

# On the Manipulability of Proportional Representation

Arkadii Slinko and Shaun White

**Abstract:** This paper presents a new model of voter behaviour under methods of proportional representation (PR). We assume that voters are concerned, first and foremost, with the distribution of power in the post-election parliament. We abstract away from rounding, and assume that a party securing  $k$  percent of the vote wins exactly  $k$  percent of the available seats. We show that, irrespective of which positional scoring rule is adopted, there will always exist circumstances where a voter would have an incentive to vote insincerely. We demonstrate that a voter's attitude toward uncertainty can influence his or her incentives to make an insincere vote. Finally, we show that the introduction of a threshold - a rule that a party must secure at least a certain percentage of the vote in order to reach parliament - creates new opportunities for strategic voting. We use the model to explain voter behaviour at the most recent New Zealand general election.

**Key words:** parliament choosing rule, proportional representation, power index, manipulability

**AMS subject classification:** 91B12

## 1 Introduction

This paper investigates opportunities for strategic voting under proportional representation. To date, research on this topic has been rather sparse. Austin-Smith and Banks [3], Baron and Diermeier [5], and De Sinopoli and Iannantuoni [11, 12, 13] have constructed multi-stage spatial models of political systems that incorporate proportional representation. In these models voters (i) have preferences over the set of policies that governments might pursue but (ii) do not necessarily vote for the party to which they are ideologically closest. Voters might support parties espousing views more extreme than their own in a bid to counteract votes from other voters whose opinions lie on the opposite side of the policy spectrum. Cox and Shugart [10] demonstrated that the need to “round off” can render proportional representation manipulable. Under methods of proportional representation, whenever the number of candidates to be elected exceeds the number of ballots cast a technique of rounding off will need to be applied (see e.g.,

[21], chapter 4). If a party is in a position where receiving a few more or a few less votes will not alter the number of seats it will take, then some of that party's supporters may peel off, and attempt to influence the distribution of the remaining seats. In all the aforementioned papers it is assumed that, at the ballot box, voters can indicate a preference for just one party (i.e., the positional scoring rule is plurality).

This paper presents a new model of voter behaviour under methods of proportional representation (PR). We assume that voters are concerned, first and foremost, with the distribution of power in the post-election parliament. We abstract away from rounding, and assume that a party securing  $k$  percent of the vote wins exactly  $k$  percent of the available seats. We show that, irrespective of which positional scoring rule is adopted, there will always exist circumstances where a voter would have an incentive to vote insincerely. We demonstrate that a voter's attitude toward uncertainty can influence his or her incentives to make an insincere vote. Finally, we show that the introduction of a threshold - a rule that a party must secure at least a certain percentage of the vote in order to reach parliament - creates new opportunities for strategic voting. All these ideas shall be made precise below.

This paper was initially motivated by a desire to explain the behaviour of voters at the New Zealand general election held September 17th, 2005. The New Zealand electoral system is mixed member proportional (MMP), similar to the system run in Germany. Anecdotal evidence has suggested that at the election some voters voted insincerely even though their doing so could have cost their most-preferred-party seats. We shall show that the model presented can account for such behaviour.

Below, Sections 2 (Parliament choosing rules), 3 (Indices of voting power), and 4 (Voters) describe our model. Theoretical results are presented in Section 5. Section 6 uses the model to explain voter behaviour at the most recent New Zealand general election, and Section 7 concludes.

## 2 Parliament Choosing Rules

We assume that a parliamentary body is to be elected, that the body contains a fixed number  $k$  of seats, and that  $m$  political parties are competing for those seats. We assume  $n$  voters are eligible to vote, and all do.

Voters have preferences on the set of political parties  $A$ . We will denote the parties by  $a_1, \dots, a_m$ . Every voter has a favourite party, a second favourite, and so on. No voter is indifferent between any two parties. Every voter's preferences can then be represented as a linear order on  $A$ . Let  $\mathcal{L}(A)$  be the set of all possible linear orders. The Cartesian product  $\mathcal{L}(A)^n$  will then represent preferences of the whole society. Elements of this Cartesian product are called *profiles*. The collection of all ballot papers will also be a profile. At the ballot box, voters do not necessarily rank the parties in the order of their sincere preference.

We assume each voter forms an expectation of what will transpire at the election. We follow Cox and Shugart and assume that these expectations "are publicly generated - by, for example, polls and newspapers analysis" of the parties' prospects - "so that diversity of opinion in the electorate is minimised" ([10], page 303).

The result of the election will be a parliament. Any parliament can be represented by a

point in the simplex

$$S^m = \left\{ (x_1, \dots, x_m) \mid \sum_{i=1}^m x_i = 1 \right\},$$

where  $x_i$  is the fraction of the seats the  $i$ th party wins at the election. In this paper we will ignore rounding and assume that a party can win any portion of the  $m$  seats. Rounding (or apportionment) is an important issue, but its necessity and its consequences have been analysed elsewhere ([10, 21]). We exclude consideration of rounding in order to focus more directly on other causes of manipulative behaviour. We presume that every party decides on a party list before the election, i.e. ranks its candidates in a certain order with no ties. After the fractions of the seats each party has won is known, the composition of the parliament is decided on the basis of those party lists. If a party is allowed to have  $k$  MPs then the first  $k$  candidates from the party list become MPs.

A *parliament choosing rule* is employed to calculate the distribution of seats in the parliament. A parliament choosing rule is a composite of a *score function* and a *seat allocation rule*.

Given a profile  $R = (R_1, \dots, R_n)$  and a set of alternatives  $A$ , a score function assigns to each  $a_i \in A$  a real number. The greater this number, the better  $a_i$  is supposed to have done. There are a wide variety of score functions ([18] has a comprehensive list of them). In this paper we will work with *normalised positional score functions*.

Let  $w_1 \geq w_2 \geq \dots \geq w_m = 0$  be  $m$  real numbers which we shall refer to as *weights*, and let  $\mathbf{w} = (w_1, \dots, w_m)$ . Let  $\mathbf{v} = (i_1, \dots, i_m)$ , where  $i_k$  indicates the number of voters that stated that they rank alternative  $a$   $k$ th best. Then, given a profile  $R = (R_1, \dots, R_n)$ , the positional score of alternative  $a$  is given by:

$$sc_{\mathbf{w}}(a) = \mathbf{w} \cdot \mathbf{v} = w_1 i_1 + \dots + w_m i_m.$$

Well known vectors of weights include:

- the Plurality score  $sc_{\mathbf{p}}(a)$ , where  $\mathbf{p} = (1, 0, \dots, 0)$ .
- the Borda score  $sc_{\mathbf{b}}(a)$ , where  $\mathbf{b} = (m - 1, m - 2, \dots, 1, 0)$ .
- the Antiplurality score  $sc_{\mathbf{a}}(a)$ , where  $\mathbf{a} = (1, \dots, 1, 0)$ .

The *vector of normalised positional scores* is given by

$$\mathbf{sc}_{\mathbf{w}} = \frac{1}{\sum_{i=1}^m sc_{\mathbf{w}}(a_i)} (sc_{\mathbf{w}}(a_1), sc_{\mathbf{w}}(a_2), \dots, sc_{\mathbf{w}}(a_m)).$$

Clearly,  $\mathbf{sc}_{\mathbf{w}} \in S^m$ . In reality, only the Plurality score has been used in the systems of proportional representation. Nevertheless we do not want to restrict our generality here as other scores may be considered in the future (e.g. Brams and Potthoff [9] suggested to use approval voting scores).

**Definition 1.** A normalised positional score function is a mapping

$$F_s: \mathcal{L}(A)^n \rightarrow S^m,$$

which assigns to every profile its vector of normalised positional scores for some fixed vector of weights  $\mathbf{w}$ .

Given a vector of scores  $\mathbf{sc} \in S^m$ , a seat allocation rule determines the distribution of seats in parliament  $(x_1, \dots, x_m)$ .

**Definition 2.** A seat allocation rule is any mapping

$$F_a: S^m \rightarrow S^m.$$

There are two main examples of such rules.

**Example 1 (Identity seat allocating rule).**  $F_a$  is the identity function, i.e.,  $F_a(\mathbf{x}) = \mathbf{x}$ .

For the next example, we fix a *threshold*, which is a positive real number  $\epsilon$  such that  $0 < \epsilon \leq 1/m$ . We define a *threshold function*  $\delta_\epsilon: [0, 1] \rightarrow [0, 1]$  so that

$$\delta_\epsilon(x) = \begin{cases} 0 & \text{if } x < \epsilon, \\ x & \text{if } x \geq \epsilon. \end{cases}$$

**Example 2 (Threshold seat allocating rule).** Let  $\epsilon$  be a positive real number such that  $0 < \epsilon \leq 1/m$ . Suppose  $\mathbf{x} \in S^m$ . Then we define  $y_i = \delta_\epsilon(x_i)$  and  $z_i = y_i / \sum_{i=1}^m y_i$ . We now set  $F_a(\mathbf{x}) = \mathbf{z}$ , where  $\mathbf{z} = (z_1, \dots, z_m)$ .

**Definition 3.** A parliament choosing rule is a composite  $F = F_a \circ F_s$  of a score function and a seat allocation rule:

$$F_a \circ F_s: \mathcal{L}(A)^n \rightarrow S^m.$$

If the Identity seat allocating rule is employed, we shall refer to the parliament choosing rule as *Pure Proportional Representation*. If a Threshold seat allocating rule is employed, we shall refer to the parliament choosing rule as *Proportional Representation with a Threshold*.

Note that there is a significant difference between parliament choosing rules and choose- $k$  rules (see [7] and references therein). A choose- $k$  rule chooses a  $k$ -element subset of the set of alternatives, which is clearly inappropriate in our context when the parties and not the candidates are alternatives. A parliament choosing rule reveals not only which parties are chosen into the parliament but also how many seats each of them gets there.

### 3 Indices of Voting Power

Choosing a parliament is effectively a fair division problem. It might be thought desirable to allocate each political party a quantity of seats in direct proportion to its support in society. Suppose we do desire this, and suppose we accept that the “support” for a party can be measured by the score it is assigned, by a score function, at an election: then PR is an obvious choice for a parliament choosing rule.

But does PR provide a satisfactory solution to the fair division problem? For sure, each party gets a (roughly) “fair” share of parliamentary representation. However, once the election is over a government has to be formed and a coalition arrangement may need to be negotiated. The political power of each player in the government formation game may not be proportional to either its score or its parliamentary representation. PR can divide seats up “fairly” but it is unlikely to divide power up “fairly.”

We will assume that the distribution of power in a parliament can be computed by a (normalised) *voting power index*. Given a parliament  $(x_1, \dots, x_m)$ , a voting power index  $P$  computes a vector of voting powers  $\mathbf{p} = (p_1, \dots, p_m)$ , where  $p_i$  denotes the proportion of power held by party  $a_i$ . To define a power index we remind to the reader the following standard definitions.

A *weighted voting game* is a simple  $m$ -person game characterised by a non-negative real vector  $(w_1, \dots, w_m)$ , where  $w_i$  represents the  $i$ th player’s voting weight and a quota  $q$  which is the quota of votes necessary to establish a winning coalition, that is, such a coalition  $C$  for which  $\sum_{i \in C} w_i > q$ .

Given the parliament, the formation of the government is a weighted voting game with weights  $x_1, \dots, x_m$  and the quota  $\frac{1}{2}$ , where the players are the parties and their weights are the fractions of their seats. Thus our set of players in this situation is the set of parties  $A = \{a_1, \dots, a_m\}$ .

Let  $N = \{1, 2, \dots, n\}$  and  $v = (N, W)$  be a simple  $n$ -person game with  $W \subseteq 2^N$  being the set of all winning coalitions. A coalition  $C$  is called a *minimal winning coalition* if  $C \in W$  and  $C \setminus \{i\} \notin W$  for all  $i \in C$ . The set of all minimal winning coalitions we will denote as  $MW$ . A party in the parliament is called *dummy* if it does not belong to any of the minimal winning coalitions.

**Definition 4.** Any mapping  $P: S^m \rightarrow S^m$  is called a *voting power index* if the following conditions hold. Suppose  $\mathbf{p} = P(\mathbf{x})$ , then

*PI1.* If  $a_i$  is a dummy, then  $p_i = 0$ ,

*PI2.* If the set of minimal winning coalitions of parliament  $\mathbf{x}$  is the same as the set of minimal winning coalitions of the parliament  $\mathbf{y}$ , then  $P(\mathbf{x}) = P(\mathbf{y})$ .

This definition follows Holler and Packel’s definition of a power index for games [17]. Allingham [2] requires also a monotonicity condition. However the Deegan-Packel index [14] and the Public Good Index [17] do not satisfy the monotonicity requirement and we do not include it.

Classic examples of voting power indices are the Banzhaf (Bz) and Shapley-Shubik (S-S) indices (see [4, 6, 22]). They count in two different ways how many times a player is critical for some winning coalition. According to Felsenthal and Machover, these two indices “have, by and large, been accepted as valid measures of a priori voting power. Some authors have a preference for one or another of these two indices; many regard them as equally valid. Although other indices have been proposed — ... — none has achieved anything like general recognition as a valid index.” [16], page 9. However, non-monotonic indices also have their justification in Riker’s “size principle” [19], which says that “... participants create coalitions just as large as they believe will ensure winning and no larger” (p. 47). Counting how many times a player is critical for some minimal winning coalition leads to the aforementioned non-monotonic indices. The book [23] provides an excellent introduction to voting power indices.

It is worth pointing out that more seats do not necessarily translate into more power. For instance, compare the parliaments  $(x_1, x_2, x_3) = (98/100, 1/100, 1/100)$  and  $(x_1, x_2, x_3) = (51/100, 48/100, 1/100)$ ; party  $a_2$  has no more power in the second than in the first. Moreover, the results of Fishburn and Brams [15] can be interpreted as showing that parties’ powers are fairly insensitive to their number of seats.

## 4 Voters

We do not assume that the  $n$  voters participating in the parliamentary election are (directly) “policy-motivated”, nor that they are (directly) concerned with the distribution of seats in the post-election parliament. Instead, we assume that voters are primarily concerned with the amount of power each of the different parties gain.

We assume that each voter has in mind one particular power index (let us say the  $i$ th voter has in mind  $P_i$ ). We assume that each voter is able to rank all possible vectors of power that the index they have in mind could produce, and that this ranking is consistent with their preferences over the set of political parties. In short, the  $i$ th voter has an order  $\succeq_i$  on  $S^m$  consistent with his or her preference order on  $A$ . More precisely, we assume that the  $i$ th voter has a vector of utilities  $\mathbf{u}_i = (u_1^{(i)}, \dots, u_m^{(i)})$ , normalised so that  $\sum_{j=1}^m u_j^{(i)} = 1$  and  $\min_j u_j^{(i)} = 0$ , such that:

- the  $i$ th voter prefers party  $a_j$  to party  $a_k$  iff  $u_j^{(i)} > u_k^{(i)}$ ;
- given any two vectors of power indices  $\mathbf{p} = (p_1, \dots, p_n)$  and  $\mathbf{q} = (q_1, \dots, q_m)$  we have  $\mathbf{p} \succeq_i \mathbf{q}$  iff  $\mathbf{p} \cdot \mathbf{u}_i \geq \mathbf{q} \cdot \mathbf{u}_i$ , where  $\cdot$  is the dot product in  $\mathbb{R}^m$ .

Hence we assume that the two preference orders (on  $A$  and on  $S^m$ ) belonging to the  $i$ th voter are encapsulated in her utility vector  $\mathbf{u}_i$ . Two voters  $i$  and  $j$  will be said to be of the same *type* iff their power indices are the same and  $\succeq_i = \succeq_j$  on the range of  $P_i = P_j$ .

**Example 3.** *If we denote the strict preference component of  $\succeq_i$  as  $\succ_i$ , and the  $i$ th voter prefers  $a_1$  to  $a_2$  to  $a_3$ , etc., then we must have*

$$(1, 0, \dots, 0) \succ_i (0, 1, \dots, 0) \succ_i \dots \succ_i (0, 0, \dots, 1). \quad (1)$$

Fix a voter  $i$ . Set  $U_i(j)$  equal to this voter’s  $j$ th largest utility. We will say that the  $i$ th voter is *uncertainty averse* if the function  $j \mapsto U_i(j)$  is concave down and *uncertainty seeking* if the function  $j \mapsto U_i(j)$  is concave up.

**Example 4.** *Consider the case where  $m = 3$  and a voter prefers  $a_1$  to  $a_2$  to  $a_3$ . Suppose this voter is comparing the vectors of power  $\mathbf{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  and  $\mathbf{q} = (0, 1, 0)$ . The vector  $\mathbf{p}$  corresponds to a post-election situation where none of the three parties has an outright majority, and a coalition government will need to be formed. If a voter anticipates, prior to the election, that  $\mathbf{p}$  will be the outcome, then she may be uncertain about the composition of the next government. The vector  $\mathbf{q}$  corresponds to a post-election situation where party  $a_2$  has total power, and can form a government by itself. A voter of the opinion that  $\mathbf{q}$  will be the outcome of the election will have no doubt as to the composition of the next government. A voter with the preferences described will rank  $\mathbf{p}$  over  $\mathbf{q}$  if she is uncertainty seeking, or  $\mathbf{q}$  over  $\mathbf{p}$  if she is uncertainty averse.*

Let now  $L$  be a linear order on the set of alternatives. We define one additional relation  $\gg_L$  on the power indices. For any two vectors  $\mathbf{p}$  and  $\mathbf{q}$  we write  $\mathbf{p} \gg_L \mathbf{q}$  if  $\mathbf{p} \succ \mathbf{q}$  for any voter whose type is consistent with the linear order  $L$ . This relation is not complete and some  $\mathbf{p}$  and  $\mathbf{q}$  will be incomparable.

**Example 5.** Consider again the case where  $m = 3$  and a voter prefers  $a_1$  to  $a_2$  to  $a_3$  (denote this linear order by  $L$ ). We will have

$$(1, 0, 0) \gg_L (0, 1, 0) \gg_L (0, 0, 1), \quad \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \gg_L (0, 0, 1).$$

The latter relation is true since all voters who prefer  $a_1$  to  $a_2$  to  $a_3$  will rank  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  over  $(0, 0, 1)$  regardless of their type.

## 5 The manipulability of proportional representation

**Definition 5.** Let  $R$  and  $R'$  be two profiles obtained through full ballots of society. Suppose that there is a group  $G = \{v_{i_1}, \dots, v_{i_k}\}$  of voters such that  $R$  is obtained when all members of this group vote sincerely, and submit the same linear order  $L$ , and  $R'$  is obtained when all members of this group are coordinating to vote insincerely and submit a linear order  $L' \neq L$ . We say that this misreporting is a weak manipulation if there is a type consistent with  $L$  for which

$$P_i(F(R')) \succ_i P_i(F(R)), \quad \text{for all } i \in \{i_1, \dots, i_k\},$$

*i.e.* if all members of the group are of a certain type, they all benefit from the misrepresentation. We say that this misreporting is a strong manipulation if

$$P_i(F(R')) \gg_L P_i(F(R)), \quad \text{for all } i \in \{i_1, \dots, i_k\},$$

*i.e.* voters of any type consistent with  $L$  will benefit from it.

In this paper we consider only susceptibility to *micro manipulation*, when a small percentage of the voters try to coordinate their efforts. This term was coined by Donald Saari and we refer the reader to [20] for more justification of the concept. Roughly speaking,  $F$  is micro (weakly or strongly) manipulable if, as  $n \rightarrow \infty$ , the manipulating group may consist of an arbitrary small fraction of the society.

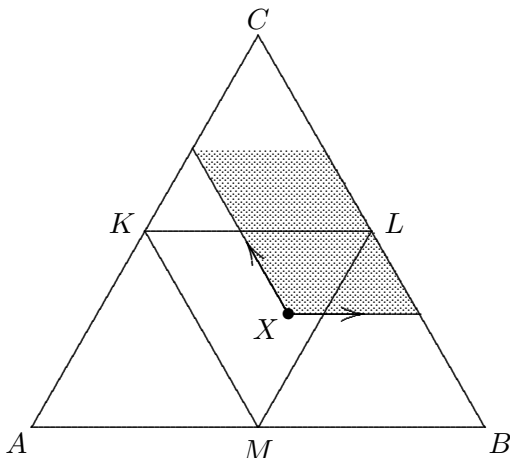
Our results are obtained for the case  $m = 3$ . In fact, this is the main case. It is clear that if we are able to demonstrate manipulability of a parliament choosing rule for  $m = 3$  parties, it will be manipulable for any  $m \geq 3$ . Austin-Smith and Banks [3], and Baron and Diemeier [5] also assume  $m = 3$ .

For ease of exposition we rename parties  $a_1, a_2, a_3$  as  $A, B, C$ , respectively. If a voter prefers  $a_1$  to  $a_2$  to  $a_3$ , we will denote this as  $A > B > C$ .

**Theorem 1.** Let the parliament choosing rule be pure PR. Then the rule is always weakly manipulable but never strongly manipulable. Moreover,

1. If  $\mathbf{w} = \mathbf{a}$ , *i.e.* for the antiplurality score, the rule is not manipulable by uncertainty averse voters.
2. If  $\mathbf{w} = \mathbf{p}$ , *i.e.* for the plurality score, the rule is not manipulable by uncertainty seeking voters.

*Proof.* Consider the triangle  $S^3$  (below). Let points in  $S^3$  be characterised by their barycentric co-ordinates. There will be a one to one correspondence between the set of all possible vectors of normalised positional scores and points in  $S^3$ . For example  $\mathbf{sc} = (1, 0, 0)$ , which would occur if party A secured all the available score, corresponds to point A. To give another example,  $\mathbf{sc} = (\frac{1}{2}, 0, \frac{1}{2})$ , which would occur if A and C secured an equal portion of score while B scored nil, corresponds to point K.



Associated with every possible vector of scores  $\mathbf{sc} = (x_1, x_2, x_3)$  (and thus with every point in  $S^3$ ) will be a parliament  $(x_1, x_2, x_3)$ , and associated with every parliament will be a vector of voting power indices,  $\mathbf{p} = (p_1, p_2, p_3)$ . For example with  $\mathbf{sc} = (1, 0, 0)$  (and point A), is associated  $(x_1, x_2, x_3) = (1, 0, 0)$  and  $\mathbf{p} = (1, 0, 0)$ . With  $\mathbf{sc} = (\frac{1}{2}, 0, \frac{1}{2})$  (and with point K) is associated  $(x_1, x_2, x_3) = (\frac{1}{2}, 0, \frac{1}{2})$  and  $\mathbf{p} = (\frac{1}{2}, 0, \frac{1}{2})$ . In what follows we will refer to a vector of scores and its corresponding point in  $S^3$  as if they were the same object.

The triangle  $S^3$  has been partitioned into regions in which the vector of voting power indices associated with each point is the same (axiom PI2). Whenever the vector of scores falls strictly inside one of the triangles  $AKM$ ,  $MKL$ ,  $KLC$  then  $\mathbf{p}$  will be  $(1, 0, 0)$ ,  $(0, 1, 0)$ , or  $(0, 0, 1)$ , respectively since two players (parties) in these cases will be dummies (axiom PI1). Should the vector of scores fall inside the inner triangle, then  $\mathbf{p}$  will equal  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  (axiom PI2 again).

It is not important for this proof but we note that should the vector of scores fall on the perimeter of the inner triangle (excluding points  $M$ ,  $K$ , and  $L$ ) the vector of power indices may depend on the index of voting power used. For example, the vector of Bz power it will be either  $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$  or some permutation thereof, and the vector of S-S power will be  $(\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$  or, again, some permutation of.

Without loss of generality, let us consider a voter with preference  $A > B > C$  who believes that if she votes sincerely, the outcome — in terms of scores — will correspond to the point  $X$ . Irrespective of the positional scoring rule, by voting insincerely she cannot improve the score of  $A$ , nor worsen the score of  $C$ . If she votes insincerely, she will expect the vector of scores to fall in the shaded area. By insincerely reporting her preferences to be  $B > A > C$ , she will move the vector of scores she expects horizontally east. This she can do so long as the score function is not antiplurality. By insincerely reporting  $A > C > B$ , she moves the vector of scores she



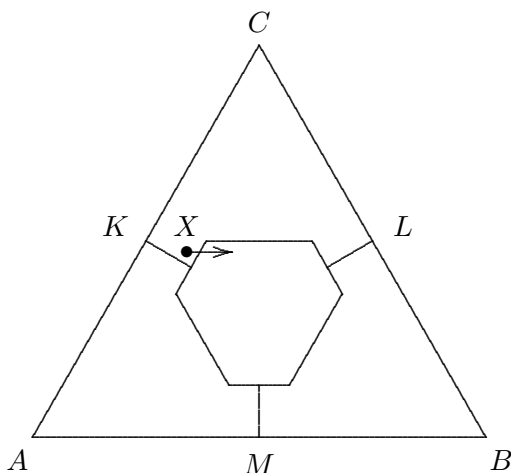
expects north west, parallel to  $BC$ , and this misrepresentation is possible except in the event the score function is plurality.

A small group of voters all of whom have preference  $A > B > C$  cannot escape from the region inside  $KLC$ . They would not wish to escape into  $KLC$ , nor out of  $AKM$ . But if they were risk averse, they would seek, by voting strategically, to move the expected vector of scores from inside  $MKL$  (or from on segment  $ML$ ) to inside  $MBL$ . If they were risk seeking, they would be keen to move the expected vector of scores the other way. In either case, if the vector of scores they expect to transpire if they vote sincerely is “close” enough to  $ML$ , and if the score function permits, an incentive to manipulate exists. It is not true, however, that all voters of a particular type will have an incentive to manipulate in the same fashion, hence the manipulative opportunities are only weak. It is interesting to note that if a group of voters with preference  $A > B > C$  expect that if they all vote sincerely the vector of scores will lie “in the vicinity of  $ML$ .”, the risk averse and risk seeking members of this group would then attempt to manipulate against each other, even though they have identical preferences on the set of parties.  $\square$

We now show that the introduction of a threshold creates opportunities for strong manipulation.

**Theorem 2.** *Let the parliament choosing rule be proportional representation rule with a threshold. Then the rule is strongly manipulable iff  $\mathbf{w} = \mathbf{a}$ .*

*Proof.* The introduction of a threshold changes the shape of the regions in which the associated vector of power indices is constant. The central region, in which  $\mathbf{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ , becomes a hexagon:



Suppose that a small group of voter with preference  $A > B > C$  believe that if they vote sincerely the outcome will correspond to the point  $X$ . At this point,  $B$  does not score highly enough to overcome the threshold. If at the election this group insincerely state their preferences to be  $B > A > C$  — which they can do so long as the score function is not antiplurality — they may be able to push  $B$  over the threshold, and move the expected vector of scores inside the hexagon. When this group votes truthfully, the vector of voting power is anticipated to be

$(0, 0, 1)$ . Untruthful voting could bring about the vector of voting power  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ . This is an unambiguously better prospect for all voters with preference  $A > B > C$ , regardless of their vector of utilities: hence the introduction of a threshold can create opportunities for strong manipulation.  $\square$

## 6 The 2005 New Zealand General Election

The election took place September 17th. The NZ electoral system is MMP, which has a PR component. If a party does not win an electoral seat, then it must win at least five percent of the “party vote” (the PR component) in order to be allocated seats. We model the election using a “PR with 5% threshold” Parliament choosing rule. We restrict our attention to four of the parties. The two opinion polls closest to the election gave the following results:

Poll	Date	Labour	National	NZ First	Greens
TVNZ Colmar Brunton	15 September	38%	41%	5.5%	5.1%
Herald Digipoll	16 September	44.6%	37.4%	4.5%	4.6%

Results of previous polls are available on [1]. At the time of the election, it was felt that if no party won an outright majority then the Greens and Labour would prefer to coalesce with each other rather than with National or NZ First. It was not clear if NZ First would prefer to enter into a coalition with Labour or National. A Labour-National coalition was thought highly unlikely.

Anecdotal evidence (reports to the authors) has suggested that some voters with preferences

Labour > Greens > National > NZ First or Labour > Greens > NZ First > National

may have cast their vote for the Greens. Their motivation for doing this was their desire “to give Labour more choices in forming a coalition government.” Clearly they wanted to increase Labour bargaining power in forming a coalition government, however, it is unlikely that they had a specific power index in mind.

At the election the Greens received 5.3% of the party vote, while Labour got 41.1%. National received 39.1% and NZ First 5.7%. In the post-election parliament the S-S powers of these four parties were Labour 0.3238, National 0.2619, NZ First 0.1429, and the Greens 0.1095. Suppose that the Greens had received just 4.9% of the party vote while Labour got 41.5%, ceteris paribus. Then the S-S powers of these four parties would have been Labour 0.3476, National 0.2476, NZ First 0.1810, and the Greens nil.

Consider a voter with one of the two above preferences, who attributes zero or negligibly small utilities to the powers of National and NZ First, and who uses the S-S power index to assess parliaments. Such a voter would prefer the actual outcome to the hypothesised one provided his utilities comply with  $0.217u_{\text{Labour}} < u_{\text{Greens}}$ . This manipulation is weak and this is why a only small percentage of voters with the above preferences voted insincerely.

What we have not explained yet is how the Labour voters manage to coordinate their votes so that not too many of them divert from sincere preference revelation. Indeed, if we assume that all Labour voters are voting power maximisers, this question will remain unexplained. In

reality it is probably only a small percentage of voters that are mindful about the voting power, with most of them being straightforward seat maximisers.

## 7 Conclusion

This paper has presented a new model of voter behaviour under methods of proportional representation. We showed that if voters are mindful of how the voting power will be distributed in the post-election parliament, then incentives to vote insincerely will exist under any method of PR. We showed that introducing a threshold could encourage greater numbers of voters to vote strategically in the same manner. We showed that attitudes to uncertainty may influence their incentives to vote insincerely.

Our main argument may be summarised thus: with each possible profile is associated a parliament, and with each parliament a vector of power indices. Let  $F = F_a \circ F_s$  be the parliament choosing rule. Let us consider a voter whose power index is  $P$ . By changing her vote, a voter can shift the value of  $F_s(R)$  by a tiny amount. However this may result in significant changes in  $P(F(R))$  since  $F_a$  may be discontinuous (threshold!). Even if  $F_a$  is continuous (as in pure PR), the power index is always discontinuous which leads to the discontinuity of the function  $R \mapsto P(F(R))$ . Where this latter function is discontinuous a small alteration to a profile may result in a large change in the vector of power indices, and this creates incentives for insincere voting.

Questions this paper raises that future research could address include: How do incentives to vote strategically vary with the choice of positional scoring rule? What if the scoring rule is not positional?

## References

- [1] New Zealand general election, 2005. Wikipedia. The Free Encyclopedia. <http://www.wikipedia.org/>
- [2] Allingham, M.G. (1975) Economic Power and Value of Games. *Zeitschrift für Nationalökonomie* **35**: 293–299.
- [3] Austin-Smith, D. and Banks, J. (1988) Elections, Coalitions, and Legislative Outcomes. *American Political Science Review* **82**: 405–422
- [4] Banzhaf, J.F. (1965) Weighted Voting does not work: A mathematical analysis. *Rutgers Law Review* **35**: 317–343
- [5] Baron, D.P. and Diermeier D. (2001) Elections, Governments, and Parliaments in Proportional Representation Systems. *The Quarterly Journal of Economics* **116**: 933–967
- [6] Brams, S.J. (1975) *Game Theory and Politics*, New York.
- [7] Brams, S.J. and Fishburn, P.C. (2002) Voting Procedures. In: Arrow, Sen and Suzumura (eds.) *Handbook of Social Choice and Welfare*, Volume 1, North-Holland.

- [8] Brams, S.J. and Fishburn, P.C. (1995) When Size is a Liability? Bargaining Power in Minimal Winning Coalitions. *Journal of Theoretical Politics* **7**: 301–316.
- [9] Brams, S. and Potthoff, R. (1998) Proportional Representation: Broadening the Options. *Journal of Theoretical Politics* **10**: 147-178.
- [10] Cox, G.W. and Shugart, M.W. (1996) Strategic Voting Under Proportional Representation. *The Journal of Law, Economics, and Organisation* **12**: 299-324.
- [11] De Sinopoli, F. and Iannantuoni, G. (2001) A Spatial Voting Model where Proportional Rule Leads to Two-Party Equilibria. Research Paper Series 31, Tor Vergata University, CEIS.
- [12] De Sinopoli, F. and Iannantuoni, G. (2002) Some Results on Strategic Voting and Proportional Representation with Multidimensional Policy Space. Universidad Carlos III De Madrid. Departamento de Economia. Working Paper 02-57. Economics Series 21.
- [13] De Sinopoli, F. and Iannantuoni, G. (2005) Extreme Voting under Proportional Representation: The Multidimensional Case. University of Cambridge. Faculty of Economics. Working Paper 0531
- [14] Deegan, J., Jr. and Packel, E.W. (1979) A New Index of Power for Simple  $n$ -Person Games. *International Journal of Game Theory* **7**:113–123.
- [15] Fishburn, P.C. and Brams S.J. (1996) Minimal winning coalitions in weighted-majority voting games. *Social Choice and Welfare* **13**: 397–417
- [16] Felsenthal, D.S. and Machover, M. (1998) *The Measurement of Voting Power*. Edward Elgar Publishing.
- [17] Holler, M.J. and Packel, E.W. (1983) Power, Luck and the Right Index. *Zeitschrift für Nationalökonomie* **43**: 21–29.
- [18] McCabe-Dansted, J.C., Slinko, A.M. (2005) Exploratory Analysis of Similarities between Social Choice Rules, Group Decision and Negotiation, Springer online, p1-31. <http://dx.doi.org/10.1007/s10726-005-9007-5>
- [19] Riker, W.H. (1962) *The Theory of Political Coalitions*, New Heaven, CT: Yale University Press.
- [20] Saari, D. (1990) Susceptibility to manipulation. *Public Choice* **64**: 21–41
- [21] Saari, D. (1994) *Geometry of Voting*. Springer-Verlag
- [22] Shapley, L.S. and Shubik, M. (1954) A method for evaluating the distribution of power in a committee system. *American Political Science Review* **48**: 787–792
- [23] Taylor, A.D. (1995) *Mathematics and politics : strategy, voting, power and proof*. Springer-Verlag