On the Order Hereditary Closure Preserving Sum Theorem

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Abstract

The main purpose of this paper is to prove the following two theorems, an order hereditary closure preserving sum theorem and an hereditary theorem:

(1) If a topological property $\mathcal{P}$ satisfies $(\sum')$ and is closed hereditary, and if $\mathcal{V}$ is an order hereditary closure preserving open cover of X and each $V \in \mathcal{V}$ is elementary and possesses $\mathcal{P}$, then X possesses $\mathcal{P}$.

(2) Let a topological property $\mathcal{P}$ satisfy $(\sum')$ and (β), and be closed hereditary. Let X be a topological space which possesses $\mathcal{P}$. If every open subset G of X can be written as an order hereditary closure preserving (in G) collection of elementary sets, then every subset of X possesses $\mathcal{P}$.

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1 Introduction

R.E.Hodel [1] obtained sum theorems and an hereditary theorem for topological spaces. S.P.Arya and M.K.Singal [1, 2] and G.Gao [4] have improved some of Hodel's sum theorems. We provide in this paper further improvements of these theorems.

A topological property $\mathcal{P}$ is said to be hereditary (closed hereditary) if when $\mathcal{P}$ is possessed by a topological space X, it is also shared by every subspace (closed subspace) of X. It is well known that covering properties such as paracompactness, subparacompactness, countable paracompactness, pointwise paracompactness, θ-refinement and collectionwise normality satisfy the following result which is denoted by (β).

(β) : If every open subset of a space X has a property $\mathcal{P}$, then every subset of X has the property $\mathcal{P}$.

Y.Katuta [6] introduced the notion of an order locally finite family of subsets of a topological space. Later G. Gao [4] also introduced the notion of an order hereditary closure preserving family of subsets of a topological space.

A family $\{A_\gamma : \gamma \in \Gamma\}$ of subsets of a topological space X is called *hereditary closure preserving relative to a subspace* A of X if for any $\Gamma' \subset \Gamma$ and any $E_\gamma \subset A_\gamma$ the following is true for all points in A.

$$\overline{\bigcup_{\gamma \in \Gamma'} E_\gamma} = \bigcup \overline{E}_\gamma.$$

Definition 1.1. *(G.Gao [4]) A family* $\{A_\alpha : \alpha < \tau\}$ *(α and τ are ordinal numbers) is defined to be order hereditary closure preserving if for every or-*

dinal number $\beta < \tau$, the family $\{A_\alpha : \alpha < \beta\}$ is hereditary closure preserving relative to A_β.

It is not difficult to see that the following implications are true for a family of subsets of a topological space. However, the converse implications are not true in general.

Proposition 1.2. *Given a family of subsets of a topological space, then*

$$\begin{array}{ccc} locally \quad finite & \Rightarrow & hereditary \quad closure \quad preserving \\ \Downarrow & & \Downarrow \\ \sigma - locally \quad finite & \Rightarrow & \sigma - hereditary \quad closure \quad preserving \\ \Downarrow & & \Downarrow \\ order \quad locally \quad finite & \Rightarrow & order \quad hereditary \quad closure \quad preserving \end{array}$$

Definition 1.3. *(R.E.Hodel [5]) Let N be the set of all positive integers. An open subset V of a topological space is called an elementary set if $V = \bigcup_{i=1}^{\infty} V_i$, where each V_i is open and $\overline{V_i} \subset V$ for all $i \in N$.*

The following two lemmas show that each open F_σ set in a normal space is exactly an elementary set.

Lemma 1.4. *Every elementary set in a topological space is an open F_σ set.*

Proof. Suppose the open subset V of a topological space is an elementary set, then $V = \bigcup_{i=1}^{\infty} V_i$, V_i is open and $\overline{V_i} \subset V$ for all $i \in N$. Hence $\bigcup_{i=1}^{\infty} \overline{V_i} \subset V$. On the other hand, $V_i \subset \overline{V_i}$ for all $i \in N$, so $V = \bigcup_{i=1}^{\infty} V_i \subset \bigcup_{i=1}^{\infty} \overline{V_i}$. Therefore, $V = \bigcup_{i=1}^{\infty} \overline{V_i}$, it follows that V is an open F_σ set. □

Lemma 1.5. *Every open F_σ subset of a normal space is an elementary set.*

Proof. Let V be an open F_σ set of a normal space X, then $V = \bigcup_{i=1}^{\infty} W_i$, W_i is closed and $W_i \subset V$ for all $i \in N$. By the normality of X, for each W_i there exists an open set V_i such that $W_i \subset V_i \subset \overline{V_i} \subset V$. Thus, $V = \bigcup_{i=1}^{\infty} W_i \subset \bigcup_{i=1}^{\infty} V_i$ and $\bigcup_{i=1}^{\infty} V_i \subset V$. That is $V = \bigcup_{i=1}^{\infty} V_i$ where each V_i is open and $\overline{V_i} \subset V$ for all $i \in N$. Therefore V is an elementary set. □

Notice that an open F_σ set may fail to be an elementary set in non-normal spaces, as the following example shows.

Example Let X be the set N of all positive integers with cofinite topology. Then X is a T_1 space which is not a normal space. Take the set $V = N/\{1,2,3\}$, then V is an open set. Furthermore, $V = \bigcup_{i=4}^{\infty}\{i\}$. Since X is a T_1 space, each singleton $\{i\}$ is a closed subset, so that V is an open F_σ set. For any subset S of X we have

$$\overline{S} = \begin{cases} S & \text{if } S \text{ is finite,} \\ X & \text{if } S \text{ is infinite.} \end{cases}$$

Since every non-empty open subset S of X is infinite, for every open subset S of V,

$$\overline{S} = X \nsubseteq V.$$

So V is not an elementary set.

We say that a topological property $\mathcal{P}$ satisfies the Locally Finite Closed Sum Theorem if the following is satisfied and denote it by $(\sum)$.

$(\sum)$: Let $\{F_\alpha : \alpha \in A\}$ be a locally finite closed cover of a topological space X and let each F_α possess a property $\mathcal{P}$, then X possesses the property $\mathcal{P}$.

We say that a topological property $\mathcal{P}$ satisfies the Hereditary Closure Preserving Closed Sum Theorem if the following is satisfied and denote it by $(\sum')$.

$(\sum')$: Let $\{F_\alpha : \alpha \in A\}$ be an hereditary closure preserving closed cover of a topological space X and let each F_α possess a property $\mathcal{P}$, then X possesses the property $\mathcal{P}$.

Observe that $(\sum') \Rightarrow (\sum)$.

For example, if the topological property $\mathcal{P}$ is one of paracompactness, sub-paracompactness, pointwise paracompactness, meso-compactness, θ-refinement, weak θ-refinement and ortho-compactness, then the property $\mathcal{P}$ satisfies $(\sum)$. If the topological property $\mathcal{P}$ is one of paracompactness and T_1 meso-compactness, then the property $\mathcal{P}$ satisfies $(\sum')$.

2 A Sum Theorem

In this section, we assume that the topological property $\mathcal{P}$ satisfies $(\sum')$ (hence $(\sum)$) and is closed hereditary.

Theorem 2.1. *Let $\mathcal{V} = \{V_\alpha : \alpha < \tau\}$ be an order hereditary closure preserving open cover of a topological space X, and let each V_α be an elementary set which possesses a topological property $\mathcal{P}$. Then X possesses the topological property $\mathcal{P}$.*

Proof. Since each V_α is an elementary set and possesses the property $\mathcal{P}$,

$$V_\alpha = \bigcup_{i=1}^{\infty} V_{\alpha,i}, \qquad \overline{V_{\alpha,i}} \subset V_\alpha, \quad \alpha < \tau, i \in N, \tag{1}$$

where each $V_{\alpha,i}$ is an open set. Then the closed set $\overline{V_{\alpha,i}}$ possesses the property $\mathcal{P}$.

For each $i \in N$, let

$$\mathcal{V}_i = \{V_{\alpha,i} : \alpha < \tau\}.$$

For each $\alpha < \tau$, let

$$F_{0,i} = \overline{V_{0,i}}, \qquad F_{\alpha,i} = \overline{V_{\alpha,i}} - \bigcup_{\beta<\alpha} V_\beta, \quad 0 < \alpha < \tau. \tag{2}$$

Then each closed set $F_{\alpha,i}$ possesses the property $\mathcal{P}$. And we claim that the family $\{F_{\alpha,i} : \alpha < \tau\}$ is an hereditary closure preserving collection.

Without loss of generality, for each $\alpha < \tau$, let $A_{\alpha,i} \subset F_{\alpha,i}$, we need to prove

$$\overline{\bigcup_{\alpha<\tau} A_{\alpha,i}} = \bigcup_{\alpha<\tau} \overline{A}_{\alpha,i}.$$

Obviously, it is enough to prove

$$\overline{\bigcup_{\alpha<\tau} A_{\alpha,i}} \subset \bigcup_{\alpha<\tau} \overline{A}_{\alpha,i}. \tag{3}$$

Suppose $x \in \overline{\bigcup_{\alpha<\tau} A_{\alpha,i}}$, since $\mathcal{V}$ is a cover of X, we may assume $x \in V_{\beta_0}$. Now the inequality (3) can be expressed in another way:

$$\left(\overline{\bigcup_{\alpha<\beta_0} A_{\alpha,i}}\right) \cup \overline{A}_{\beta_0,i} \cup \left(\overline{\bigcup_{\beta_0<\alpha<\tau} A_{\alpha,i}}\right)$$
$$\subset \left(\bigcup_{\alpha<\beta_0} \overline{A}_{\alpha,i}\right) \cup \overline{A}_{\beta_0,i} \cup \left(\bigcup_{\beta_0<\alpha<\tau} \overline{A}_{\alpha,i}\right). \tag{4}$$

According to (2), $V_{\beta_0} \cap F_{\alpha,i} = \emptyset$, $\beta_0 < \alpha < \tau$. So $X - V_{\beta_0} \supset \bigcup_{\beta_0<\alpha<\tau} F_{\alpha,i}$. Since $X - V_{\beta_0}$ is a closed set, then $X - V_{\beta_0} \supset \overline{\bigcup_{\beta_0<\alpha<\tau} F_{\alpha,i}}$, that is $x \notin \overline{\bigcup_{\beta_0<\alpha<\tau} F_{\alpha,i}}$.

Therefore $x \notin \overline{\bigcup_{\beta_0 < \alpha < \tau} A_{\alpha,i}}$. If $x \in A_{\beta_0,i}$, the inequality (4) is satisfied. We may assume $x \in \overline{\bigcup_{\alpha < \beta_0} A_{\alpha,i}}$. Since $\mathcal{V}$ is order hereditary closure preserving, $\{V_\alpha : \alpha < \beta_0\}$ is hereditary closure preserving at every point of V_{β_0}. Notice that $x \in V_{\beta_0}$, thus $x \in \bigcup_{\alpha < \beta_0} \overline{A}_{\alpha,i}$. So the inequality (2) is proved.

Let $F_i = \bigcup_{\alpha < \tau} F_{\alpha,i}$, then F_i possesses the property $\mathcal{P}$ by applying $(\sum')$, for all $i \in N$.

For each $i \in N$, let

$$\mathcal{V}_i^* = \bigcup_{\alpha < \tau} \{V_{\alpha,i}\},$$

then $\mathcal{V}_i^* \subset F_i$ by the well order property. Hence $\{\mathcal{V}_i^*\}$ and $\{F_i\}$ are open covers and closed covers of the space X respectively.

Finally, let

$$H_1 = F_1, \qquad H_i = F_i - \bigcup_{j=1}^{i-1} \mathcal{V}_j^*, \quad i = 2, 3, ...$$

then $\{H_i\}$ is a locally finite closed cover of X and each H_i possesses the property $\mathcal{P}$. It follows from $(\sum)$ that X possesses the property $\mathcal{P}$. □

Apply Proposition 1.2 to Theorem 2.1 above, to obtain the following two corollaries.

Corollary 2.2. *(S.P.Arya and M.K.Singal [2]) Let $\mathcal{V}$ be a σ-hereditary closure preserving cover of a topological space X and each $V \in \mathcal{V}$ be an elementary set which possesses a topological property $\mathcal{P}$, then X possesses the property $\mathcal{P}$.*

Corollary 2.3. *(R.E.Hodel [5]) Let $\mathcal{V}$ be a σ-locally finite cover of a topological space X and each $V \in \mathcal{V}$ be an elementary set which possesses a* *topological property $\mathcal{P}$, then X possesses the property $\mathcal{P}$.*

3 Two Hereditary Theorems

We assume that the topological property $\mathcal{P}$ in this section satisfies $(\sum')$ (hence $(\sum)$), (β) and is closed hereditary.

Theorem 3.1. *Let X be a topological space which possesses a topological property $\mathcal{P}$. If every open subset G of X can be written as an order hereditary closure preserving (in G) collection of elementary sets, then every subset of X possesses the property $\mathcal{P}$.*

Proof. Let $\mathcal{V} = \{V_\alpha : \alpha < \tau\}$ be order hereditary closure preserving at every point of G, and let $\mathcal{V}^* = \bigcup_{\alpha<\tau} V_\alpha = G$, where each $V_\alpha, \alpha < \tau$ is an elementary subset of X. We may assume

$$V_\alpha = \bigcup_{i=1}^{\infty} V_{\alpha,i}, \quad \overline{V_{\alpha,i}} \subset V_\alpha, \quad \alpha < \tau$$

where each $V_{\alpha,i}$ is an open set. Let

$$F_{\alpha,1} = \overline{V_{\alpha,1}}, \quad F_{\alpha,i} = \overline{V_{\alpha,i}} - \bigcup_{j<i} V_{\alpha,j}, \quad i = 2, 3, ...$$

Then $\{F_{\alpha,i}\}$ is a locally finite cover of V_α, so it is an hereditary closure preserving cover of V_α. Since each $F_{\alpha,i}$ is a closed subset of X and the property $\mathcal{P}$ is closed hereditary, then each $F_{\alpha,i}$ possesses the property $\mathcal{P}$. According to $(\sum')$, each subspace V_α possesses the property $\mathcal{P}$. Apply Theorem 2.1 to the subspace G, then G possesses the property $\mathcal{P}$. Since (β) holds, then every subset of X possesses the property $\mathcal{P}$. □

Since Lemmas 1.4 and 1.5 state that open F_σ sets are equivalent to elementary sets in a normal space, they give the following theorem immediately.

Theorem 3.2. *Let a normal space X possess a topological property $\mathcal{P}$. If every open subset G of X can be written as an order hereditary closure preserving (in G) collection of open F_σ sets, then every subset of X possesses the property $\mathcal{P}$.*

Definition 3.3. *(C.H.Dowker [3]) A normal space X is totally normal if every open subset G of X can be written as a locally finite (in G) collection of open F_σ sets of X.*

Finally, Theorem 3.2 and Proposition 1.2 imply the following corollary.

Corollary 3.4. *(R.E.Hodel [5]) Let X be a totally normal space and X have a topological property $\mathcal{P}$, then every subset of X has the property $\mathcal{P}$.*

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