GENERALIZED CONVEXITY AND INEQUALITIES

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ABSTRACT. Several convexity properties are studied, with applications to power series, in particular to hypergeometric and related functions.

1. INTRODUCTION

In this paper we study several convexity and monotonicity properties of certain functions and deduce sharp inequalities. We deduce analogous results for certain power series, especially hypergeometric functions. This work continues studies in [ABRVV], [B1], and [B2].

The following result [HVV, Theorem 4.3], a variant of a result by Biernacki and Krzyż [BK], will be very useful in studying convexity and monotonicity of certain power series.

1.1. *Lemma.* For $0 < R \le \infty$, let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two real power series converging on the interval $(-R, R)$. If the sequence $\{a_n/b_n\}$ is increasing (decreasing), and $b_n > 0$ for all n, then the function

$$f(x) = \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n}$$

is also increasing (decreasing) on $(0, R)$. In fact, the function

$$f'(x) \left(\sum_{n=0}^{\infty} b_n x^n \right)^2$$

has positive Maclaurin coefficients.

1.2. *Notation.* If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are two power series, where $b_n > 0$ for all n, we let $T_n = T_n(f(x), g(x)) = a_n/b_n$. We will use $F = F(a, b; c; x)$ to denote the Gaussian hypergeometric function

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n) n!} x^n, \quad |x| < 1,$$

2000 *Mathematics Subject Classification.* Primary 33C05, 33C20. Secondary 26015, 26A51.

Key words and phrases. Convexity, monotonicity, power series, hypergeometric function, generalized hypergeometric series.

where (a,n) denotes the product $a(a+1)(a+2)\cdots(a+n-1)$ when $n \geq 1$, and $(a,0)=1$ if $a \neq 0$. The expression $(0,0)$ is not defined. By including more numerator and denominator parameters we can consider the *generalized hypergeometric function*

$$_pF_q(x) = {}_pF_q(a_1,\ldots,a_p;b_1,\ldots b_q;x) = 1+\sum_{n=1}^{\infty}\frac{\prod_{k=1}^{p}(a_k,n)}{\prod_{k=1}^{q}(b_k,n)}\frac{x^n}{n!}, \quad |x|<1,$$

where no denominator parameter b_k is zero or a negative integer.

Particularly interesting hypergeometric functions are the *complete elliptic integrals of the first and second kind*, defined respectively by

$$\mathcal{K}(x) = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;x^2\right), \quad \mathcal{K}'(x) = \mathcal{K}(x'),$$

and

$$\mathcal{E}(x) = \frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;x^2\right), \quad \mathcal{E}'(x) = \mathcal{E}(x'),$$

for $x \in (0,1)$.

Throughout this note, for $x \in (0,1)$ we denote $x' = \sqrt{1-x^2}$.

2. Generalized convexity

The notions of convexity and concavity of a real function of a real variable are well known [RV]. In this section we study certain generalizations of these notions for a positive function of a positive variable.

2.1. *Definition.* A function $M : (0,\infty)\times(0,\infty)\to(0,\infty)$ is called a *Mean function* if

(1) $M(x,y) = M(y,x)$,
(2) $M(x,x) = x$,
(3) $x < M(x,y) < y$, whenever $x < y$,
(4) $M(ax,ay) = aM(x,y)$ for all $a > 0$.

2.2. *Examples.* [VV]

(1) $M(x,y) = A(x,y) = (x+y)/2$ is the *Arithmetic Mean.*
(2) $M(x,y) = G(x,y) = \sqrt{xy}$ is the *Geometric Mean.*
(3) $M(x,y) = H(x,y) = 1/A(1/x,1/y)$ is the *Harmonic Mean.*
(4) $M(x,y) = L(x,y) = (x-y)/(\log x - \log y)$ for $x \neq y$, and $L(x,x) = x$, is the *Logarithmic Mean.*
(5) $M(x,y) = I(x,y) = (1/e)(x^x/y^y)^{1/(x-y)}$ for $x \neq y$, and $I(x,x) = x$, is the *Identric Mean.*

2.3. *Definition.* Let $f : I \to (0, \infty)$ be continuous, where I is a subinterval of $(0, \infty)$. Let M and N be any two Mean functions. We say f is MN-convex (concave) if $f(M(x, y)) \le (\ge)\ N(f(x), f(y))$, for all $x, y \in I$.

Note that this definition reduces to usual convexity (concavity) when $M = N = A$. We now show that for $M, N = A, G, H$, the nine possible MN-convexity (concavity) properties reduce to ordinary convexity (concavity) by a simple change of variable.

2.4. *Theorem.* Let I be an open subinterval of $(0, \infty)$ and let $f \colon I \to (0, \infty)$ be continuous. In parts $(4) - (9)$, let $I = (0, b)$, $0 < b < \infty$.

(1) f is AA-convex (concave) if and only if f is convex (concave).
(2) f is AG-convex (concave) if and only if $\log f$ is convex (concave).
(3) f is AH-convex (concave) if and only if $1/f$ is concave (convex).
(4) f is GG-convex (concave) on I if and only if $\log f(be^{-t})$ is convex (concave) on $(0, \infty)$.
(5) f is GA-convex (concave) on I if and only if $f(be^{-t})$ is convex (concave) on $(0, \infty)$.
(6) f is GH-convex (concave) on I if and only if $1/f(be^{-t})$ is concave (convex) on $(0, \infty)$.
(7) f is HA-convex (concave) on I if and only if $f(1/x)$ is convex (concave) on $(1/b, \infty)$.
(8) f is HG-convex (concave) on I if and only if $\log f(1/x)$ is convex (concave) on $(1/b, \infty)$.
(9) f is HH-convex (concave) on I if and only if $1/f(1/x)$ is concave (convex) on $(1/b, \infty)$.

Proof. (1) This follows by definition.

(2)

$$\begin{aligned} f(A(x,y)) &\le (\ge)\ G(f(x), f(y)) \\ &\iff f\left(\frac{x+y}{2}\right) \le (\ge)\ \sqrt{f(x)f(y)} \\ &\iff \log f\left(\frac{x+y}{2}\right) \le (\ge)\ \frac{1}{2}\left(\log f(x) + \log f(y)\right), \end{aligned}$$

hence the result.

(3)

$$\begin{aligned} f(A(x,y)) &\le (\ge)\ H(f(x), f(y)) \\ &\iff f\left(\frac{x+y}{2}\right) \le (\ge)\ 2/(1/f(x) + 1/f(y)) \\ &\iff 1/f\left((x+y)/2\right) \ge (\le)\ \frac{1}{2}(1/f(x) + 1/f(y)), \end{aligned}$$

hence the result.

(4) With $x = be^{-r}$ and $y = be^{-s}$,

$$f(G(x,y)) \le (\ge)\ G(f(x), f(y))$$
$$\iff \log f(be^{-(r+s)/2}) \le (\ge)\ \frac{1}{2}(\log f(be^{-r}) + \log f(be^{-s})),$$

hence the result.

(5) With $x = be^{-r}$ and $y = be^{-s}$,

$$f(G(x,y)) \le (\ge)\ A(f(x) + f(y))$$
$$\iff f\left(be^{-(r+s)/2}\right) \le (\ge)\ \frac{1}{2}\left(f(be^{-r}) + f(be^{-s})\right),$$

hence the result.

(6) With $x = be^{-r}$ and $y = be^{-s}$,

$$f(G(x,y)) \le (\ge)\ H(f(x), f(y))$$
$$\iff 1/f(be^{-(r+s)/2}) \ge (\le)\ \frac{1}{2}(1/f(be^{-r}) + 1/f(be^{-s})),$$

hence the result.

(7) Let $g(x) = f(1/x)$, and let $x, y \in (1/b, \infty)$, so that $1/x, 1/y \in (0, b)$. Then f is HA-convex (concave) on $(0, b)$ if and only if

$$f\left(\frac{2}{x+y}\right) \le (\ge)\ (1/2)(f(1/x) + f(1/y))$$
$$\iff g\left(\frac{x+y}{2}\right) \le (\ge)\ \frac{1}{2}(g(x) + g(y)),$$

hence the result.

(8) Let $g(x) = \log f(1/x)$, and let $x, y \in (1/b, \infty)$, so $1/x, 1/y \in (0, b)$. Then f is HG-convex (concave) on $(0, b)$ if and only if

$$f\left(\frac{2}{x+y}\right) \le (\ge)\ \sqrt{f(1/x)f(1/y)}$$
$$\iff \log f((2/(x+y))) \le (\ge)\ (1/2)(\log f(1/x) + \log f(1/y))$$
$$\iff g\left(\frac{x+y}{2}\right) \le (\ge)\ \frac{1}{2}(g(x) + g(y)),$$

hence the result.

(9) Let $g(x) = 1/f(1/x)$, and let $x, y \in (1/b, \infty)$, so $1/x, 1/y \in (0, b)$. Then f is HH-convex (concave) on $(0, b)$ if and only if

$$f\left(\frac{2}{x+y}\right) \le (\ge)\ 2/(1/f(1/x) + 1/f(1/y))$$
$$\iff 1/f(2/(x+y)) \ge (\le)\ \frac{1}{2}(1/f(1/x) + 1/f(1/y))$$
$$\iff g\left(\frac{x+y}{2}\right) \ge (\le)\ \frac{1}{2}(g(x) + g(y)),$$

hence the result.

□

The next result is an immediate consequence of Theorem 2.4.

2.5. *Corollary.* Let I be an open subinterval of $(0,\infty)$ and let $f\colon I \to (0,\infty)$ be differentiable. In parts $(4)-(9)$, let $I=(0,b)$, $0<b<\infty$.

(1) f is AA-convex (concave) if and only if $f'(x)$ is increasing (decreasing).
(2) f is AG-convex (concave) if and only if $f'(x)/f(x)$ is increasing (decreasing).
(3) f is AH-convex (concave) if and only if $f'(x)/f(x)^2$ is increasing (decreasing).
(4) f is GG-convex (concave) if and only if $xf'(x)/f(x)$ is increasing (decreasing).
(5) f is GA-convex (concave) if and only if $xf'(x)$ is increasing (decreasing).
(6) f is GH-convex (concave) if and only if $xf'(x)/f(x)^2$ is increasing (decreasing).
(7) f is HA-convex (concave) if and only if $x^2f'(x)$ is increasing (decreasing).
(8) f is HG-convex (concave) if and only if $x^2f'(x)/f(x)$ is increasing (decreasing).
(9) f is HH-convex (concave) if and only if $x^2f'(x)/f(x)^2$ is increasing (decreasing).

2.6. *Remark.* Since $H(x,y)\le G(x,y)\le A(x,y)$, it follows that

(1) f is AH-convex $\Longrightarrow$ f is AG-convex $\Longrightarrow$ f is AA-convex.
(2) f is GH-convex $\Longrightarrow$ f is GG-convex $\Longrightarrow$ f is GA-convex.
(3) f is HH-convex $\Longrightarrow$ f is HG-convex $\Longrightarrow$ f is HA-convex.

Further, if f is increasing (decreasing) then AN-convex (concave) implies GN-convex (concave) implies HN-convex (concave), where $N = A,G,H$. For concavity, the implications in (1), (2), and (3) are reversed. These implications are strict, as shown by the examples below.

2.7. *Examples.*

(1) $f(x)=\cosh x$ is AG-convex, but not AH-convex, nor GH-convex, nor HH-convex.
(2) $f(x)=\sinh x$ is AA-convex, but not AG-convex on $(0,\infty)$.
(3) $f(x)=\sec x$ is AH-convex on $(0,\pi/2)$.
(4) $f(x)=\tan x$ is AA-convex, but not AG-convex on $(0,\pi/2)$.

3. Applications to power series

3.1. *Theorem.* Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_n > 0$ for $n = 0, 1, 2, \ldots$, be convergent on $(-R, R)$, $0 < R < \infty$. Then the following convexity results hold.

(1) f is AA-convex, GG-convex, GA-convex, HA-convex, and HG-convex on $(0, R)$.

(2) If the sequence $\{(n+1)a_{n+1}/a_n\}$ is increasing (decreasing), then the function $f'(x)/f(x)$ is increasing (decreasing) on $(0, R)$, so that the function $\log f(x)$ is convex (concave) on $(0, R)$. In particular,
$$f\left(\frac{x+y}{2}\right) \le (\ge)\ \sqrt{f(x)f(y)}$$
for all $x, y \in (0, R)$, with equality if and only if $x = y$.

(3) Let $b_n = \sum_{k=0}^{n} a_k a_{n-k}$. If the sequence $(n+1)a_{n+1}/b_n$ is increasing (decreasing), then f is AH-convex (concave) on $(0, R)$.

(4) Let $b_n = \sum_{k=0}^{n} a_k a_{n-k}$. If the sequence na_n/b_n is increasing, then f is GH-convex and HH-convex on $(0, R)$.

(5) If the sequence $\{R(n+1)a_{n+1}/a_n - n\}$ is increasing (decreasing), then the function $(R-x)f'(x)/f(x)$ is increasing (decreasing) on $(0, R)$, so that the function $\log f(R(1-e^{-t}))$ is convex (concave) on $(0, \infty)$. In particular,
$$\sqrt{f(x)f(y)} \ge (\le)\ f(R - \sqrt{(R-x)(R-y)}\ )$$
for all $x, y \in (0, R)$, with equality if and only if $x = y$.

(6) If the sequence $\{na_n R^n\}$ is increasing (decreasing), then the function $(R-x)f'(x)$ is increasing (decreasing) on $(0, R)$, so that the function $f(R(1-e^{-t}))$ is convex (concave) as a function of t on $(0, \infty)$. In particular,
$$f(R - \sqrt{(R-x)(R-y)}\ ) \le (\ge)\ \frac{f(x)+f(y)}{2},$$
for all $x, y \in (0, R)$, with equality if and only if $x = y$.

(7) The function $xf'(x)/f(x)$ is increasing on $(0, R)$, so that the function $\log f(Re^{-t})$ is convex on $(0, \infty)$. In particular,
$$f(\sqrt{xy}) \le \sqrt{f(x)f(y)},$$
for all $x, y \in (0, R)$, with equality if and only if $x = y$.

(8) If the sequence $\{na_n R^n\}$ is increasing and if also the sequence $\{n! a_n R^n/(1/2, n)\}$ is decreasing, then the function $1/f(Re^{-t})$ is concave on $(0, \infty)$. In particular,
$$f(\sqrt{xy}) \le \frac{2f(x)f(y)}{f(x)+f(y)},$$
for all $x, y \in (0, 1)$, with equality if and only if $x = y$.

Proof.

(1) This is obvious.

(2) $(d/dx)\log f(x) = f'(x)/f(x)$, so that

$$T_n(f'(x), f(x)) = (n+1)a_{n+1}/a_n,$$

which is increasing (decreasing). Thus, by Lemma 1.1 and Corollary 2.5(2) the assertion follows.

(3) Since $Tn(f'(x), f(x)^2) = (n+1)a_{n+1}b_n$, the result follows by Lemma 1.1 and Corollary 2.5(3).

(4) Since $T_n(xf'(x), f(x)^2) = na_n/b_n$, the result follows by Lemma 1.1 and Corollary 2.5(6).

(5)

$$\frac{d}{dt}\log f(R(1-e^{-t})) = Re^{-t}\frac{f'(R(1-e^{-t}))}{f(R(1-e^{-t}))} = (R-x)\frac{f'(x)}{f(x)},$$

where $x = R(1-e^{-t})$. Then

$$T_n((R-x)f'(x), f(x)) = R(n+1)a_{n+1}/a_n - n,$$

which is increasing (decreasing), so that the assertion follows by Lemma 1.1.

(6) $(d/dt)f(R(1-e^{-t})) = Re^{-t}f'(R(1-e^{-t})) = (R-x)f'(x) = f'(x)/(1/(R-x))$, where $x = R(1-e^{-t})$. Then,

$$T_n(f'(x), 1/(R-x)) = (n+1)a_{n+1}R^{n+1},$$

which is increasing (decreasing) by hypothesis, so that the assertion follows by Lemma 1.1.

(7) First,

$$\frac{d}{dt}\log f(Re^{-t}) = -Re^{-t}\frac{f'(Re^{-t})}{f(Re^{-t})} = -x\frac{f'(x)}{f(x)} = -h(x),$$

say, with $x = Re^{-t}$. Next, $T_n(xf'(x), f(x)) = n$, which is trivially increasing. Thus $h(x)$ is increasing in x on $(0,1)$, and $(d/dt)\log f(Re^{-t})$ is increasing in t on $(0,\infty)$. Thus, $\log f(Re^{-t})$ is strictly convex in t on $(0,\infty)$. In particular,

$$\log f\left(Re^{-(r+s)/2}\right) \le \frac{1}{2}\left(\log f(Re^{-r}) + \log f(Re^{-s})\right),$$

for all $r, s \in (0,\infty)$, with equality if and only if $r = s$. If we set $x = Re^{-r}$, $y = Re^{-s}$, this simplifies to

$$f(\sqrt{xy}) \le \sqrt{f(x)f(y)},$$

for all $x, y \in (0, R)$, with equality if and only if $x = y$.

(8)

$$\frac{d}{dt}\frac{1}{f(Re^{-t})} = \frac{Re^{-t}f'(Re^{-t})}{(f(Re^{-t}))^2} = \frac{xf'(x)}{(f(x))^2},$$

where $x = Re^{-t}$. Now $(R-x)f'(x) = f'(x)/[1/(R-x)]$, so that

$$T_n(f'(x), 1/(R-x)) = (n+1)a_{n+1}R^{n-1},$$

which is increasing by hypothesis. Hence, $(R-x)f'(x)$ is increasing on $(0, R)$, by Lemma 1.1. Since

$$\sqrt{R-x}f(x) = f(x)/(R-x)^{-1/2},$$

we have

$$T_n(f(x), (R-x)^{-1/2}) = \frac{n! a_n R^{n+1/2}}{(1/2, n)},$$

which is decreasing by hypothesis. Hence, $\sqrt{R-x}f(x)$ is also decreasing on $(0, R)$, by Lemma 1.1. Thus $(d/dt)[1/f(R(e^{-t}))]$ is decreasing on $(0, \infty)$, proving the required assertion.

□

3.2. *Theorem.* Let $F(x) = F(a, b; c; x)$, for $a, b, c > 0$ and $|x| < 1$. Then the following results hold.

(1) If $ab/(a+b+1) < c$, then $\log F(x)$ is convex on $(0, 1)$. In particular, $F((x+y)/2) \le \sqrt{F(x)F(y)}$, for all $x, y \in (0, 1)$, with equality if and only if $x = y$.
(2) If $(a-c)(b-c) > 0$, then $\log F(1-e^{-t})$ is concave on $(0, \infty)$. In particular, $\sqrt{F(x)F(y)} \le F(1 - \sqrt{(1-x)(1-y)})$, for all $x, y \in (0, 1)$, with equality if and only if $x = y$.
(3) If $a + b \ge c$, then $F(1-e^{-t})$ is convex on $(0, \infty)$. In particular, $F(1 - \sqrt{(1-x)(1-y)}) \le (F(x) + F(y))/2$ for all $x, y \in (0, 1)$, with equality iff $x = y$.

Proof. (1) $T_n(F'(x), F(x)) = (n+1)a_{n+1}/a_n = (a+n)(b+n)/(c+n)$. Hence,

$$\begin{aligned}
T_{n+1} - T_n > 0 &\iff (a+n+1)(b+n+1)(c+n) \\
&\qquad - (a+n)(b+n)(c+n+1) > 0 \\
&\iff (a+n)(c+n) + (b+n)(c+n) + (c+n) \\
&\qquad - (a+n)(b+n) > 0 \\
&\iff n^2 + n(2c+1) + (ac+bc+c-ab) > 0 \\
&\iff ab/(a+b+1) < c.
\end{aligned}$$

Hence, the assertion follows from Theorem 3.1(1).

(2)

$$\begin{aligned} T_n((1-x)F'(x), F(x)) &= \frac{(n+1)a_{n+1}}{a_n} - n \\ &= \frac{(a+n)(b+n)}{c+n} - n \\ &= a+b-c+\frac{(a-c)(b-c)}{c+n}, \end{aligned}$$

which is decreasing if and only if $(a-c)(b-c) > 0$, so that the assertion follows from Theorem 3.1(2).

(3) $T_{n-1}(F'(x), 1/(1-x)) = na_n = (a,n)(b,n)/[(c,n)(n-1)!]$. Hence,

$$\begin{aligned} T_n - T_{n-1} &= \frac{(a,n+1)(b,n+1)}{(c,n+1)n!} - \frac{(a,n)(b,n)}{(c,n)(n-1)!} \\ &= \frac{(a,n)(b,n)}{(c,n+1)n!}[(a+n)(b+n) - n(c+n)] > 0 \\ &\iff n(a+b-c) + ab > 0 \text{ for all } n \\ &\iff a+b \geq c. \end{aligned}$$

Hence, the assertion follows from Theorem 3.1(3).

□

3.3. *Theorem.* (cf. [BPV, Lemma 2.1]). Let $F(x) = F(a,b;c;x)$, with $c = a+b, a, b > 0$, and $|x| < 1$. Then

(1) $\log F(x)$ is convex on $(0,1)$,
(2) $\log F(1-e^{-t})$ is concave on $(0,\infty)$,
(3) $F(1-e^{-t})$ is convex on $(0,\infty)$.

In particular,

$$F\left(\frac{x+y}{2}\right) \leq \sqrt{F(x)F(y)} \leq F(1-\sqrt{(1-x)(1-y)}) \leq \frac{F(x)+F(y)}{2}$$

for all $x, y \in (0,1)$, with equality if and only if $x = y$.

Proof. This result follows immediately from Theorem 3.2. □

3.4. *Theorem.* Let $a, b, c > 0$, $a, b \in (0,1)$ and $a < c, b < c$. Let F and F_1 be the conjugate hypergeometric functions on $(0,1)$ defined by $F = F(x) = F(a,b;c;x)$ and $F_1 = F_1(x) = F(1-x)$. Then the function f defined by $f(x) = x(1-x)F(x)F_1(x)$ is increasing on $(0,1/2]$ and decreasing on $[1/2,1)$.

Proof. Since $f(x) = f(1-x)$, it is enough to prove the assertion on $(0, 1/2]$. Following Rainville [R, p. 51] we let $F(a-) = F(a-1, b; c; x)$ and $F_1(a-) = F_1(a-1, b; c; x)$. Now, since

$$x(1-x)F'(x) = (c-a)F(a-) + (a-c+bx)F$$

[AQVV, Theorem 3.12(2)], we have

$$\begin{aligned} f'(x) &= x(1-x)[F'(x)F_1(x) + F(x)F_1'(x)] + (1-2x)F(x)F_1(x) \\ &= [(c-a)F(a-)F_1 - (c-a-bx)FF_1] \\ &\quad - [(c-a)F_1(a-)F - (c-a-b(1-x))F_1F] \\ &\quad + (1-2x)FF_1 \\ &= (c-a)[F(a-)F_1 - F_1(a-)F] + (1-2x)(1-b)FF_1. \end{aligned}$$

Since $F_1(a-)F$ is increasing on $(0, 1)$, it follows that $f'(x)$ is positive on $(0, 1/2)$ and negative on $(1/2, 1)$. □

3.5. *Corollary.* Let $f(x) = x^2 x'^2 \mathcal{K}(x)\mathcal{K}(x')$. Then $f(x)$ is increasing on $(0, 1/\sqrt{2}]$ and decreasing on $[\sqrt{1/2}, 1)$, with

$$\max_{0<x<1} f(x) = f\left(\sqrt{1/2}\right) = \frac{1}{4}\left(\mathcal{K}\left(\sqrt{1/2}\right)\right)^2 = 0.859398\ldots.$$

Proof. This follows from Theorem 3.4, if we take $a = b = 1/2, c = 1$, and replace x by x^2. □

3.6. *Theorem.* Let $F(x) = F(a, b; c; x)$, with $a, b, c > 0$ and $|x| < 1$. If $a+b \geq c \geq 2ab$ and $c > a+b-1/2$, then $1/F(e^{-t})$ is concave on $(0, \infty)$. In particular,

$$F(\sqrt{xy}) \leq \frac{2F(x)F(y)}{F(x) + F(y)},$$

for all $x, y \in (0, 1)$, with equality if and only if $x = y$.

Proof. Here a_n, the coefficient of x^n, is $(a, n)(b, n)/[(c, n)n!]$, so that

$$na_n = (a, n)(b, n)/[(c, n)(n-1)!].$$

This is increasing if and only if $n(a + b - c) + ab > 0$, which is true if $a + b \geq c$. Next,

$$T_n(f(x), 1/\sqrt{1-x}) = \frac{n!a_n}{(1/2, n)} = \frac{2^n(a, n)(b, n)}{(c, n)\cdot 1\cdot 3\cdots(2n-1)},$$

which is decreasing if and only if $2n(a + b - c - 1/2) + (2ab - c) < 0$, which is satisfied if $a + b - 1/2 < c$ and $2ab \leq c$. Hence, the assertion follows from Theorem 3.1(e). □

3.7. *Theorem.* (cf. [B2, (1.12) and Remark 1.13] Let $F(x)$ denote the hypergeometric function $F(a,b;a+b;x)$, with $a,b\in(0,1]$ and $|x|<1$. Then $1/F(e^{-t})$ is concave on $(0,\infty)$. In particular,

$$F(\sqrt{xy})\leq\frac{2F(x)F(y)}{F(x)+F(y)},$$

for all $x,y\in(0,1)$, with equality if and only if $x=y$.

Proof. In this case $c=a+b$, and $c-2ab=a(1-b)+b(1-a)\geq 0$, so that the assertion follows from Theorem 3.6. □

The next result improves [B2, Theorem 1.25].

3.8. *Theorem.* Let $f(x)=\sum_{n=0}^{\infty}b_nx^n$, where $b_n=(-c/4)^n/[n!(k,n)]$, $k=p+(b+1)/2$, as in [B2], be the generalized-normalized Bessel function of the first kind of order p. Let $c<0$ and $k>0$. Then

(1) $\log f(Re^{-t}))$ is convex on $(0,\infty)$, so that $f(\sqrt{xy})\leq\sqrt{f(x)f(y)}$ for all $x,y\in(0,\infty)$, with equality if and only if $x=y$.
(2) $\log f(x)$ is concave on $(0,\infty)$, so that $\sqrt{f(x)f(y)}\leq f((x+y)/2)$ for all $x,y\in(0,\infty)$, with equality if and only if $x=y$.
(3) $f(x)$ is convex on $(0,\infty)$, so that

$$f((x+y)/2)\leq(1/2)(f(x)+f(y))$$

for all $x,y\in(0,\infty)$, with equality if and only if $x=y$.
(4) If $k>-1-cR/4$, then $f(R(1-e^{-t}))$ is concave on $(0,\infty)$, so that $(f(x)+f(y))/2\leq f(R-\sqrt{(R-x)(R-y)})$ for all $x,y\in(0,R)$, with equality if and only if $x=y$.

Proof. By the ratio test, the radius of convergence of the series for $f(x)$ is ∞.

(1) This follows from Theorem 3.1(4).
(2) $T_n(f'(x),f(x))=(n+1)b_{n+1}/b_n=(-c/4)/(k+n)$, which is decreasing; hence the result follows from Theorem 3.1(1).
(3) This is obvious, since $b_n>0$ for all n.
(4) Since

$$T_n(f'(x),1/(R-x))=(n+1)b_{n+1}R^{n+1}=(-cR/4)^{n+1}/[n!(k,n+1)]$$

and $k>-1-cR/4$, we have

$$\frac{T_{n+1}}{T_n}=(-cR/4)/[(n+1)(k+n+1)]<1.$$

Hence, the result follows from Theorem 3.1(3).

□

3.9. *Remark.* For $0 < x < y$, let $y/x = \exp(2\sqrt{t})$, $t \in (0, \infty)$, and let

$$G(x,y) = \sqrt{xy},\ L(x,y) = \frac{y-x}{\log y - \log x},\ \text{and } A(x,y) = \frac{x+y}{2},$$

denote the *Geometric Mean*, *Logarithmic Mean*, and *Arithmetic Mean* of x and y, respectively. Then

$$\frac{L(x,y)}{G(x,y)} = \frac{\sinh(\sqrt{t})}{\sqrt{t}} = \sum_{n=0}^{\infty} \frac{t^n}{(2n+1)!},$$

$$\frac{A(x,y)}{G(x,y)} = \cosh(\sqrt{t}) = \sum_{n=0}^{\infty} \frac{t^n}{(2n)!}.$$

Hence, it will be interesting to study the convexity properties of these two functions.

3.10. *Corollary.* (cf. [B2, Corollary 1.26])

(1)

$$\begin{aligned} \cosh(\sqrt{xy}) &\le \sqrt{(\cosh x)(\cosh y)} \\ &\le \cosh\left(\sqrt{(x^2+y^2)/2}\right) \\ &\le \frac{1}{2}(\cosh x + \cosh y) \end{aligned}$$

for all $x, y \in (0, \infty)$, with equality if and only if $x = y$.

(2)

$$\frac{1}{2}(\cosh x + \cosh y) \le \cosh\left(\sqrt{R - \sqrt{(R-x^2)(R-y^2)}}\right)$$

for $0 < R < 6$ and all $x, y \in (0, \sqrt{R})$, with equality if and only if $x = y$.

(3)

$$\begin{aligned} \frac{\sinh(\sqrt{xy})}{\sqrt{xy}} &\le \sqrt{\frac{\sinh x}{x} \cdot \frac{\sinh y}{y}} \\ &\le \frac{\sinh\sqrt{\frac{1}{2}(x^2+y^2)}}{\sqrt{\frac{1}{2}(x^2+y^2)}} \\ &\le \frac{1}{2}\left(\frac{\sinh x}{x} + \frac{\sinh y}{y}\right) \end{aligned}$$

for all $x, y \in (0, \infty)$, with equality if and only if $x = y$.

(4)

$$\frac{1}{2}\left(\frac{\sinh x}{x}+\frac{\sinh y}{y}\right) \leq \frac{\sinh\left(\sqrt{R-\sqrt{(R-x^2)(R-y^2)}}\right)}{\sqrt{R-\sqrt{(R-x^2)(R-y^2)}}}$$

for $0 < R < 10$ and all $x, y \in (0, \sqrt{R})$, with equality if and only if $x = y$.

Proof. (1) In Theorem 3.8, let $b = 1$, $c = -1$, and $p = -1/2$. Then $f(x^2) = \cosh x$, and the result follows from Theorem 3.8(1),(2),(3) if we replace x and y by x^2 and y^2, respectively.

(2) In Theorem 3.8, take $b = 1$, $c = -1$, and $p = -1/2$. Then $T_n(f'(x), 1/(R-x)) = (n+1)a_{n+1}R^{n+1}$, so

$$\frac{T_{n+1}}{T_n} = \frac{(n+2)R(2n+2)!}{(n+1)(2n+4)!} = \frac{R}{2(n+1)(2n+3)},$$

which is less than 1 if and only if $R < \min_{n\geq 0} 2(n+1)(2n+3) = 6$. So the result follows from Theorem 3.8(4) if we replace x and y by x^2 and y^2, respectively.

(3) In Theorem 3.8, take $b = 1$, $c = -1$, and $p = 1/2$. Then $f(x^2) = (\sinh x)/x$, and the result again follows from Theorem 3.8(1),(2),(3), if we replace x and y by x^2 and y^2, respectively.

(4) In Theorem 3.8, take $b = 1$, $c = -1$, and $p = 1/2$. Then, with $T_n(f'(x), 1/(R-x)) = (n+1)a_{n+1}R^{n+1}$, we have $T_{n+1}/T_n = R/[2(n+1)(2n+5)]$, which is less than 1 for $n \geq 0$ if and only if $R < \min_{n\geq 0} 2(n+1)(2n+5) = 10$. So the result follows from Theorem 3.8(4) if we replace x and y by x^2 and y^2, respectively. □

3.11. *Remark.* Computer experiments show that in Corollary 3.10(2) the bound $R < 6$ cannot be replaced by $R < 7$ and that in Corollary 3.10(4) the bound $R < 10$ cannot be replaced by $R < 11$.

3.12. *Theorem.* For $0 < R < \infty$, let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $a_n > 0$, be convergent on $(-R, R)$. Let $m = m_f$ be the function defined by $m(x) = f(R - x^2/R)/f(x^2/R)$. If the sequence $\{R(n+1)a_{n+1}/a_n - n\}$ is decreasing, then

$$\frac{1}{m(\sqrt[4]{(R^2-x^2)(R^2-y^2)})} \leq \sqrt{m(x)m(y)} \leq m(\sqrt{xy})$$

for all $x, y \in (0, R)$, with equality if and only if $x = y$.

Proof. By Theorem 3.1(2),(4), we have

$$f(\sqrt{xy}) \le \sqrt{f(x)f(y)} \le f(R - \sqrt{(R-x)(R-y)})$$

for all $x, y \in (0, R)$, with equality if and only if $x = y$. If we change x, y to (i) x^2/R, y^2/R and (ii) $R - x^2/R$, $R - y^2/R$, respectively, then

$$\begin{aligned} f(xy/R) &\le \sqrt{f(x^2/R)f(y^2/R)} \\ &\le f(R - \sqrt{(R - x^2/R)(R - y^2/R)}) \end{aligned}$$

and

$$\begin{aligned} f(\sqrt{(R - x^2/R)(R - y^2/R)}) &\le \sqrt{f(R - x^2/R)f(R - y^2/R)} \\ &\le f(R - xy/R), \end{aligned}$$

for all $x, y \in (0, R)$, with equality if and only if $x = y$. The result follows if we divide the second chain of inequalities by the first. $\square$

We may extend some of the previous results on log-convexity to the generalized hypergeometric function. For non-negative integers p, q, let $a_1, \ldots a_p, b_1, \ldots, b_q$ be positive numbers and let $F(x) = {}_pF_q(x)$ be the generalized hypergeometric function defined on $(-1, 1)$ as in Section 1.2 (Notations).

3.13. *Theorem.* (1) If $p = q = 0$, then $F(x) = e^x$, which is trivially log-convex.

(2) Let $p = q \ge 1$. If $a_k \le b_k$ for each k, with at least one strict inequality, then F is strictly log-convex on $(0, 1)$. If $a_k \ge b_k$, with at least one strict inequality, then F is strictly log-concave on $(0, 1)$.

(3) If $p > q$ and $a_k \le b_k$, with at least one strict inequality, for $k = 1, 2, \ldots, q$, then F is strictly log-convex on $(0, 1)$.

(4) If $1 \le p < q$ and $a_k \ge b_k$, with at least one strict inequality, for $k = 1, 2, \ldots, p$, then F is strictly log-concave on $(0, 1)$.

(5) If $p = 0$, and $q \ge 1$, then F is log-concave.

Proof. For (1), $F(x) = \sum_{n=0}^{\infty} x^n/n! = e^x$, hence the result.

In case (2),

$$T_n(F'(x), F(x)) = \frac{B}{A}\frac{a_1 + n}{b_1 + n} \cdots \frac{a_p + n}{b_p + n},$$

where $A = a_1 \cdots a_p$ and $B = b_1 \cdots b_p$. Clearly, a ratio of the form $(a + n)/(b + n)$ is increasing or decreasing in n according as $a < b$ or $a > b$. Hence, $F'(x)/F(x)$ is increasing or decreasing as asserted, and the result follows.

(3) As in case (2), if $p > q \ge 1$, each ratio of the form $(a_k+n)/(b_k+n)$ is increasing, with at least one strictly, hence so is $F'(x)/F(x)$. Next, if $q = 0$ and $p > 0$, then $T_n(F'(x), F(x)) = (a_1+n)(a_2+n) \cdots (a_p+n)$,

which is clearly increasing, so that $F'(x)/F(x)$ is also increasing on $(0,1)$. Thus F is log-convex.

(4) As in case (2), each ratio of the form $(a_k+n)/(b_k+n)$ is decreasing, with at least one strictly, hence so is $F'(x)/F(x)$.

(5) Here, $T_n(F'(x), F(x)) = 1/[(n+b_1)(n+b_2)...(n+b_q)]$, which is clearly decreasing. □

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