 & -1/3 \\ -1/3 & 1/3 & 0 & 0 \\ -1/3 & 1/3 & 0 & 0 \end{bmatrix},$$

where $\mathfrak{L}_{ij} = cov(M_{ij}, M_{12})$.

Definition 5.20 *Define Y to be a collection of random normal variables indexed by i, j for $1 \leq i < j \leq m$ each with mean of 0, and covariance matrix Ω, where*

$$\Omega_{ij,rs} \quad = \quad cov(Y_{ij}, Y_{rs}) = cov(M_{ij}, M_{rs}),$$

We may use the fact that $i < j$, $r < s$ implies $i \neq j$, $r \neq s$, $(s = i \Rightarrow r \neq j)$ and $(r = j \Rightarrow s \neq i)$ to *simplify the definition of* Ω *as shown below:*

$$\Omega_{ij,rs} = \begin{cases} 1 & \textit{if} \quad (r,s) = (i,j) \\ 1/3 & \textit{if} \quad r = i, s \neq j \textit{ or } s = j, r \neq i \\ -1/3 & \textit{if} \quad s = i \textit{ or } r = j \\ 0 & \textit{if} \quad i,j,r,s \textit{ are all distinct} \end{cases},$$

i.e. if i, j, r, s *are all distinct then*

$$\begin{aligned} \Omega_{ij,ij} &= 1, \\ \Omega_{ij,rj} = \Omega_{ij,is} &= 1/3, \\ \Omega_{ij,ri} = \Omega_{ij,js} &= -1/3, \\ \Omega_{ij,rs} &= 0. \end{aligned}$$

Lemma 5.21 *As* n *approaches infinity,* $\sum_{i=1}^{n} M_i/\sqrt{n}$ *converges in distribution to*

$$\begin{bmatrix} 0 & Y_{12} & Y_{13} & \cdots & Y_{1m} \\ -Y_{12} & 0 & Y_{23} & \cdots & Y_{2m} \\ -Y_{13} & -Y_{23} & 0 & \cdots & Y_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -Y_{1m} & -Y_{2m} & -Y_{3m} & \cdots & 0 \end{bmatrix},$$

where M_i *is the adjacency matrix for the* i^{th} *linear order in a profile* P*, and recall that* Y *is be a collection of random normal variables indexed by* i, j *for* $1 \leq i < j \leq m$ *each with mean of 0, and covariance matrix* Ω*, where*

$$\Omega_{ij,rs} = \mathit{cov}(Y_{ij}, Y_{rs}) = \mathit{cov}(M_{ij}, M_{rs}).$$

Proof. As $M_1, M_2, \ldots, M_n$ are independent identically-distributed (i.i.d.) random variables, we know from the multivariate central limit theorem (see e.g. Anderson, 1984; p81) that $\sum_{i=1}^{n} M_i/\sqrt{n}$ converges in distribution to the multivariate normal distribution with the same mean and covariance as M_1. As $M^T = -M$ and $M_{ii} = 0$, we have the result. □

Lemma 5.22 Ω *is non-singular.*

Proof. Consider Ω^2:

$$(\Omega^2)_{ij,kl} = \sum_{1 \leq r < s \leq m} \Gamma_{ij,kl}(r,s),$$

where $\Gamma_{ij,kl}(r,s) = \Omega_{ij,rs}\Omega_{rs,kl}$.

For i, j, r, s distinct then

$$\begin{aligned}
\Gamma_{ij,ij}(i,j) = \Omega_{ij,ij}\Omega_{ij,ij} &= (1)(1) = 1, \\
\Gamma_{ij,ij}(r,j) = \Omega_{ij,rj}\Omega_{rj,ij} &= ({}^1\!/\!_3)({}^1\!/\!_3) = {}^1\!/\!_9, \\
\Gamma_{ij,ij}(i,s) = \Omega_{ij,is}\Omega_{is,ij} &= ({}^1\!/\!_3)({}^1\!/\!_3) = {}^1\!/\!_9, \\
\Gamma_{ij,ij}(r,i) = \Omega_{ij,ri}\Omega_{ri,ij} &= (-{}^1\!/\!_3)(-{}^1\!/\!_3) = {}^1\!/\!_9, \\
\Gamma_{ij,ij}(j,s) = \Omega_{ij,js}\Omega_{js,ij} &= (-{}^1\!/\!_3)(-{}^1\!/\!_3) = {}^1\!/\!_9, \\
\Gamma_{ij,ij}(r,s) = \Omega_{ij,rs}\Omega_{ij,rs} &= 0.
\end{aligned}$$

Case $(i, j) = (k, l)$:

If $(i, j) = (k, l)$ then

$$\begin{aligned}
\Gamma_{ij,ij}(r,s) &= \Omega_{ij,rs}\Omega_{rs,ij} \\
&= \begin{cases} (1)^2 & \text{if} \quad (r,s) = (i,j) \\ ({}^1\!/\!_3)^2 & \text{if} \quad r = i, s \neq j \text{ or } s = j, r \neq i \\ (-{}^1\!/\!_3)^2 & \text{if} \quad s = i, (r \neq j) \text{ or } r = j, (s \neq i) \\ 0 & \text{if} \quad i, j, r, s \text{ are all distinct} \end{cases},
\end{aligned}$$

Recall that $r < s$, $i < j$ and $r, s \in [1, m]$. Let us consider for how many values of (r, s) each of the above cases occur:

- $(r, s) = (i, j)$: This occurs for exactly one value of (r, s).

- $r = i, s \neq j$: Combining the fact that $r < s$ and $r = i$ we get $i < s$. Thus $s \in (i, j) \cup (j, m]$, and there are $(j - i - 1) + (m - j) = (m - i - 1)$ possible values of s. As there is only one possible value of r this means that there are also $(m - i - 1)$ possible values of (r, s).

- $s = j, r \neq i$: Combining the fact that $r < s$ and $s = j$ we get $r < j$. Thus $r \in [1, i) \cup (i, j)$, and there are $(i - 1) + (j - i - 1) = (j - 2)$ possible values of (r, s).

- $s = i$: Here we want $r \neq j$, however $r < s = i < j$, so explicitly stating $r \neq j$ is redundant. Combining the fact that $r < s$ and $s = i$ we get $r < i$. Hence $r \in [1, i]$ and there are $i - 1$ possible values for (r, s).

- $r = j$: Here we want $s \neq i$, however $i < j = r < s$, so explicitly stating that $r \neq j$ is redundant. From here on we will not state redundant inequalities. Combining the fact that $r < s$ and $r = j$ we get $j < s$. Hence $s \in (j, m]$ and there are $m - j$ possible values for (r, s).

hence,

$$\begin{aligned}\sum_{1\le r<s\le m}\Gamma_{ij,ij}(r,s) &= (1)(1)+((m-i-1)+(j-2))\left(\frac{1}{3}\right)^2+((i-1)+(m-j))\left(\frac{-1}{3}\right)^2\\ &= 1+(m+j-i-3)\left(\frac{1}{9}\right)+(m+i-j-1)\left(\frac{1}{9}\right)\\ &= (9+(m+j-i-3)+(m+i-j-1))/9\\ &= \frac{2m+5}{9}.\end{aligned}$$

Case $i=k,\ j\neq l$: then,

$$\begin{aligned}\Gamma_{ij,il}(r,s) &= \Omega_{ij,rs}\Omega_{rs,il}=\\ &= \begin{cases} 1\Omega_{rs,il} & \text{if} \quad (r,s)=(i,j)\\ {}^1\!/\!_3\Omega_{rs,il} & \text{if} \quad r=i, s\neq j \text{ or } s=j, r\neq i\\ -{}^1\!/\!_3\Omega_{rs,il} & \text{if} \quad s=i \text{ or } r=j\\ 0 & \text{if} \quad i,j,r,s \text{ are all distinct}\end{cases},\end{aligned}$$

more precisely,

$$\Gamma_{ij,il}(r,s) = \begin{cases} (1)({}^1\!/\!_3) = {}^1\!/\!_3 & \text{if} \quad (i,j)=(r,s)\\ ({}^1\!/\!_3)(1) = {}^1\!/\!_3 & \text{if} \quad r=i, s=l\neq j\\ ({}^1\!/\!_3)({}^1\!/\!_3) = {}^1\!/\!_9 & \text{if} \quad r=i, s\neq j, s=\neq l\\ ({}^1\!/\!_3)(0) = 0 & \text{if} \quad s=j\neq l, r\neq i\\ (-{}^1\!/\!_3)(-{}^1\!/\!_3) = {}^1\!/\!_9 & \text{if} \quad s=i\\ (-{}^1\!/\!_3)({}^1\!/\!_3) = -{}^1\!/\!_9 & \text{if} \quad r=j, s=l\\ (-{}^1\!/\!_3)(0) = 0 & \text{if} \quad r=j, s\neq l\\ 0 = 0 & \text{if} \quad i,j,r,s \text{ are all distinct}\end{cases},$$

hence,

$$\begin{aligned}\sum_{1\le r<s\le m}\Gamma(r,s) &= \frac{1}{3}+\frac{1}{3}+\sum_{1\le r<s\le m, r=i, s\neq j, s=\neq l}\frac{1}{9}+\sum_{1\le r<s\le m, s=i}\frac{1}{9}-\frac{1}{9}\\ &= \frac{1}{3}+\frac{1}{3}+\sum_{i<s\le m}\frac{1}{9}-\frac{2}{9}+\sum_{1\le r<i}\frac{1}{9}-\frac{1}{9}\\ &= \frac{1}{3}+(m-i)\frac{1}{9}+(i-1)\frac{1}{9}\\ &= \frac{m+2}{9}.\end{aligned}$$

Similarly for $i \neq k,\ j = l$, we may show $(\Omega^2)_{ij,kj} = \frac{m+2}{9}$. If $j = k$ then

$$\begin{aligned}(\Omega^2)_{ij,kl} &= -\frac{1}{3} - \frac{1}{3} + \frac{1}{9} - \sum_{1 \leq r < i, r \neq i} \frac{1}{9} - \sum_{j < s \leq m, s \neq l} \frac{1}{9}, \\ &= -\frac{m+2}{9},\end{aligned}$$

similarly for $l = i$. If i, j, k, l are all distinct, $(\Omega^2)_{ij,kl}$ equals 0. Consequently

$$\Omega^2 = \left(\frac{m+2}{3}\right)\Omega - \left(\frac{m+1}{9}\right)I$$

Now, when a matrix Ω satisfies $\Omega^2 = \alpha\Omega + \beta I$ with $\beta \neq 0$ it has an inverse as shown below,

$$\Omega\left(\frac{\Omega - \alpha}{\beta}\right) = I,$$

and hence Ω is not singular. □

Theorem 5.23 *The probability that the Tideman winner and Dodgson Quick winner coincide converges asymptotically to* 1 *as* $n \to \infty$.

Proof. The Tideman winner is the alternative $a \in A$ with the minimal value of

$$G(a) = \sum_{b \in A} \mathrm{adv}(b, a),$$

while the DQ-winner has minimal value of

$$F(a) = \sum_{b \in A} \left\lceil \frac{\mathrm{adv}(b, a)}{2} \right\rceil.$$

Let a_T be the Tideman winner and a_Q be the DQ-winner. Note that $G - m \leq 2F \leq G$. If for some b we have $G(b) - m > G(a_T)$, then $2F(b) \geq G(b) - m > G(a_T) \geq 2F(a_T)$ and so b is not a DQ-winner. Hence, if $G(b) - m > G(a_T)$ for all alternatives b distinct from a, then a_T is also the DQ-winner a_Q. Thus,

$$\begin{aligned}P(a_T \neq a_Q) &\leq P(\exists_{a,b}|G(a) - G(b)| \leq 2m \wedge a \neq b) \\ &= P\left(\exists_{a,b}\left|\frac{G(a) - G(b)}{\sqrt{n}}\right| \leq \frac{2m}{\sqrt{n}} \wedge a \neq b\right),\end{aligned}$$

thus for any $\epsilon > 0$ and sufficiently large n, we have

$$P(a_T \neq a_Q) \leq P\left(\exists_{a,b}\left|\frac{G(a) - G(b)}{\sqrt{n}}\right| \leq \epsilon \wedge a \neq b\right)$$

We will show that the right-hand side of the inequality above converges to 0 as n tends to ∞. All probabilities are non-negative so $0 \leq P(a_T \neq a_Q)$. From these facts and the sandwich theorem it follows that $\lim_{n\to\infty} P(a_T \neq a_Q) = 0$. We let,

$$G_j = \sum_{i<j} (Y_{ij})^+ + \sum_{k>j} (-Y_{jk})^+ ,$$

and so,

$$\lim_{n\to\infty} P\left(\exists_{a,b} \left|\frac{G(a)-G(b)}{\sqrt{n}}\right| \leq \epsilon \wedge a \neq b\right) = P\left(\exists_{i,j} |G_i - G_j| \leq \epsilon \wedge i \neq j\right)$$

Since $\epsilon > 0$ is arbitrary, the infimum is $\epsilon = 0$. Thus,

$$\lim_{n\to\infty} P(a_T \neq a_Q) \leq P(\exists_{i,j} G_i = G_j \wedge i \neq j).$$

For fixed $i < j$ we have

$$G_i - G_j = -Y_{ij} + \sum_{k<i} (-Y_{ki})^+ + \sum_{k>i, k\neq i} (Y_{ik})^+ - \sum_{k<j, k\neq i} (Y_{kj})^+ - \sum_{k>j} (-Y_{jk})^+$$

Define v so that $G_i - G_j = -Y_{ij} + v$. Then $P(G_i = G_j) = P(Y_{ij} = v) = E[P(Y_{ij} = v|v)]$. Since Y has a multivariate normal distribution with a non-singular covariance matrix Ω, it follows that $P(Y_{ij} = v|v) = 0$. That is, $P(G_i = G_j) = 0$ for any i, j where $i \neq j$. Hence $P(\exists_{i,j} G_i = G_j \wedge i \neq j) = 0$. As discussed previously in this proof, we may now use the sandwich theorem to prove that $\lim_{n\to\infty} P(a_T \neq a_Q) = 0$. □

6 Numerical Results

Although we have proven theorems on the rate of convergence, tables of figures can help illustrate the nature of the convergence. In this section we present tables demonstrating the rate fast convergence of our Dodgson Quick rule in comparison to the Tideman rule. We also study the asymptotic limit of the probability that the Simpson winner is the Dodgson winner as we increase the number of agents.

As shown below the convergence of the Tideman winner to the Dodgson Winner occurs much slower than the exponential convergence of the DQ-Winner.

Table 1 reports the probability that these rules pick the same winner after we break ties according to the preferences of the first agent. It was generated by averaging 10,000 simulations, so the figures are only approximate, however the trends are clearly significant.

Since the DQ-Winner and Tideman Winner seem to closely approximate Dodgson's Rule we may wish to also look at the probability that these rules pick the same set of tied winners, presented in Table 2.

From Theorem 4.8 we know that the Dodgson Quick winner converges to the Dodgson winner at an exponentially fast rate. These figures confirm that for a large number of agents the simple Dodgson Quick rule provides a very good approximation to the Dodgson rule.

Another question is how well does Dodgson Quick approximate the Dodgson rule with other numbers of alternatives, or if the number of agents is not large in comparison to the number of agents. From 3, it appears that our approximation is still reasonably accurate under these conditions. This table was generated by averaging 10,000 simulations, and splitting ties according to the preferences of the first agent.

To give meaning to these figures, let us compare them with the figures in Tables 4 and 5. We see that even where the number of agents is not very large, the Dodgson Quick rule seems to do a slightly better job of approximating the Dodgson rule than Tideman's approximation. We also see that Simpson's rule does a particularly poor job of approximating frequency of the Tideman approximation picking the Dodgson winner when the number of candidates is large.

As an aside, it would appear that Simpson's rule is not a very accurate approximation of Dodgson's Rule. The probability that the Simpson winner does not equal the Dodgson winner is much

Table 1: Number of occurrences per 1000 Elections with 5 alternatives that the Dodgson Winner was not chosen

Voters	3	5	7	9	15	17	25	85	257	1025
DQ	1.5	1.9	1.35	0.55	0.05	0.1	0	0	0	0
Tideman	1.5	2.3	2.7	3.95	6.05	6.85	7.95	8.2	5.9	2.95
Simpson	57.6	65.7	62.2	57.8	48.3	46.6	41.9	30.2	23.4	21.6

Table 2: Number of Occurrences per 1000 Elections with 5 Alternatives that the Set of Tied Dodgson Winners is Not Chosen

Voters	3	5	7	9	15	17	25	33	85	257	1025
DQ-Winners	4.31	4.41	3.21	1.94	0.27	0.08	0.04	0	0	0	0
Tideman	4.31	5.57	7.31	8.43	12.73	13.15	15.46	16.35	15.18	10.2	5.4

Table 3: Frequency that the DQ-Winner is the Dodgson Winner

# Alternatives \ # Agents	3	5	7	9	15	25	85
3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.9984	0.9976	0.9980	0.9992	0.9999	1.0000	1.0000
7	0.9902	0.9875	0.9879	0.9933	0.9980	0.9995	1.0000
9	0.9792	0.9742	0.9778	0.9837	0.9924	0.9978	0.9999
15	0.9468	0.9327	0.9338	0.9412	0.9571	0.9743	0.9988
25	0.8997	0.8718	0.8661	0.8731	0.8971	0.9265	0.9840

greater than for Tideman or DQ. We may ask, does the Simpson rule eventually converge to the Dodgson rule as we increase the number of voters, and if not, how close does it get?

6.1 Asymptotic Behaviour of Simpson's Rule

From Lemma 4.4 and Theorem 5.23 we know that the Dodgson winner, Dodgson Quick winner, and Tideman winner all asymptotically converge as we increase the number of agents. Hence we may compute the asymptotic probability that the Simpson winner is equal to the Dodgson winner, by computing the asymptotic probability that the Simpson winner equals the Tideman winner.

From Lemma 5.21 we know that the matrix of advantages converges to a multivariate normal distribution as we increase the number of agents. If we had a multivariate normal random vector generator, we could use this model to perform simulations and count in how many simulations the Simpson winner is equal to the Tideman winner. We decided to use a slightly different model so that we could use a univariate normal random number generator.

Table 4: Frequency that the Tideman winner is the Dodgson winner

# Alternatives \ # Agents	3	5	7	9	15	25	85
3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.9984	0.9974	0.9961	0.9972	0.9936	0.9917	0.9930
7	0.9902	0.9864	0.9852	0.9868	0.9845	0.9805	0.9847
9	0.9792	0.9730	0.9724	0.9731	0.9718	0.9760	0.9815
15	0.9468	0.9292	0.9263	0.9273	0.9379	0.9485	0.9649
25	0.8997	0.8691	0.8620	0.8625	0.8833	0.9113	0.9534

Table 5: Frequency that the Simpson Winner is the Dodgson Winner

# Alternatives \ # Agents	3	5	7	9	15	25	85
3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.9433	0.9307	0.9339	0.9398	0.9493	0.9575	0.9714
7	0.8734	0.8627	0.8689	0.8786	0.9018	0.9153	0.9404
9	0.8256	0.8153	0.8167	0.8251	0.8562	0.8808	0.9124
25	0.5895	0.5772	0.6147	0.6322	0.7114	0.7529	0.7957

Table 6: Number of Occurrences per 1000 Elections that the Simpson Winner is Not the Dodgson Winner. (Limit as $n \to \infty$)

#Alternatives	3	4	5	6	7	8
#(DO $\neq$ SI) per 1000	0	6.81	17.18	27	39.33	50.18

Let $\mathcal{P}$ be our profile with m alternatives A and n agents. Say a and b are two distinct alternatives in the set A. Say $V = (v_1, v_2, \ldots, v_{m!/2})$ is an ordered set of possible linear orders where a is ranked above b. Note that $\{v_1, \bar{v}_1, v_2, \bar{v}_2, \ldots, v_{m!/2}, \bar{v}_{m!/2}\}$ is the set $\mathcal{L}(A)$ of all possible linear orders of A. We define a random vector X on a randomly selected random linear order $\mathbf{v}$ such that

$$X_i = \begin{cases} 1 & \text{if } \mathbf{v} = v_i \\ -1 & \text{if } \mathbf{v} = \bar{v}_i \\ 0 & \text{otherwise} \end{cases}$$

We likewise define an ordered set $\mathcal{X} = \{X^1, X^2, \ldots, X^n\}$, where X^i is the random vector defined on the i^{th} linear order in $\mathcal{P}$. The random vectors are independently identically distributed (i.i.d.) with means of 0, and covariance matrix $\Omega = rI$ where r is some real number greater than 0 and I is the identity matrix. By the multivariate central limit theorem, we know that $Y = \sum_{i=1}^{n} X^i/\sqrt{n}$ converges to an $N(0, rI)$ multivariate normal distribution. Hence we may easily model $Y_1, Y_2, \ldots, Y_n$ as i.i.d. univariate normally distributed variables.

Using this model we performed 100,000 simulations and generate Table 6.

Note that as the number of agents approaches infinity, the probability of a tie approaches 0, and so tie breaking is irrelevant in this table. In Table 6, we see that even with an infinite number of voters, the Simpson rule is not especially close to the Dodgson rule.

7 Conclusion

In Section 1 we discussed the history of voting theory and various important theorems. One of these theorems, the McGarvey theorem is very useful in the study of C1 voting rules. Unfortunately this theorem was of no use to us since we consider only C2 and C3 rules. In Section 2 we defined the Social Choice Functions and related concepts needed for this papers.

In Section 3 we presented a proof of a McGarvey theorem for weighted tournaments; we proved that a weighted tournament is the weighted majority relation of some society if and only if all the weights have the same parity. Although this result was probably originally proven by Debord (1987), we presented an independent proof as we did not have access to Debord's thesis. This generalisation was very useful in simplifying the proofs in Section 5. It appears that this generalisation is as useful in the study of C2 voting rules as the original McGarvey theorem is in studying C1 voting rules.

In Section 4 we proved that under the impartial culture assumption, the probability that our new approximation Dodgson Quick chose the Dodgson winner converged to 1 exponentially fast as we increase the number of agents. In Section 5 we proved that the Tideman rule also converged to the Dodgson rule, but would not converge exponentially fast unless the number of agents was even.

In Section 6 we presented numerical results demonstrating the rapid convergence of our new approximation Dodgson Quick to the true Dodgson rule, as we increase the number of agents. Although we suspected that the Simpson rule would also converge to the Dodgson rule, we showed numerically that the Simpson rule does not converge to the Dodgson rule. For a large number of alternative, the Simpson rule is a particularly poor approximation to the Dodgson rule. (Simpson's rule was not intended as an approximation to Dodgson's rule)

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