HOLOMORPHIC FORMS OF GEOMETRICALLY FORMAL KÄHLER MANIFOLDS

J.-F. GROSJEAN AND P.-A. NAGY

Abstract. We investigate harmonic forms of geometrically formal metrics, which are defined as those having the exterior product of any two harmonic forms still harmonic. We prove that a formal Sasakian metric can exist only on a real cohomology sphere and that holomorphic forms of a formal Kähler metric are parallel w.r.t. the Levi-Civita connection. In the general Riemannian case a formal metric with maximal second Betti number is shown to be flat . Finally we prove that a six-dimensional manifold with $b_1 \neq 1, b_2 \geq 3$ and not having the cohomology algebra of $\mathbb{T}^3 \times S^3$ carries a symplectic structure as soon as it admits a formal metric.

Contents

1. Introduction

Let (M^n, g) be a compact oriented Riemannian manifold. We denote by $\Lambda^p(M), 0 \leq p \leq n$ the space of smooth, real-valued, differential p-forms of M. We have then a differential complex

$$\ldots \rightarrow \Lambda^p(M) \xrightarrow{d} \Lambda^{p+1}(M) \ldots$$

where d is the exterior derivative. The p-th cohomology group of this complex, known as the p-th deRham cohomology group will be denoted by $H^p_{DR}(M)$. Now, the Riemannian metric g induces a scalar product at the level of differential forms, hence one can consider also the operator $d^\star$, the formal adjoint of d. For $0 \leq p \leq n$ we define the space of harmonic p-forms by setting

$$\mathcal{H}^p(M, g) = \{\alpha \in \Lambda^p(M) : \Delta\alpha = 0\}.$$

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Here the Laplacian Δ is defined by

$$\Delta = dd^\star + d^\star d.$$

Classical Hodge theory produces an isomorphism

$$H^p_{DR}(M) \equiv \mathcal{H}^p(M, g) \tag{1.1}$$

for all $0 \le p \le n$. Whilst $H^\star(M) = \bigoplus\limits_{p \geqslant 0} H^p_{DR}(M)$ is a graded algebra, generally $\mathcal{H}^\star = \bigoplus\limits_{p \geqslant 0} \mathcal{H}^p(M, g)$ is not an algebra with respect to the wedge product operation for there is no reason the isomorphism 1.1 descends to the level of harmonic forms. Our next definition is related to this fact.

Definition 1.1. *Let (M^n, g) be a compact and oriented Riemannian manifold.*

(i) *The metric g is p-formal for some $1 \le p \le n-1$ if the product of any harmonic p-forms remains harmonic.*
(ii) *The metric g is formal if it is p-formal for all $1 \le p \le n-1$.*

A closely related notion is that of topological formality (see [2] for instance), which implies that the rational homotopy type of the manifold is a formal consequence of its cohomology ring [13]. From the existence of a formal metric it follows that the underlying manifold is topologically formal, and this provides obstructions to the existence to such metrics; for instance they cannot exist on nilmanifolds since those have non-trivial Massey products, fact which is in itself an obstruction to formality [2, 16]. On the other hand, simply connected, compact manifolds of dimension not execeding 6 are topologically formal [7, 9].
Now the existence of formal metrics is more directly related to the geometry of the ambient manifold and known obstructions are related to the length of harmonic forms.

Theorem 1.1. [6]*Let (M^n, g) be compact and oriented such that g is a formal metric.*

(i) *The inner product of any two harmonic forms is a constant function.*
(ii) $b_p(M) \le \binom{n}{p}$ *for all $1 \le p \le n$.*
(iii) *If in (ii) equality occurs for $p = 1$ then g is a flat metric.*

Standard examples of formal metrics are provided by compact symmetric spaces for in this case all harmonic forms must be parallel with respect to the Levi-Civita connection. D. Kotschik proved that in dimension 4 every geometrically formal manifold has the cohomology algebra of a compact symmetric space. One of the current questions related to the notion of geometric formality is then to examine up to what extent this is true in general.
In this paper we shall be mainly concerned with making obstructions to the existence of a geometrically formal, Kähler metric. Here the topological formality is no longer an issue since any Kähler manifold is known to have this property [2]. At first, we show that, in the Kähler context, the p-formality assumption on the metric is enough to imply the constancy of the length of harmonic p-forms.

Theorem 1.2. *Let (M^{2n}, g) be a compact Kähler manifold.*

(i) *If g is p-formal, then the inner product of any two harmonic p-forms is a constant function.*
(ii) *If g is $2p$-formal then it is $2r$-formal for any $r \leq p$.*
(iii) *g is formal if and only if it is $n-1$ and n-formal.*

When trying to investigate similar issues for Kähler related, odd dimensional geometries such as Sasakian geometry it turns out that the whole cohomology algebra can be described. We prove

Theorem 1.3. *Let (M^{2n+1}, g) be a compact Sasakian manifold. If g is a formal metric then M is a real cohomology sphere.*

Next we study properties of holomorphic forms of a formal Kähler metric and obtain

Theorem 1.4. *Let (M^{2n}, g, J) be a Kähler manifold which is geometrically formal (or only p-formal). Then every harmonic form Ω of real type $(p, 0) + (0, p)$ (hence every holomorphic p-form) is parallel with respect to the Levi-Civita connection. Moreover Ω induces in a canonical way a local splitting of M as the Riemannian product of two compact Kähler manifolds M_1 and M_2 so that Ω is zero on M_1, non-degenerate on M_2 which is Ricci flat.*

Remark 1.1. *(i) Theorem 1.4 was already proved in* [8] *for $p = 2$, using arguments relying heavily on the algebraic structure of the space of harmonic 2-forms. For higher degree forms, such results are no longer available.*
(ii) If in Theorem 1.4 we furthermore assume the metric being locally irreducible and not symmetric, it follows from Berger's holonomy classification theorems (see [12]*) that the only cases when we can have a non-vanishing holomorphic form are when $Hol(g) = Sp(m)(n = 2m)$ or $Hol(g) = SU(n)$.*
(iii) From the above it also follows that if M admits a locally irreducible Kähler and formal metric which is not Ricci flat then $Td(M) = 1$.

In the second part of the paper we are concerned with giving a characterisation of geometrically formal Riemannian manifold with maximal second Betti number. We prove :

Theorem 1.5. *Let (M^n, g) be a geometrically formal Riemannian manifold with $n \geqslant 3$. If $b_2(M)$ is maximal, that is equal to $b_2(M) = \binom{n}{2}$, then the metric g is flat.*

This clarifies the equality case in Theorem 1.1, (iii) for degree 2-forms. Note that the assertion in Theorem 1.5 is straightforward when n is odd for if $n = 2k + 1$ the formality and the maximality of b_2 imply that b_{2k} is maximal. Hodge duality implies then the maximality of b_1 and hence the flatness of the metric. When n is even, our point of departure consists in observing that the metric must admit a compatible almost Kähler structure and then work out this situation within the same circle of arguments which have led to proving Theorem 1.4. Along the way, as a necessary ingredient to our proof, we show that any harmonic 2-form w.r.t. a formal metric diagonalises with constant eigenvalues and constant rank eigendistributions. This is extending results from [8] to the general Riemannian case and can also be used to give sufficient conditions, essentially phrased in terms of Betti numbers lower bounds, for a formal metric to admit a compatible symplectic form, at least in dimension 6.

Theorem 1.6. *Let (M^6, g) be geometrically formal. If $b_1(M) \neq 1$ and $b_2(M) \geq 3$ and moreover M has not the real cohomology algebra of $\mathbb{T}^3 \times S^3$ then M carries a g-compatible symplectic form.*

It would be interesting to see have results similar to Theorem 1.6 in arbitrary even dimensions and of course to give necessary but also sufficient conditions for a geometrically formal metric to admit a compatible symplectic structure. In doing so, the difficulties one faces are related to understanding, at the algebraic level, the constraints imposed by geometric formality on forms of degree ≥ 3.

2. Length of harmonic forms

Let (M^{2n}, g, J) be a compact oriented Kählerian manifold of dimension $2n$. Let $L : \Lambda^\star(M) \to \Lambda^\star(M)$ be the exterior multiplication with the Kähler form $\omega = g(J\cdot, \cdot)$. Recall that the space of primitive forms, $\Lambda_0^\star(M)$, is given as the kernel of $L^\star$, the adjoint of L, w.r.t. the metric g. Moreover let us extend the complex structure to $J : \Lambda^\star(M) \to \Lambda^\star(M)$ by setting

$$(J\alpha)(X_1, \ldots, X_p) = \alpha(JX_1, \ldots, JX_p)$$

for all α in $\Lambda^p(M)$ and $X_1, \ldots, X_p$ in TM. Now, let us consider the operators $P_k : \Lambda^r(M) \times \Lambda^s(M) \to \Lambda^{r+s-2k}(M)$ defined by

$$P_k(\alpha, \beta) := \sum_{i_1 \ldots i_k} (e_{i_1} \lrcorner \ldots e_{i_k} \lrcorner \alpha) \wedge (Je_{i_1} \lrcorner \ldots Je_{i_k} \lrcorner \beta)$$

Proposition 2.1. *For any $\alpha \in \Lambda^r(M)$ and $\beta \in \Lambda^s(M)$, we have*

(i) $L^\star(\alpha \wedge \beta) = L^\star\alpha \wedge \beta + \alpha \wedge L^\star\beta + (-1)^{r-1} P_1(\alpha, \beta)$.
(ii) $L^\star P_k(\alpha, \beta) = P_k(L^\star\alpha, \beta) + P_k(\alpha, L^\star\beta) + (-1)^{r-k-1} P_{k+1}(\alpha, \beta)$.
(iii) $(L^\star)^p(\alpha \wedge \beta) = (-1)^{\frac{p(p-1)}{2}} p! < \alpha, J\beta >$ *for any primitive p-forms α and β.*

Proof. Let $\alpha \in \Lambda^r(M)$ and $\beta \in \Lambda^s(M)$. Then

$$\begin{aligned} L^\star(\alpha \wedge \beta) &= \frac{1}{2} \sum_i Je_i \lrcorner e_i \lrcorner (\alpha \wedge \beta) \\ &= \frac{1}{2} \sum_i Je_i \lrcorner ((e_i \lrcorner \alpha) \wedge \beta) + \frac{1}{2}(-1)^r \sum_i Je_i \lrcorner (\alpha \wedge (e_i \lrcorner \beta)) \\ &= L^\star\alpha \wedge \beta + \frac{1}{2}(-1)^{r-1} \sum_i (e_i \lrcorner \alpha) \wedge (Je_i \lrcorner \beta) \\ &+ \frac{1}{2}(-1)^r \sum_i (Je_i \lrcorner \alpha) \wedge (e_i \lrcorner \beta) + \alpha \wedge L^\star\beta \\ &= L^\star\alpha \wedge \beta + \alpha \wedge L^\star\beta + (-1)^{r-1} P_1(\alpha, \beta). \end{aligned}$$

This completes the proof of (i). Now

$$\begin{aligned}
L^\star P_k(\alpha,\beta) &= \frac{1}{2}\sum_{ii_1\dots i_k} Je_i\lrcorner e_i\lrcorner((e_{i_1}\lrcorner\dots e_{i_k}\lrcorner\alpha)\wedge(Je_{i_1}\lrcorner\dots Je_{i_k}\lrcorner\beta))\\
&= \frac{1}{2}\sum_{ii_1\dots i_k} Je_i\lrcorner((e_i\lrcorner e_{i_1}\lrcorner\dots e_{i_k}\lrcorner\alpha)\wedge(Je_{i_1}\lrcorner\dots Je_{i_k}\lrcorner\beta))\\
&+ \frac{1}{2}(-1)^{r-k}\sum_{ii_1\dots i_k} Je_i\lrcorner((e_{i_1}\lrcorner\dots e_{i_k}\lrcorner\alpha)\wedge(e_i\lrcorner Je_{i_1}\lrcorner\dots Je_{i_k}\lrcorner\beta))\\
&= P_k(L^\star\alpha,\beta)+\frac{1}{2}(-1)^{r-k-1}\sum_{i_1\dots i_{k+1}}(e_{i_1}\lrcorner\dots e_{i_{k+1}}\lrcorner\alpha)\wedge(Je_{i_1}\lrcorner\dots Je_{i_{k+1}}\lrcorner\beta))\\
&+ \frac{1}{2}(-1)^{r-k}\sum_{i_1\dots i_{k+1}}(Je_{i_1}\lrcorner e_{i_2}\lrcorner\dots e_{i_{k+1}}\lrcorner\alpha)\wedge(e_{i_1}\lrcorner Je_{i_2}\lrcorner\dots Je_{i_{k+1}}\lrcorner\beta))+P_k(\alpha,L^\star\beta)
\end{aligned}$$

and the assertion (ii) is proved. To prove (iii) we first obtain by induction from (ii) that $L^\star)^p(\alpha\wedge\beta)=(-1)^{\frac{p(p-1)}{2}}P_p(\alpha,\beta)$. To finish the proof it is enough to directly use the definition of P_p to get $P_p(\alpha,\beta)=p!<\alpha,J\beta>$. □

Lemma 2.1. *Let (M^{2n},g,J) be a $2p$-formal Kähler manifold. For any $\alpha\in\mathcal{H}^{2r}(M)$, $\beta\in\mathcal{H}^{2s}(M)$ with $r,s\le p$ and for any $k\le\min(2r,2s)$, $P_k(\alpha,\beta)$ is harmonic. More generally, if (M^{2n},g,J) is formal then for any $\alpha\in\mathcal{H}^p(M)$ and $\beta\in\mathcal{H}^q(M)$ and any $k\leqslant\min(p,q)$, $P_k(\alpha,\beta)$ is harmonic.*

Proof. Let $\alpha\in\mathcal{H}^{2r}(M)$ and $\beta\in\mathcal{H}^{2s}(M)$. By the relation 2. of the proposition 2.1, it is obvious that $P_1(\alpha,\beta)$ is harmonic. By induction, if we assume that for $k\le 1$, $P_k(\alpha,\beta)$ is harmonic for any $\alpha\in\mathcal{H}^{2r}(M)$ and $\beta\in\mathcal{H}^{2s}(M)$ with $r,s\le p$, the relation 3. of the proposition 2.1 says that $P_{k+1}(\alpha,\beta)$ is harmonic. The proof is the same for the second assertion. □

Proof of Theorem 1.2 (i) It is enough to work only with primitive forms in virtue of the Lefschetz decomposition of $\Lambda^\star(M)$, (see [5]) which basically says that any form on M can be manufactured out of primitive forms. If α be harmonic in $\Lambda_0^p(M)$, then $J\alpha$ is still harmonic hence by hypothesis $\alpha\wedge J\alpha$ is still harmonic. Since g is a Kähler metric $(L^\star)^p(\alpha\wedge J\alpha)$ is harmonic too, hence a constant function. The proof now follows by Proposition 2.1, (iii), given that $J^2=(-1)^p id$ on p-forms. □

Part of the algebraic facts developed above can be also used to describe completely the cohomology algebra of a geometrically formal, Sasakian metric. For an introduction to Sasakian geometry, the odd dimensional analogue of Kähler geometry, we refer the reader to [4].

Theorem 2.1. *Let (M^{2n+1},g) be a Sasakian manifold. If the metric g is formal then $b_p(M)=0$ for all $1\le p\le 2n+1$, in other words M is a homology sphere.*

Proof. Recall that the tangent bundle of M splits as $TM=\mathcal{V}\oplus H$ an orthogonal direct sum where $\mathcal{V}$ is spanned by the so-called Reeb vector field, to be denoted by ζ. the contact distribution H admits a g-compatible complex structure $J:H\to H$

which moreover satisfies $d\theta = \omega$ where is the 1-form dual to ζ and $\omega = g(J\cdot, \cdot)$. We call a differential p-form horizontal, and denote the corresponding space by $\Lambda^p(H)$ if the interior product with ζ vanishes. Now let $d_H : \Lambda^\star(H) \to \Lambda^\star(H)$ be the projection of the usual exterior derivative d onto H. If $d_H^\star$ is its formal adjoint w.r.t. to the restriction of g on H, we have on $\Lambda^p(M) = \Lambda^p(H) \oplus \left[\Lambda^{p-1}(H) \wedge \theta\right]$

$$d^\star = \begin{pmatrix} d_H^\star & (-1)^p \mathcal{L}_\zeta \\ (-1)^p L^\star & \mathcal{L}_\zeta \end{pmatrix} \tag{2.1}$$

where $\mathcal{L}_\zeta$ denotes the Lie derivative. As a last reminder, we mention that the operator J of $\Lambda^\star(H)$, defined in analogy with the Kähler case preserves the space of harmonic forms.

Let now α be a harmonic form on M. It is known fact that if $0 \leq p \leq n$, every harmonic form α on M is horizontal, invariant by the Reeb vector field. Moreover, α must be primitive, that is $L^\star \alpha = 0$. Using the formality assumption on g we obtain that $\alpha \wedge J\alpha$ is still harmonic. Since this a horizontal form, invariant by the Reeb field it follows from (2.1) that $L^\star(\alpha \wedge J\alpha) = 0$. We conclude again to the vanishing of α by means of Proposition 2.1, (iii). $\square$

3. Some algebraic facts

Let (V^{2n}, g, J) be a Hermitian vector space and let $\Lambda^\star$ be its exterior algebra. Consider the operator $\mathcal{J} : \Lambda^p \to \Lambda^p$ acting on a p-form α as

$$(\mathcal{J}\alpha)(v_1, \ldots, v_p) = \sum_{k=1}^{p} \alpha(v_1, \ldots, Jv_k, \ldots v_p)$$

for all $v_1, \ldots v_p$ in V. $\mathcal{J}$ acts as a derivation on $\Lambda^\star$ and gives the complex bi-grading of the exterior algebra in the following sense. Let $\lambda^{p,q}$ be given as the $-(p-q)^2$-eigenspace of $\mathcal{J}^2$. Then

$$\Lambda^s = \sum_{p+q=s} \lambda^{p,q}$$

an orthogonal, direct sum. Note that $\lambda^{p,q} = \lambda^{q,p}$. Of special importance in our discussion are the spaces $\lambda^p = \lambda^{p,0}$; forms α in λ^p are such that $(X_1, \ldots, X_p) \to \alpha(JX_1, X_2, \ldots, X_p)$ is still an alternating form which equals $p^{-1}\mathcal{J}\alpha$. Let $\lambda^p \otimes_1 \lambda^q$ be the space of tensors $Q \in \lambda^p \otimes \lambda^q$ which satisfy $[(\mathbb{J}Q)(X_1, ...X_p)](Y_1, ..., Y_q) = -[\mathbb{J}(Q(X_1, ...X_p))](Y_1, ..., Y_q)$ (here $\mathbb{J}$ as a map of λ^p stands in fact for $p^{-1}\mathcal{J}$). We also define $\lambda^p \otimes_2 \lambda^q$ as the space of tensors $Q : \lambda^p \to \lambda^q$ such that $Q\mathbb{J} = \mathbb{J}Q$.

Lemma 3.1. *Let $a : \lambda^p \otimes \lambda^q \to \Lambda^{p+q}$ be the total antisymmetrisation map. Then*

(i) *The image of the restriction of a to $\lambda^p \otimes_1 \lambda^q \to \Lambda^{p+q}$ is contained in $\lambda^{p,q}$.*
(ii) *The image of the restriction of a to $\lambda^p \otimes_2 \lambda^q \to \Lambda^{p+q}$ is contained in λ^{p+q}.*

Proof. We shall provide a direct proof, but only for (i), that of (ii) being similar. Pick Q in $\lambda^p \otimes_1 \lambda^q$. Then

$$a(Q) = \sum_{I=(i_1, \ldots i_p)} e_{i_1}^\star \wedge \ldots e_{i_p}^\star \wedge Q(e_{i_1}, \ldots e_{i_p})$$

where for v in V we denote by $v^\star$ the dual, w.r.t to the metric, 1-form. Then

$$\begin{aligned}\mathcal{J}(a(Q)) = & \sum_{I=(i_1,\dots i_p)} \mathcal{J}(e_{i_1}^\star \wedge \dots e_{i_p}^\star) \wedge Q(e_{i_1}, \dots e_{i_p}) \\ & + \sum_{I=(i_1,\dots i_p)} e_{i_1}^\star \wedge \dots e_{i_p}^\star \wedge \mathcal{J}Q(e_{i_1}, \dots e_{i_p}).\end{aligned}$$

For any $1 \leq r \leq p$ we compute

$$\begin{aligned} & \sum_{I=(i_1,\dots i_p)} e_{i_1}^\star \wedge \dots Je_{i_r}^\star \dots \wedge e_{i_p}^\star \wedge Q(e_{i_1}, \dots e_{i_p}) \\ = & - \sum_{I=(i_1,\dots i_p)} e_{i_1}^\star \wedge \dots (Je_{i_r})^\star \dots \wedge e_{i_p}^\star \wedge Q(e_{i_1}, \dots e_{i_p}) \\ = & \sum_{I=(i_1,\dots i_p)} e_{i_1}^\star \wedge \dots e_{i_r}^\star \dots \wedge e_{i_p}^\star \wedge Q(e_{i_1}, \dots Je_{i_r} \dots e_{i_p}) \\ = & \sum_{I=(i_1,\dots i_p)} e_{i_1}^\star \wedge \dots e_{i_r}^\star \dots \wedge e_{i_p}^\star \wedge Q(Je_{i_1}, \dots e_{i_r} \dots e_{i_p}) \\ = & \sum_{I=(i_1,\dots i_p)} Je_{i_1}^\star \wedge \dots e_{i_r}^\star \dots \wedge e_{i_p}^\star \wedge Q(e_{i_1}, \dots e_{i_r} \dots e_{i_p}).\end{aligned}$$

On the other side we have $\mathcal{J}Q(e_{i_1}, \dots e_{i_r} \dots e_{i_p}) = q\mathbb{J}Q(e_{i_1}, \dots e_{i_r} \dots e_{i_p}) = - qQ(Je_{i_1}, \dots e_{i_r} \dots e_{i_p})$ and putting all these together we arrive easily at

$$\mathcal{J}(a(Q)) = (p-q) \sum_{I=(i_1,\dots i_p)} Je_{i_1}^\star \wedge \dots e_{i_p}^\star \wedge Q(e_{i_1}, \dots e_{i_p}).$$

Applying $\mathcal{J}$ once more time while passing through the same steps yields $\mathcal{J}^2 a(Q) = -(p-q)^2 a(Q)$ and the proof is completed. $\square$

The main technical observation in this section is

Proposition 3.1. *The following hold*

(i) *The total alternation map $a : \lambda^p \otimes_1 \lambda^q \to \Lambda^{p+q}$ is injective for $p \neq q$.*
(ii) *The kernel of $a : \lambda^p \otimes \lambda^q \to \Lambda^{p+q}$ is contained in $\lambda^p \otimes_2 \lambda^q$.*

Proof. (i) If Q belongs to $\lambda^p \otimes_1 \lambda^q$ and X is in V we define Q_X and Q^X in $\lambda^{p-1} \otimes_1 \lambda^q$ and $\lambda^p \otimes_1 \lambda^{q-1}$ respectively by

$$Q_X = Q(X, \quad) \text{ and } Q^X = X \lrcorner Q.$$

It is easy to see that those are well defined. Assume now that $a(Q)=0$. Then

$$
\begin{aligned}
0 = X\lrcorner a(Q) &= \sum_{i_1,...,i_p} (X\lrcorner(e_{i_1}^{\star}\wedge ... \wedge e_{i_p}^{\star}))\wedge Q(e_{i_1},...,e_{i_p}) \\
&\quad + (-1)^p \sum_{i_1,...,i_p} (e_{i_1}^{\star}\wedge ... \wedge e_{i_p}^{\star})\wedge (X\lrcorner Q(e_{i_1},...,e_{i_p})) \\
&= p \sum_{i_1,...,i_{p-1}} (e_{i_1}^{\star}\wedge ... \wedge e_{i_{p-1}}^{\star})\wedge Q(X, e_{i_1},...,e_{i_{p-1}}) \\
&\quad + (-1)^p \sum_{i_1,...,i_p} (e_{i_1}^{\star}\wedge ... \wedge e_{i_p}^{\star})\wedge (Q^X(e_{i_1},...,e_{i_{p-1}})) \\
&= pa(Q_X) + (-1)^p a(Q^X)
\end{aligned}
$$

By the previous Lemma $a(Q_X)$ is in $\lambda^{p-1,q}$ whilst $a(Q^X)$ belongs to $\lambda^{p,q-1}$ hence both must vanish since elements of distinct spaces as $p \neq q$. Now an induction argument leads directly to the proof of the Proposition.
(ii) □

4. Holomorphic forms with harmonic squares

Let (M^{2n}, g, J) be a compact Kähler manifold and consider a harmonic p-form Ω in λ^p, that is of type $(0,p)+(p,0)$. It is a well now fact, see [5] for instance, that Ω must be holomorphic, that is

$$\nabla_{JX}\Omega = \nabla_X(\mathbb{J}\Omega) \tag{4.1}$$

for all X in TM. Together with Ω comes $S:\Lambda^{p-1}\to\Lambda^1$ defined$S(X_1,...,X_{p-1}) = \Omega(X_1,...,X_{p-1},\cdot)$. That Ω has real type $(0,p)+(p,0)$ translates into

$$(S(JX_1,...,X_{p-1}))^{\sharp} = -J(S(X_1,...,X_{p-1}))^{\sharp} \tag{4.2}$$

whenever $X_1,...,X_{p-1}$ belong to TM and where for any 1-form θ, $\theta^{\sharp}$ denotes the associated vector field with respect to the metric. Let now $Q:\Lambda^{p-1}\to\lambda^p$ be given by

$$Q(X_1,...,X_{p-1}) = \nabla_{(S(X_1,...,X_{p-1}))^{\sharp}}\Omega$$

for all $X_1,...,X_{p-1}$ in TM. The next Lemma provides information about the complex type of$Q$.

Lemma 4.1. *The tensor Q belongs to $\lambda^{p-1}\otimes_1\lambda^p$.*

Proof. Follows immediately from (4.1) and (4.2). □

Proposition 4.1. *Let Ω in λ^p be a harmonic form. If the metric g is p-formal*

$$\nabla_{(S(X_1,...,X_{p-1}))^{\sharp}}\Omega = 0 \tag{4.3}$$

holds, for all $X_1,...,X_{p-1}$ in TM.

Proof. Let $\{e_i\}$ be a geodesic frame at a point m in M. If p is even $\Omega \wedge \Omega$ is harmonic hence and then we have at m

$$\begin{aligned} 0 = & - d^\star(\Omega \wedge \Omega) = \sum_i e_i \lrcorner \nabla_{e_i}(\Omega \wedge \Omega) \\ = & 2\sum_i e_i \lrcorner (\nabla_{e_i}\Omega \wedge \Omega) = 2\sum_i \nabla_{e_i}\Omega \wedge (e_i \lrcorner \Omega) \end{aligned}$$

since Ω is itself co-closed. In other words $a(Q) = 0$ and we conclude by means of Lemma 4.1 and Proposition 3.1 that $Q = 0$. If p is odd the harmonicity of $\Omega \wedge \mathbb{J}\Omega$ gives

$$\begin{aligned} 0 = -d^\star(\Omega \wedge \mathbb{J}\Omega) &= \sum_{i=1}^{2n} e_i \lrcorner (\nabla_{e_i}\Omega \wedge \mathbb{J}\Omega + \Omega \wedge \nabla_{e_i}\mathbb{J}\Omega) \\ &= \sum_{i=1}^{2n} -(\nabla_{e_i}\Omega) \wedge (e_i \lrcorner \mathbb{J}\Omega) + (e_i \lrcorner \Omega) \wedge \nabla_{e_i}(\mathbb{J}\Omega) \end{aligned}$$

where we took into account the co-closedeness of Ω and $\mathbb{J}\Omega$. Now $\nabla_{e_i}\mathbb{J}\Omega = \nabla_{Je_i}\Omega$ hence

$$\begin{aligned} 0 = & \sum_{i=1}^{2n} -\nabla_{e_i}\Omega \wedge (Je_i \lrcorner \Omega) + (e_i \lrcorner \Omega) \wedge \nabla_{Je_i}\Omega \\ = & -2\sum_{i=1}^{2n} \nabla_{e_i}\Omega \wedge (Je_i \lrcorner \Omega). \end{aligned}$$

From this we see that $a(Q) = 0$ and Lemma 4.1 together with Proposition 3.1 lead then to the vanishing of Q and hence to the claimed result. □

Remark 4.1. *From the proof of the result above we see that it actually holds for harmonic forms Ω in λ^p such that $\Omega \wedge \Omega$ (p even) resp. $\Omega \wedge \mathbb{J}\Omega$ (p odd).*

We need now recall some facts about the algebraic structure of harmonic forms of type $(1,1)$.

Proposition 4.2. [8] *Let (M^{2n}, g, J) be a compact Kähler manifold such that the metric g is formal. If $\alpha = g(F\cdot, \cdot)$ is harmonic in $\lambda^{1,1}$ then we have an orthogonal and J-invariant splitting*

$$TM = \bigoplus_{i=0}^{p} E_i$$

which is preserved by F and such that $F = \lambda_i J_i$ on E_i, for all $0 \le i \le p$. Here J_i are almost complex structures on E_i and λ_i are real constants, for $0 \le i \le p$.

Now we would like to conclude from Proposition 4.1 that Ω is actually parallel. This is eventually seen to be the case if Ω is non-degenerate at every point of the manifold. To rule out the general case we must study the null distribution of Ω. For each m in M define $\mathcal{V}_m = \{X \in T_mM : X \lrcorner \Omega = 0\}$. Our first concern is to show that $m \to \mathcal{V}_m$ gives a smooth, *constant* rank distribution on M.

Lemma 4.2. *Suppose that (M^{2n}, g, J) is p-formal and let Ω in λ^p be harmonic. Then $\mathcal{V}$ is of constant rank.*

Proof. Let α_Ω be defined by $\alpha_\Omega(X, Y) = \langle JX \lrcorner \Omega, Y \lrcorner \Omega \rangle$ for all X, Y in TM. This is harmonic since proportional to $P_{p-1}(\Omega, \Omega)$ and α_Ω is in $\lambda^{1,1}$ by Lemma 2.1. Since g is p-formal every harmonic form in $\lambda^{1,1}$ has constant rank by Proposition 4.2; but the nullity of α_Ω coincides with that of Ω and the Lemma is proved. □

It is easy to see that $\mathcal{V}$ is actually J-invariant and since it has constant rank we obtain all over M an orthogonal, J-invariant splitting

$$TM = \mathcal{V} \oplus H$$

where H is the orthogonal complement of $\mathcal{V}$ in TM.

Lemma 4.3. *Both distributions $\mathcal{V}$ and H are integrable and H is totally geodesic.*

Proof. By the definition of $\mathcal{V}$ the distribution H is spanned by $S(X_1, ..., X_{p-1})$ with $X_1, ..., X_{p-1}$ in TM hence $\nabla_X \Omega = 0$ for all X in $\Gamma(H)$. Taking now a direction, say V in $\Gamma(\mathcal{V})$ gives that $\nabla_X V$ belongs to $\mathcal{V}$ and this shows the total geodesicity hence the integrability of H. The integrability of $\mathcal{V}$ is an easy consequence of the closedeness of Ω. Indeed, taking $X_1, ..., X_{p-1}$ in H and V, W in $\mathcal{V}$, we have

$$\begin{aligned} 0 = d\Omega(X_1, ..., X_{p-1}, V, W) = & \sum_{1 \leqslant i \leqslant p-1} (-1)^{i+1} (\nabla_{X_i} \Omega)(X_1, ..., \widehat{X_i}, ..., X_{p-1}, V, W) \\ & - (\nabla_V \Omega)(X_1, ..., X_{p-1}, W) + (\nabla_W \Omega)(X_1, ..., X_{p-1}, V) \\ & = \Omega(X_1, ..., X_{p-1}, [V, W]) \end{aligned}$$

□

To prove the parallelism of Ω, which amounts to having $\mathcal{V}$ totally geodesic we need to establish one more fact. Recall [11] that the transversal Ricci tensor Ric^H of the totally geodesic H is defined as

$$Ric^H(X, Y) = \sum_i R(X, e_i, Y, e_i)$$

for all X, Y in H and frames $\{e_i\}$ in H. When $\mathcal{V}$ integrates to give a Riemannian submersion, which is always true locally, Ric^H corresponds to the usual Ricci tensor of the base manifold.

Lemma 4.4. *The transversal Ricci tensor Ric^H of the distribution H vanishes.*

Proof. Since M is p-formal with p even, M is also 2-formal. Let S_Ω be the symmetric J-invariant $(1, 1)$-tensor defined by $\langle S_\Omega X, Y \rangle = -\alpha_\Omega(JX, Y)$. Using proposition 3.1 of [8] we have an orthogonal J-invariant integrable decomposition $TM = \bigoplus_{i=0}^{N} E_i$ where E_i are the eigenspaces of S_Ω corresponding respectively to (the pairwise distinct) eigenvalues of S_Ω. Choosing $\lambda_0 = 0$, we have $\mathcal{V} = E_0$ and $H = \bigoplus_{i=1}^{N} E_i$.

From [8], we know that the forms $\alpha_{\Omega,k}$ defined by $\alpha_{\Omega,k}(X,Y) = \langle S_\Omega^k JX, Y\rangle$ are belonging to $\lambda^{1,1}$ and

$$\alpha_{\Omega,k} = \sum_{i=1}^{N} \lambda_i^k \omega_i \tag{4.4}$$

where ω_i are the orthogonal projection of the Kähler form ω on E_i. Now, the restriction of α_Ω to an integral distribution of H is parallel and then $\alpha_{\Omega,k}$ too. By a similar argument to the proof of Proposition 3.1 of [8], we deduce from (4.4) that each ω_i is parallel on H and then each E_i is totally geodesic. Now, let Z be in H. Since H is totally geodesic, we have $\nabla\Omega = 0$ where ∇ is the Levi-Civita connection. Then

$$\begin{aligned}
0 &= \langle R(Z,JZ)\Omega, \mathbb{J}\Omega\rangle \\
&= -\frac{1}{p!}\sum_{k,i_1,\dots,i_p} R(Z,JZ,e_{i_j},e_k)\Omega(e_{i_1},\dots,\widehat{e_{i_j}},e_k,\dots,e_{i_p})(\mathbb{J}\Omega)(e_{i_1},\dots,e_{i_p}) \\
&= -\frac{1}{(p-1)!}\sum_{k,l,i_1,\dots,i_{p-1}} R(Z,JZ,e_l,e_k)\Omega(e_k,e_{i_1},\dots,e_{i_{p-1}})(\mathbb{J}\Omega)(e_l,e_{i_1},\dots,e_{i_{p-1}}) \\
&= \sum_{k,l}(R(JZ,e_l,Z,e_k)+R(e_l,Z,JZ,e_k))\langle e_k\lrcorner\Omega, Je_l\lrcorner\Omega\rangle \\
&= -2\sum_{k,l} R(Z,e_k,Z,e_l)\langle S_\Omega e_k, e_l\rangle
\end{aligned}$$

Then $\sum_{k,l} R(Z,e_k,Z,e_l)\langle S_\Omega e_k,e_l\rangle = 0$. Since S_Ω is non-degenerate and each E_i is totally geodesic for $i \geqslant 1$, it follows that $\mathrm{Ric}^{E_i} = 0$ for any $i \geqslant 1$ and then $\mathrm{Ric}^H = 0$. □

Remark 4.2. *The vanishing of Ric^H continues to hold when (M^{2n}, g) is supposed to be only p-formal with p even. For this implies 2-formality and therefore the harmonicity of α_Ω and the proof continues as above..*

We have proved the nullity of Ric^H, a situation well described by the following

Theorem 4.1. [8] *Let (M^{2n}, g, J) be a compact Kähler manifold equipped with a Riemannian foliation with complex leaves. If the the foliation is transversally totally geodesic with nonnegative transversal Ricci tensor then it has to be locally a Riemannian product.*

Proof of Theorem 1.4 Since Ric^H vanishes, it follows by Theorem 4.1 that $\mathcal{V}$ is totally geodesic, fact which implies immediately the parallelism of Ω. □

5. Harmonic 2-forms

We shall develop in this section the general Riemannian counterpart of Proposition 4.2. From now on, we use the metric to identify a 2-form α with a skew-symmetric endomorphism A of TM; explicitly $\alpha = g(A\cdot,\cdot)$. Moreover, the space $\mathcal{A}$

is the space of skew-symmetric endomorphisms of TM which are associated to an element of $\mathcal{H}^2(M)$.

Proposition 5.1. *Let (M^n, g) be a compact geometrically formal manifold. We have :*

$$A_2A_1A_3 + A_3A_1A_2 \in \mathcal{A}$$

whenever $A_i, 1 \leq i \leq 3$ belong to $\mathcal{A}$.

Proof. Let α belong to $\mathcal{H}^2(M)$. Denote by $L_\alpha : \Lambda^\star \to \Lambda^\star$ the exterior multiplication by α. Since g is formal and $L_\alpha^\star$ is up to sign equal to $\star L_\alpha \star$ it follows that both L_α and $L_\alpha^\star$ preserve the space of harmonic forms of (M, g). Therefore, if $\alpha_i, 1 \leq i \leq 3$ belong to $\mathcal{H}^2(M)$ then $L_{\alpha_1}^\star L_{\alpha_2}\alpha_3$ is an element of $\mathcal{H}^2(M)$. Let $A_i, 1 \leq i \leq 3$ the skew-symmetric endomorphisms associated to the forms $\alpha_i, 1 \leq i \leq 3$ and let $\{e_i, 1 \leq i \leq n\}$ be a local orthonormal basis in TM. We shall now compute

$$L_{\alpha_1}^\star L_{\alpha_2}\alpha_3\alpha_1 = \sum_{i,j=1}^n \alpha_1(e_i, e_j) e_j \lrcorner \left[e_i \lrcorner (\alpha_2 \wedge \alpha_3) \right]$$

But

$$e_j \lrcorner \left[e_i \lrcorner (\alpha_2 \wedge \alpha_3) \right] = \alpha_2(e_i, e_j)\alpha_3 - (e_i \lrcorner \alpha_2) \wedge (e_j \lrcorner \alpha_3) + (e_j \lrcorner \alpha_2) \wedge (e_i \lrcorner \alpha_3) + \alpha_3(e_i, e_j)\alpha_2.$$

Further computation yields, after some elementary manipulations

$$\alpha_1 \lrcorner (\alpha_2 \wedge \alpha_3) = \langle \alpha_1, \alpha_2 \rangle \alpha_3 + \langle \alpha_1, \alpha_3 \rangle \alpha_2 + 2\langle A_3A_1A_2 + A_2A_1A_3 \cdot, \cdot \rangle$$

□

Proposition 5.2. *Let (M^n, g) be a compact geometrically formal and α belong to $\mathcal{H}^2(M)$ with associated A. Then :*

(i) *The eigenvalues of A^2 are constant with eigenbundles of constant rank.*

(ii) *Let $-\lambda_i^2$ be (the pairwise distinct) eigenvalues of A^2, with $\lambda_0 = 0$ and let E_i be eigenspaces of A^2 corresponding to $-\lambda_i^2$. Then for $1 \leqslant i \leqslant p$, E_i is of even dimension and we have an orthogonal decomposition*

$$\alpha = \sum_{i=1}^p \lambda_i \omega_i$$

where ω_i belongs to $\mathcal{H}^2(M)$, $\lambda_i\omega_i$ is the orthogonal projection of α on E_i and ω_i is a complex structure on E_i.

Proof. (i) From the Proposition 5.1 it follows by induction that A^{2k+1} belongs to $\mathcal{A}$ whenever A is in $\mathcal{A}$. Since $\mathcal{A}$ is finite dimensional, there exists $P \in \mathbb{R}[X]$ so that $P(A^2) = 0$. Since A^2 is symmetric, P can be supposed to have only real and simple roots. Let $\mu_i = -\lambda_i^2$, $0 \leqslant i \leqslant p$ be these (pairwise distinct) roots and let m_i be the dimension of the corresponding eigenbundle. To see that m_i, $0 \leqslant i \leqslant p$ are constant over M, we use the fact that A^{2k+1} belongs to $\mathcal{A}$ for any $k \in \mathbb{N}$ and from the formality of M we deduce that $Tr(A^{2k}) = -\langle A^{2k-1}, A \rangle = c_k$ for some constant c_k and for any integer k. Solving this Vandermonde system leads to the constancy of the functions m_i, $1 \leqslant i \leqslant p$.

(ii) The orthogonal projection of α on E_i is given by $\lambda_i\omega_i$ for $1 \leqslant i \leqslant p$ where

ω_i is a complex structure on E_i. More precisely for $1 \leqslant i \leqslant p$ the dimension m_i is even ($m_i = 2d_i$) and there exists an orthonormal adapted basis $(e_{ij})_{1 \leqslant j \leqslant d_i}$ so that $\omega_i = \sum\limits_{1 \leqslant j \leqslant d_i} e^{\star}_{ij} \wedge e^{\star}_{ij+d_i}$. From the proposition 5.1, it follows that $\sum\limits_{i=1}^{p} \lambda_i^{2k+1} \omega_i$ is harmonic and by a similar argument use in the proof of the proposition 3.1 of [8] we deduce that ω_i belong to $\mathcal{H}^2(M)$. $\square$

The technical advantage of Proposition 5.2 is that all distributions algebraically out of harmonic forms are of constant rank over the manifold, and in this respect they can -as we shall in the next section-treated as algebraic objects.

5.1. **6-dimensions.** We shall present here a geometric application of the algebraic facts from the previous section. More precisely, we are going to obtain sufficient conditions for a geometrically formal 6-manifold to admit a compatible symplectic structure. Note this is well known in dimension 4, see [6]. We need first making a number of preliminary results.

Lemma 5.1. *Let (M^n, g) be geometrically formal and let α be a harmonic 2-form with kernel $\mathcal{V}$ and such that on $H = \mathcal{V}^{\perp}, \alpha = g(J\cdot, \cdot)$ for some almost complex structure J of H. Then for any ϕ in $\mathcal{H}^p(M)$ we have that ϕ^{ij} belongs to $\mathcal{H}^p(M)$ where for any i, j with $i + j = p$ we have denoted by ϕ^{ij} the orthogonal projection of ϕ onto $\Lambda^i(\mathcal{V}) \otimes \Lambda^j(H) \subseteq \Lambda^p(M)$.*

Proof. We first note that

$$L^{\star}_{\omega_J}(\psi \wedge \omega_J) = -dimH \cdot \psi + (L^{\star}_{\omega_J}\psi) \wedge \omega_J + 2(-1)^{p+1} Q\psi$$

whenever ψ is a p-form on M, where the operator Q is given by $Q\psi = \sum\limits_{e_i \in H} (e_i \lrcorner \psi) \wedge e^i$ for an arbitrary local frame $\{e_i\}$ in H. Hence Q preserves the space of harmonic forms and on the hand a standard computation shows that the eigenvalues of Q on $\Lambda^p(M)$ are $(-1)^i j$ with corresponding eigenbundles $\Lambda^i(\mathcal{V}) \otimes \Lambda^j(H)$. The usual raising to power argument gives now the wanted result. $\square$

Lemma 5.2. *Let (M^6, g) be geometrically formal. If g doesn't admit a compatible symplectic form then every non-zero harmonic 2-form on M has 4-dimensional kernel.*

Proof. Let α belong to $\mathcal{H}^2(M)$. We must only show that it cannot have 2-dimensional kernel. Proceeding by contradiction, let us suppose that $\mathcal{V} = ker(\alpha)$ is 2-dimensional, so that $H = \mathcal{V}^{\perp}$ is of dimension 4. Moreover, on H from α we get following **??** a harmonic $\alpha' = g(J\cdot, \cdot)$ for some almost complex J on H. Then $\alpha' + \star(\alpha' \wedge \alpha')$ gives a globally defined symplectic form on M, compatible with g, hence the desired contradiction. $\square$

Proposition 5.3. *Let (M^6, g) be geometrically formal with $b_1 = 0$ and $b_2 \geq 2$. If g does not admit a compatible symplectic form we must have $b_2 = 2, b_3 = 6$ and moreover M is a parallelisable manifold.*

Proof. Let α be a non-zero harmonic 2-form on M. By Lemma 5.2 the distribution $\mathcal{V} = ker(\alpha)$ must be 4-dimensional, so after constant rescaling α can be written as

$\alpha = g(J\cdot, \cdot)$ where J is an almost complex structure on the plane distribution $H = \mathcal{V}^{\perp}$. We now note there are no non-zero harmonic 2-forms contained in $\Lambda^2(\mathcal{V})$, for by Lemma 5.2 any such form must have 4-dimensional kernel and hence must vanish. It follows then from Lemma 5.1 that $\mathcal{H}^2(M)$ is contained in $(\Lambda^1(\mathcal{V}) \otimes \Lambda^1(H)) \oplus \mathbb{R}\omega_J$. Further, because $b_2 \geq 2$, there must be β in $\Lambda^1(\mathcal{V}) \otimes \Lambda^1(H)$, and again by Lemma 5.2 this has 4-dimensional kernel which we shall denote by $\mathcal{V}'$. By rescaling if necessary we shall also assume that β is of unit length. Consider the orthogonal splitting $\mathcal{V}' = E_1 \oplus E_2$ obtained by orthogonally projecting elements of $\mathcal{V}'$ onto $\mathcal{V}$ and H respectively. Similarly, we split $H' = (\mathcal{V}')^{\perp}$ as an orthogonal sum $H' = F_1 \oplus F_2$ with F_1 and F_2 subspaces of $\mathcal{V}$ and H respectively. We now reason by counting dimensions : E_2 is not the zero space because that would imply $\mathcal{V}' \subseteq \mathcal{V}$, actually $\mathcal{V}' = \mathcal{V}$ because both distributions are of rank 4 therefore the vanishing of β since this an element of $\Lambda^1(\mathcal{V}) \otimes \Lambda^1(H)$. We cannot have $F_2 = (0)$ neither : it would imply eventually that $H' \subseteq \mathcal{V}$ and $\mathcal{V} = E_1 \oplus H'$ which is easily seen to lead to M having a compatible symplectic structure given by $\alpha + \beta + \star(\alpha + \beta)^2$, an absurdity. We showed that both of E_2 and F_2 have rank at least 1, and their orthogonality (following again from β being orthogonal to $\Lambda^2(\mathcal{V}) \oplus \Lambda^2(H)$), together with having H of rank 2 leads to having E_2, F_2 of rank 1. Since the manifold is oriented, every real line bundle over M is trivial and this leads to the existence of a globally defined frame $\{e_1, e_2\}$ on H, spanning E_2 and F_2 respectively. Similar arguments imply the global existence of a unit ζ in $\mathcal{V}$ spanning F_1 and it is straightforward to see that $\beta = \zeta \wedge e^2$.
Pick now a non-zero harmonic 3-form T on M. By Lemma 5.1 the components T^{11} in $\Lambda^3(\mathcal{V}), T^{12}$ in $\Lambda^1(\mathcal{V}) \otimes \Lambda^2(H) = L_{\alpha}\Lambda^1(\mathcal{V})$ of T are harmonic, but we conclude to their vanishing since those are easily seen to produce harmonic 1-forms. Hence T can be written as

$$T = \omega_1 \wedge e^1 + \omega_2 \wedge e^2$$

with $\omega_k, k = 1, 2$ in $\Lambda^2(\mathcal{V})$. Moreover, $L_{\phi}T$ resp. $L_{\phi}^{\star}T$ vanish for any harmonic 2-form ϕ because $b_1 = 0$. Hence from $L_{\beta}T = 0$ and $L_{\beta}^{\star}T = 0$ we get that $\zeta \wedge \omega_1 = 0, \zeta \lrcorner \omega_2 = 0$. It follows easily that harmonic 3-forms on M are contained in a rank 6 sub-bundle of $\Lambda^3(M)$, thus using that scalar products of harmonic 3-forms are (pointwisely) constant we obtain that $b_3(M) \leq 6$. Since M has nowhere vanishing vector fields, it has vanishing Euler characteristic, and from $b_1 = 0, b_2 \geq 2$ we get

$$b_3 = 2(1 + b_2) \geq 6$$

showing that actually $b_2(M) = 2$ and $b_3(M) = 6$. □

Theorem 5.1. *Let (M^6, g) be geometrically formal with $b_1(M) \neq 1$ and $b_2(M) \geq 2$. If g does not admit a compatible symplectic form then either :*
(i) M has the cohomology algebra of $\mathbb{T}^3 \times S^3$
or
(ii) $b_1 = 0, b_2 = 2, b_3 = 6$ and M is a parallelisable manifold.

Proof. In view of the previous results it suffices to treat the cases when $b_1 \neq 0$. Again, we do a case by case discussion. Let $\mathcal{V}$ be the distribution spanned by the harmonic 1-forms. say $\zeta_k, 1 \leq k \leq b_1$ be a frame in $\mathcal{V}$, and consider the orthogonal splitting $TM = \mathcal{V} \oplus H$ where $H = \mathcal{V}^{\perp}$. As an immediate consequence of Lemma 5.1

and of the fact that H doesn't contain, by definition, harmonic 1-forms it follows that harmonic 2-forms are contained in $\Lambda^2(\mathcal{V}) \oplus \Lambda^2(H)$.
If $b_1 = 2$, H is of rank 4 and since $b_2(M) \geq 2$ there must be a non-zero harmonic 2-form contained in $\Lambda^2(H)$. In view of Lemma 5.2 it has rank 4 kernel and therefore vanishes, a contradiction.
Suppose now that $b_1 = 3$ so that H is of rank 3. Then $\mathcal{H}^2(M) \subseteq \Lambda^2(\mathcal{V})$ and as before, by using Lemma 5.1 we have that $\mathcal{H}^3(M) \subseteq \Lambda^3(\mathcal{V}) \oplus \Lambda^3(H)$. It is now straightforward that M has the cohomology algebra of $\mathbb{T}^3 \times S^3$.
If $b_1 = 4$, then $\zeta_1 \wedge \zeta_2 + \zeta_3 \wedge \zeta_4 + \star(\zeta_1 \wedge \zeta_2 \wedge \zeta_3 \wedge \zeta_4)$ is a symplectic form, a contradiction. Now we cannot have $b_1 = 5$ ([6]) and when $b_1 = 6$ the metric is flat hence there is a compatible symplectic structure, a contradiction. This finishes the proof of the theorem. □

The proof of Theorem 1.6 follows now immediately from the above.

Remark 5.1. *(i) The proof of Proposition 5.3 can also be adapted to show that if (M^6, g) is formal with $b_1 = 0, b_2 = 1$ and g does not admit a compatible symplectic structure then $b_3 \leq 6$.*
(ii) While it is unlikely that alternative (ii) in Theorem 5.1 is possible, for the time being we felt short ruling this case out.

6. The case when b_2 is maximal

We study in this section geometrically formal manifolds (M^n, g) having maximal second Betti number, i.e. $b_2(M) = \binom{n}{2}$. To prove Theorem 1.5, we split our discussion into two cases accordingly to the parity of n.

Proposition 6.1. *Let (M^n, g) be geometrically formal. If the $b_p(M)$ is maximal for some $1 \leq p \leq n-1$ and $(p, n) = 1$ then g is a flat metric.*

Proof. Formality tells us that $b_{pq}(M)$ is maximal for any natural q and since $(p, n) = 1$ we arrive at b_1 maximal, and it follows that g is flat. □

Hence, when n is odd and $b_2(M)$ is maximal, the metric is flat and we need only consider the case when n is even.

6.1. Reduction to the symplectic case.

As an immediate consequence of Proposition 5.2 we have :

Proposition 6.2. *Let (M^{2n}, g) be a compact geometrically formal manifold such that b_2 is maximal. Then there exists an almost Kähler structure, that is an almost complex structure J, which is compatible with g and so that the 2-form $g(J\cdot, \cdot)$ is closed.*

Proof. From the maximality of b_2, there exists a harmonic 2-form α which is non-degenerate at a given point x. It is known that α is non-degenerate if and only if $\alpha^n \neq 0$. Since M is geometrically formal, α^n is harmonic and of constant norm. It follows that α is non-degenerate all over M. Now, if we denote by A the skew-symmetric endomorphism associated to α, we deduce from Proposition 5.2 that A^2

diagonalises with constant eigenvalues $-\lambda_i^2$ and constant rank associated eigenbundles E_i. Moreover $\alpha = \sum_{i=1}^{p} \lambda_i \omega_i$ (with $\lambda_i \neq 0$ since ω is non-degenerate) where ω_i belong to $\mathcal{H}^2(M)$ and ω_i is a complex structure on each E_i. Therefore $\omega = \sum_{i=1}^{p} \omega_i$ is an almost Kähler structure which is compatible with g and so that the 2-form ω is closed. □

6.2. **Proof of flatness.** We consider hereafter a geometrically formal almost-Kähler manifold (M^n, g, J) $(n = 2k)$ so that $b_2(M) = \binom{n}{2}$. Let $\omega = g(J\cdot, \cdot)$ be the so-called Kähler form of the almost Kähler structure. We first remark that the bi-type splitting of Λ^2 is preserved at the level of harmonic forms (note, by contrast with the Kähler case that this need no longer be true in the case of an arbitrary almost Kähler manifold).

Lemma 6.1. *Any harmonic 2-form splits as $\alpha = \alpha_1 + \alpha_2$ where the harmonic α_1, α_2 are in $\lambda^{1,1}$ and λ^2 respectively.*

Proof. Pick α in Λ^2, which splits as $\alpha = \alpha_1 + \alpha_2$ with α_1 in $\lambda^{1,1}$ and α_2 in λ^2. Because of formality we can assume w.l.o.g. that α is primitive. Again the formality tells us that $L^\star_\alpha(\omega \wedge \omega)$ is harmonic and a straightforward computation shows the latter is proportional to $\alpha_1 - \alpha_2$. This eventually proves the Lemma. □

Therefore, if b_2 is maximal, both $\lambda^{1,1}$ and λ^2 are spanned by harmonic forms. We need now see what geometric properties a harmonic 2-form in λ^2 must satisfy. To do so, recall that the first canonical Hermitian connection $\overline{\nabla}$ of the almost Kähler (g, J) is given by

$$\overline{\nabla}_X = \nabla_X + \eta_X$$

for all X in TM. Here ∇ is the Levi-Civita connection of g and $\eta_X = \frac{1}{2}(\nabla_X J)J$. It must perhaps be noted that η actually gives the intrinsic torsion of the $U(n)$-structure induced by (g, J). The connection $\overline{\nabla}$ is metric and Hermitian, that is it preserves both the metric and the almost-complex structure. The almost Kähler condition i.e. that $d\omega = 0$ is given then given in terms of the intrinsic torsion tensor as

$$\langle \eta_X Y, Z \rangle + \langle \eta_Y Z, X \rangle + \langle \eta_Z X, Y \rangle = 0 \tag{6.1}$$

for all X, Y, Z in TM. The latter also implies that (g, J) is quasi-Kähler :

$$\eta_{JX} = \eta_X J \tag{6.2}$$

for all X in TM. Moreover we have

$$\eta_X J = -J\eta_X \tag{6.3}$$

in other words η belongs to $\lambda^1 \otimes_1 \lambda^2$. The relations (6.1), (6.2) and (6.3) will be used implicitly in subsequent computations.

Lemma 6.2. *Let (M^{2k}, g, J) be an almost-Kähler manifold and let $\alpha = g(F\cdot, \cdot)$ be harmonic in λ^2. Then*

$$(\overline{\nabla}_{JX}F)JY + (\overline{\nabla}_X F)Y = -2\eta_{FX}Y \tag{6.4}$$

for all X, Y in TM.

Proof. From $d\alpha = 0$ we have that $a(\nabla\alpha) = 0$. But $\nabla_X\alpha = \overline{\nabla}_X\alpha + < [F, \eta_X]\cdot, \cdot >$ for all X in TM and moreover a simple computation based on (6.1) shows that

$$a((X, Y, Z) \to < [F, \eta_X]Y, Z >) = a((X, Y, Z) \to < \eta_{FX}Y, Z >).$$

Therefore $a(\overline{\nabla}\alpha + \eta_{F\cdot}) = 0$ and since the tensor under alternation belongs to $\lambda^1 \otimes \lambda^2$ we use Lemma 3.2 to conclude that it is actually in $\lambda^1 \otimes_2 \lambda^2$ and the proof of the claimed result follows by using the relations (6.2), (6.3). □

If Q is an endomorphism of M, let us define the action $Q \bullet \eta$ as

$$(Q \bullet \eta)(X, Y, Z) = \sigma_{X,Y,Z}\langle \eta_{QX}Y, Z\rangle$$

for all X, Y, Z in TM. Note that this action is different from the usual action of $End(TM)$.

Lemma 6.3. *Let (M^{2k}, g, J) be an almost-Kähler manifold and let $\alpha = g(F\cdot, \cdot)$ be harmonic in λ^2 with harmonic square. Then*

$$F^2 \bullet \eta = 0. \tag{6.5}$$

Proof. That $d^\star(\alpha \wedge \alpha) = 0$ translates after a calculation overlapping that in the proof of Proposition 4.1 into

$$\sigma_{X,Y,Z}\langle(\nabla_{FX}F)Y, Z\rangle = 0$$

for all X, Y, Z in TM, where σ stands for the cyclic sum. Rewritten by means of the canonical Hermitian connection and using (6.1) this gives

$$\begin{aligned} &\langle(\overline{\nabla}_{FX}F)Y, Z\rangle + \langle(\overline{\nabla}_{FY}F)Z, X\rangle + \langle(\overline{\nabla}_{FZ}F)X, Y\rangle \\ &+ \langle\eta_X FY, FZ\rangle + \langle\eta_Y FZ, FX\rangle + \langle\eta_Z FX, FY\rangle = 0 \end{aligned} \tag{6.6}$$

We shall exploit now the algebraic symmetries of the above. Changing (Y, Z) in (JY, JZ) and substracting from the original equation implies

$$\begin{aligned} &2\langle(\overline{\nabla}_{FX}F)Y, Z\rangle - 2\langle\eta_X FZ, FY\rangle \\ &+ \langle(\overline{\nabla}_{FY}F)Z + (\overline{\nabla}_{JFY}F)JZ, X\rangle - \langle(\overline{\nabla}_{FZ}F)Y + (\overline{\nabla}_{JFZ}F)JY, X\rangle = 0 \end{aligned}$$

or further, after using the relation 6.4

$$\begin{aligned} &\langle(\overline{\nabla}_{FX}F)Y, Z\rangle - \langle\eta_X FZ, FY\rangle \\ &- \langle\eta_{F^2Y}Z, X\rangle + \langle X, \eta_{F^2Z}Y\rangle = 0. \end{aligned} \tag{6.7}$$

Now taking the cyclic sum and using (6.6) we get the desired result. □

Remark 6.1. *On an almost Kähler manifold (M^{2n}, g, J) a harmonic form α in $\lambda^2(M)$ with harmonic wedge powers need not vanish. This happens for instance when $\alpha = g(I\cdot, \cdot)$ for a g-compatible almost complex structure I, and hence defines a complex-symplectic structure on M. Simple examples of the latter situation, which are not hyperkähler, can be displayed on certain classes of nilmanifolds* [3].

From the Lemma above we find by J-polarisation that

$$[F, G] \bullet \eta = 0$$

for all F, G dual to harmonic forms in λ^2. Recall now the standard fact that the splitting $\mathfrak{so}(2k) = \mathfrak{u}(k) \oplus \mathfrak{m}$, where $\mathfrak{m}$ consists in elements of $\mathfrak{so}(2k)$ anti-commuting with J, is such that $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{u}(k)$ for $k \geq 2$. Therefore, if g is a formal metric on M^{2k} and $b_2(M)$ is maximal, we get that $F \bullet \eta = 0$ for all F dual to forms in $\lambda^{1,1}$ provided that $dim M \geq 6$.

Lemma 6.4. *If dim $M \geqslant 6$, the intrinsic torsion η must vanish identically.*

Proof. It is enough to prove the statement at an arbitrary point m of M. Pick an arbitrary unit vector V in T_mM and let F be the skew-symmetric, J-invariant endomorphism of TM which is J on $E = \langle\{V, JV\}\rangle$ and vanishes on $H = E^{\perp}$. That $F \bullet \eta = 0$ says

$$\langle \eta_{FX} Y, Z\rangle + \langle \eta_{FY} Z, X\rangle + \langle \eta_{FZ} X, Y\rangle = 0$$

for all X, Y, Z in TM. It follows that $\langle \eta_V X, Y\rangle = 0$ for all X, Y in H, hence $\eta_V X$ is in$E$ for any $X \in H$. Moreover, since dim $M \geqslant 6$, there exists a unit vector $U \in TM$ so that (V, JV, U, JU, X, JX) is an orthogonal system. Let us consider the skew-symmetric, J-invariant endomorphism G of TM defined by $GV = U$, $GJV = JU$, $GU = -V$, $GJU = -JV$ and G vanishes on $E'^{\perp}$ where $E' = \langle\{V, JV, U, JU\}\rangle$. Then

$$\langle \eta_{GU} X, V\rangle + \langle \eta_{GX} V, U\rangle + \langle \eta_{GV} U, X\rangle = 0$$

This implies that $\langle \eta_V X, V\rangle = -\langle \eta_U X, U\rangle$. Changing V in JV and using the anti-J-invariance of η we get $\langle \eta_V X, V\rangle = 0$. Then

$$\eta_V X = 0$$

for all $X \in H$ and $\eta_V X = \langle X, V\rangle \eta_V V + \langle X, JV\rangle \eta_V JV$ for all $X \in TM$. But from (6.2) it follows that $\eta_V V = \eta_V JV = 0$ and $\eta_V X = 0$ for all $X \in TM$. $\square$

In other words (g, J) is a Kähler structure and the flatness of the metric follows now from [8]. To finish the proof of Theorem 4.1 it remains to treat the case when $n = 4$. In this situation, we notice that the bundles $\Lambda^{\pm}$ of (anti) self-dual forms are trivialised by almost-Kähler structures satisfying the quaternionic identities and using the well-known Hitchin lemma [10] we obtain that $\Lambda^{\pm}$ both contain a hyper-Kähler structure which leads routineously to the flatness of the metric.

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