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CHARACTER TABLES OF PARABOLIC SUBGROUPS OF THE CHEVALLEY GROUPS OF TYPE G_2

Jianbei An and Shih-Chang Huang
Department of Mathematics, University of Auckland, Auckland, New Zealand

The main aim of this paper is to construct the character tables of the parabolic subgroups of the Chevalley groups $G_2(q)$, where q is a power of a prime $p > 3$.

Key Words: ■. AQ1

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1. INTRODUCTION

Let q be a power of a prime p and $G_2(q)$ the Chevalley group of type G_2 defined over a field of q elements. The character table of $G_2(q)$ has been determined by Enomoto and Yamada (1986) for $p = 2$, by Enomoto (1976) for $p = 3$, and by Chang and Ree (1974) for $p > 3$. The main aim of this article is to construct the character tables of the parabolic subgroups of $G_2(q)$ in the case $p > 3$. Using these tables and the table of Chang and Ree (1974), the second author (Huang, 2005b) verified Dade invariant conjecture for $G_2(q)$ in the case $p > 3$. Together with the work by the first author (An, 2003, Submitted), this completes the verification of the Dade (final) conjecture for $G_2(q)$ in any characteristic.

The approach is similar to that used by Enomoto and Yamada (1986) and by Enomoto (1976), but the calculations are more complicated. Let B be a Borel subgroup, and P, Q maximal parabolic subgroups of $G_2(q)$ containing B. Then an irreducible character of B is either lifted from a linear character of $B/O_p(B)$ or induced by a linear character of a subgroup of B. If $X = P$ or Q, then an irreducible character of X is either induced by a character of B or a linear combination of induced characters from characters of B or two other subgroups of X.

The outline of this article is as follows. We introduce notation and preliminaries in Section 2, and construct irreducible characters of B in Section 3, of P in Section 4, and of Q in Section 5. The conjugacy classes and character tables of B, P, and Q are summarized in tabular form in Appendix A at the end of the article. In these tables, zeros are replaced by dots.

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Address correspondence to Jianbei An, Department of Mathematics, University of Auckland, Auckland, New Zealand; Fax: +64-9-3737457; E-mail: an@math.auckland.ac.nz AQ3

2. NOTATION AND PRELIMINARIES

Let q be a power of a prime $p > 3$, $\mathbb{F}_q$ the field of q elements, and $\mathbb{F}_q^*$ the multiplicative group of $\mathbb{F}_q$. Choose $\epsilon = 1$ or -1 so that $q \equiv \epsilon \pmod 3$ and set $\zeta = (-1)^{\frac{1}{2}(q-1)}$, so that $q \equiv \zeta \pmod 4$. Let κ be a fixed generator of $\mathbb{F}_{q^6}^*$,

$$\sigma = \kappa^{(q+1)(q^3-1)}, \qquad \tau = \kappa^{(q-1)(q^3+1)}, \qquad \theta = \kappa^{q^4+q^2+1},$$
$$\eta = \theta^{q-1}, \qquad \gamma = \theta^{q+1}, \qquad \omega = \theta^{(q^2-1)/3}.$$

Then $\mathbb{F}_{q^2}^* = \langle \theta \rangle$, $\mathbb{F}_q^* = \langle \gamma \rangle$, and $\omega = \eta^{(q-\epsilon)/3}$. Fix an isomorphism from $\mathbb{F}_{q^6}^*$ into the multiplicative group $\mathbb{C}^*$ of complex numbers $\mathbb{C}$, and let $\tilde{x}$ be the image of $x \in \mathbb{F}_{q^6}^*$. Put $\alpha_k = \tilde{\gamma}^k + \tilde{\gamma}^{-k}$, $\beta_k = \tilde{\eta}^k + \tilde{\eta}^{-k}$, $\omega_k = \tilde{\omega}^k + \tilde{\omega}^{-k}$ for an integer k. So $\omega_k = 2$ or -1 according as $k \equiv 0 \pmod 3$ or not.

We summarize some properties of $G = G_2(q)$ from Ree (1961). Let $\Sigma = \{\pm\xi_i, \xi_i - \xi_j \mid 1 \le i, j \le 3, i \ne j\}$ (where $\xi_1 + \xi_2 + \xi_3 = 0$) be the set of roots of a simple Lie algebra of type G_2, and choose $a = \xi_2$, $b = \xi_1 - \xi_2$ for a fundamental system of roots. Then the set Σ^+ of positive roots can be arranged in increasing order such as: $a = \xi_2$, $b = \xi_1 - \xi_2$, $a + b = \xi_1$, $2a + b = -\xi_3$, $3a + b = \xi_2 - \xi_3$, and $3a + 2b = \xi_1 - \xi_3$. For each $r \in \Sigma$, $x_r(t)$, $x_{-r}(t)$ and ω_r are defined in Ree (1961, p. 434). Denote the element $h(\chi)$ of Ree (1961, p. 434) by $h(z_1, z_2, z_3)$ and $h(*^i, *^j, *^{-i-j})$ by $h_*(i, j, -i-j)$, where $\chi(\xi_i) = z_i$ with $z_1z_2z_3 = 1$. Let $X_r = \{x_r(t) \mid t \in \mathbb{F}_q\}$ be the root subgroup of G corresponding to $r \in \Sigma$,

$$U = \prod_{r \in \Sigma^+} X_r = X_a X_b X_{a+b} X_{2a+b} X_{3a+b} X_{3a+2b},$$

$B = N_G(U) = U \times H$, where $H = \{h(z_1, z_2, z_3) \mid z_i \in \mathbb{F}_q^*, z_1z_2z_3 = 1\}$. Then U is a Sylow p-subgroup and B is a Borel subgroup of G. Set $P = \langle B, \omega_a \rangle$ and $Q = \langle B, \omega_b \rangle$. Then P and Q are the maximal parabolic subgroups of G containing B.

The commutator relations are given as follows (cf. Ree, 1961, p. 443):

$$\begin{aligned}
[x_a(t), x_b(u)] &= x_{a+b}(-tu)x_{2a+b}(-t^2u)x_{3a+b}(t^3u)x_{3a+2b}(-2t^3u^2),\\
[x_a(t), x_{a+b}(u)] &= x_{2a+b}(-2tu)x_{3a+b}(3t^2u)x_{3a+2b}(3tu^2),\\
[x_a(t), x_{2a+b}(u)] &= x_{3a+b}(3tu),\\
[x_b(t), x_{3a+b}(u)] &= x_{3a+2b}(tu),\\
[x_{a+b}(t), x_{2a+b}(u)] &= x_{3a+2b}(3tu),\\
[x_r(t), x_s(u)] &= 1, \qquad \text{for all other pairs of roots } r, s \in \Sigma^+.
\end{aligned} \tag{2.1}$$

The actions of $h(z_1, z_2, z_3)$, ω_a, ω_b on $x_r(t)$ are as follows (see Enomoto and Yamada, 1986, p. 324):

$$\begin{array}{cccc}
x & x^{h(z_1,z_2,z_3)} & x^{\omega_a} & x^{\omega_b}\\
x_a(t) & x_a(z_2^{-1}t) & x_{-a}(-t) & x_{a+b}(t)\\
x_b(t) & x_b(z_1^{-1}z_2t) & x_{3a+b}(t) & x_{-b}(-t)\\
x_{a+b}(t) & x_{a+b}(z_1^{-1}t) & x_{2a+b}(t) & x_a(-t)
\end{array} \tag{2.2}$$

$$\begin{array}{llll}
x_{2a+b}(t) & x_{2a+b}(z_3 t) & x_{a+b}(-t) & x_{2a+b}(t) \\
x_{3a+b}(t) & x_{3a+b}(z_2^{-1} z_3 t) & x_b(-t) & x_{3a+2b}(t) \\
x_{3a+2b}(t) & x_{3a+2b}(z_1^{-1} z_3 t) & x_{3a+2b}(t) & x_{3a+b}(-t).
\end{array}$$

For a finite group H and $L \leq H$, let $\langle \cdot, \cdot \rangle_H$ be the scalar product of class functions of H, denote by ${}_L\chi$ a character χ of L and by ${}_L^H\chi$ the induced character from L to H.

Our notation for the parameter sets of conjugacy classes and characters of B, P, and Q is similar to that of Enomoto and Yamada (1986, pp. 328–330). Here $I = J \bmod \{\alpha \equiv \beta\}$ means that I is the set of equivalence classes of J under the relation $\equiv$. In the following definition $c = \frac{1}{3}(q-1)$ and $d = \frac{1}{3}(q+1)$. The parameter set $R_0 = \mathbb{Z}/(q-1)\mathbb{Z}$ is considered to be $\{0, 1, \ldots, q-2\}$. Similar convention is used for S_0 and T_0.

$S = \{\gamma^{2i}\}$ (the set of square elements in $\mathbb{F}_q^*$),

$N = \{\gamma^{2i+1}\}$ (the set of non-square elements in $\mathbb{F}_q^*$),

$$R_0 = \mathbb{Z} \bmod \{i \equiv i + (q-1)\}, \qquad |R_0| = q-1,$$

$$R_1 = R_0 - \left\{0, \frac{1}{2}(q-1)\right\}, \qquad |R_1| = q-3,$$

$$R_1^* = \begin{cases} R_1 & \text{if } \epsilon = -1, \\ R_1 - \{c, 2c\} & \text{if } \epsilon = 1, \end{cases} \qquad |R_1^*| = q - 4 - \epsilon,$$

$${}^2R_1 = R_1 \bmod \{i \equiv -i\}, \qquad |{}^2R_1| = |R_1|/2,$$

$${}^2R_1^* = R_1^* \bmod \{i \equiv -i\}, \qquad |{}^2R_1^*| = |R_1^*|/2,$$

$$R_2 = \{(i,j) \in R_0 \times R_0 \mid i - j \not\equiv 0 \ (\bmod\, q-1)\}, \qquad |R_2| = (q-1)(q-2),$$

$${}^2R_2 = R_2 \bmod \{(i,j) \equiv (j,i)\}, \qquad |{}^2R_2| = |R_2|/2,$$

$$R_3 = \{(i,j) \in R_0 \times R_0 \mid i - 2j \not\equiv 0 \ (\bmod\, q-1)\}, \qquad |R_3| = (q-1)(q-2),$$

$${}^2R_3 = R_3 \bmod \{(i,j) \equiv (i, i-j)\}, \qquad |{}^2R_3| = |R_3|/2,$$

$$R_4 = \{(i,j) \in R_0 \times R_0 \mid i, j, i-j \not\equiv 0 \ (\bmod\, q-1)\}, \qquad |R_4| = (q-2)(q-3),$$

$$R_5 = \{(i,j) \in R_4 \mid i+j, 2i+j, i+2j \not\equiv 0 \ (\bmod\, q-1)\}, \qquad |R_4| = q^2 - 8q + 17 + 2\epsilon,$$

$${}^2R_5 = R_5 \bmod \{(i,j) \equiv (i+j, -j)\}, \qquad |{}^2R_5| = |R_5|/2,$$

$${}^2R_5' = R_5 \bmod \{(i,j) \equiv (j,i)\}, \qquad |{}^2R_5'| = |R_5|/2,$$

$$R_7 = \{(i,j) \in R_0 \times R_0 \mid i-j, 2i+j, i+2j \not\equiv 0 \ (\bmod\, q-1)\}, \qquad |R_7| = q^2 - 5q + 8 + 2\epsilon,$$

$$S_0 = \mathbb{Z} \bmod \{i \equiv i + (q+1)\}, \qquad |S_0| = q+1,$$

$$S_1 = S_0 - \left\{0, \frac{1}{2}(q+1)\right\}, \qquad |S_1| = q-1,$$

$$S_1^* = \begin{cases} S_1 - \{d, 2d\} & \text{if } \epsilon = -1, \\ S_1 & \text{if } \epsilon = 1, \end{cases} \qquad |S_1^*| = q - 2 + \epsilon,$$

$${}^2S_1 = S_1 \bmod \{i \equiv -i\}, \qquad |{}^2S_1| = |S_1|/2,$$

$${}^2S_1^* = S_1^* \bmod \{i \equiv -i\}, \qquad |{}^2S_1^*| = |S_1^*|/2,$$

$$
\begin{aligned}
&T_0 = \mathbb{Z} \bmod \{i \equiv i + (q^2 - 1)\}, && |T_0| = q^2 - 1,\\
&T_1 = \{i \in T_0 \mid i \not\equiv 0 \ (\mathrm{mod}\, q-1)\}, && |T_1| = q(q-1),\\
&{}^2T_1 = T_1 \bmod \{i \equiv qi\}, && |{}^2T_1| = |T_1|/2,\\
&T_2 = \{i \in T_0 \mid i \not\equiv 0 \ (\mathrm{mod}\, q-1)\}, && |T_2| = (q+1)(q-2),\\
&T_3 = T_1 \cap T_2, && |T_3| = (q-1)^2,\\
&{}^2T_3 = T_3 \bmod \{i \equiv qi\}, && |{}^2T_3| = |T_3|/2.
\end{aligned}
$$

Using an approach similar to that of Chang (1968), we classify the conjugacy classes of B, P, Q and list them in Tables A.1, A.3, and A.6 (for details, see Huang, 2005a, 2.2–2.4). The symbols y_i, $i = 0, 1, 2$, in the tables are defined by

$$y_i = \begin{cases} x_b(\eta^i)x_{3a+b}(\eta^{-i})x_{3a+2b}(\omega) & \text{if } \epsilon = -1,\\ x_b(1)x_{3a+b}(\gamma^i) & \text{if } \epsilon = 1. \end{cases}$$

3. THE CHARACTER TABLE OF *B*

In this section, we construct the irreducible characters of B. The $q^2 + 5q + 12 + 4\epsilon$ conjugacy classes of B are given in Table A.1, and the classes $B_0(i)$, $B_1(i)$, $B_2(i)$, $B_3(i)$, and $B_4(i, j)$ occur only when $\epsilon = 1$. Each irreducible character of B is either induced from a linear character of a subgroup of B or lifted from a linear character of $B/O_p(B)$. The character table of B is given by Table A.2. Apart from the values of the character ${}_B\chi_6(k)$ at the class $A_{52}(0)$ are distinct in the two cases $\epsilon = -1$ and 1, the other values of the characters of B are identical in both cases.

We would like to point out that we are not able to describe all values of the characters ${}_B\theta(x)$ ($x \in \mathbb{F}_q^*$ and $\epsilon = 1$), ${}_B\theta(x)$ ($x \in \mathbb{F}_q$ and $\epsilon = -1$), and ${}_B\theta_2(k, l)$ generically, because an inspection of small values of q ($q = 13$) shows, that the values of these characters on the unipotent conjugacy classes A_{51}, $A_{52}(i)$, $A_{53}(t)$, A_{54}, and A_{55} of B depend on x or l and we do not have a generic description of these classes.

Fix a nontrivial linear character $\phi : \mathbb{F}_q \to \mathbb{C}^*$, so (see Isaacs, 1976, Problem (2.1))

$$\sum_{z \in \mathbb{F}_q} \phi(z) = 0 \qquad \text{and} \qquad \sum_{z \in \mathbb{F}_q^*} \phi(z) = -1.$$

For $k, l \in R_0$, let ${}_B\chi_1(k, l)$ be the linear character of B defined by

$$h_\gamma(i, j, -i-j)x_a(t_1)x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6) \mapsto \tilde{\gamma}^{ik+jl}.$$

For $k \in R_0$, define a linear character of the subgroup $C_H(X_a)U$ by

$$h_\gamma(i, 0, -i)x_a(t_1)x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6) \mapsto \tilde{\gamma}^{ik}\phi(t_1).$$

Inducing this character to B, we obtain ${}_B\chi_2(k)$. Similarly, ${}_B\chi_3(k)$, $k \in R_0$, is defined to be the induced character of the following linear character of the subgroup $C_H(X_b)U$:

$$h_\gamma(i, i, -2i)x_a(t_1)x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6) \mapsto \tilde{\gamma}^{ik}\phi(t_2).$$

Let $U_1 = X_{a+b}X_{2a+b}X_{3a+b}X_{3a+2b}$, and let ${}_B\chi_4(k)$, $k \in R_0$ be the characters of B induced by the following linear character of $C_H(X_{a+b})X_bU_1$:

$$h_\gamma(0, i, -i)x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6) \mapsto \tilde{\gamma}^{ik}\phi(t_3).$$

Let ${}_B\chi_5(k)$, $k \in R_0$, be the character of B induced by the following linear character of $C_H(X_{2a+b})X_bU_1$:

$$h_\gamma(i, -i, 0)x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6) \mapsto \tilde{\gamma}^{ik}\phi(t_4).$$

Let ${}_B\chi_6(k)$, $k \in R_0$, be the character of B induced by the following linear character of $C_H(X_{3a+b})X_bU_1$:

$$h_\gamma(-2i, i, i)x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6) \mapsto \tilde{\gamma}^{ik}\phi(t_5).$$

Let ${}_B\chi_7(k)$, $k \in R_0$, be the character of B induced by the following linear character of $C_H(X_{3a+2b})X_aX_{2a+b}X_{3a+b}X_{3a+2b}$:

$$h_\gamma(i, -2i, i)x_a(t_1)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6) \mapsto \tilde{\gamma}^{ik}\phi(t_6).$$

It is no difficult to compute the values of the above characters. We demonstrate how to determine the values of ${}_B\chi_6(k)$, $k \in R_0$, on the conjugacy classes $A_{53}(t)$ of B. The set $\{h(z, 1, z^{-1})x_a(w) : z \in \mathbb{F}_q^*, \omega \in \mathbb{F}_q\}$ is a set of representatives for the right coset of $C_H(X_{3a+b})X_bU_1$ in B and the element $h(z, 1, z^{-1})x_a(w)$ transforms $x_b(t)x_{2a+b}(1)x_{3a+b}(1)$ into

$$x_b(tz^{-1})x_{a+b}(tz^{-1}w)x_{2a+b}(w^2tz^{-1} + z^{-1})x_{3a+b}(z^{-1}(-tw^3 - 3w + 1))x_{3a+2b}(-2w^3z^{-1})$$

by conjugation.

For $0 \leq i \leq 3$, let Ω_i be the subset of $\mathbb{F}_q^*$ consisting of all elements t such that the number of distinct roots of $X^3t + 3X - 1$ in $\mathbb{F}_q$ is i. Then by Huang (2005a, Lemma 2.5.1),

$$|\Omega_i| = \begin{cases} \dfrac{q-\epsilon}{3} & \text{if } t \in \Omega_0, \\ \dfrac{q-2+\epsilon}{2} & \text{if } t \in \Omega_1, \\ 1 & \text{if } t \in \Omega_2, \\ \dfrac{q-6-\epsilon}{6} & \text{if } t \in \Omega_3. \end{cases}$$

It follows that

$${}_B\chi_6(k)(A_{53}(t)) = \sum_{w\in\mathbb{F}_q, z\neq 0} \phi(-z^{-1}(tw^3 + 3w - 1)) = q\delta(t) = \begin{cases} -q & \text{for } t \in \Omega_0, \\ 0 & \text{for } t \in \Omega_1, \\ q & \text{for } t \in \Omega_2, \\ 2q & \text{for } t \in \Omega_3, \end{cases}$$

where $\delta(t) = i - 1$ for $t \in \Omega_i$. Now $\langle {}_B\chi_j(k), {}_B\chi_j(k)\rangle_B = 1$ for $k \in R_0$, $2 \le j \le 7$, so ${}_B\chi_j(k)$ are irreducible characters of B.

Let $\epsilon = -1$ and $x \in \mathbb{F}_q$. Define a linear character of $X_b U_1$ by

$$x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6) \mapsto \phi(t_2 + xt_3 + t_5).$$

Inducing this character to B, we obtain ${}_B\theta(x)$ $(x \in \mathbb{F}_q)$. Since

$$\left\langle \sum_{x\in\mathbb{F}_q} {}_B\theta(x), \sum_{x\in\mathbb{F}_q} {}_B\theta(x) \right\rangle_B = q,$$

it follows that ${}_B\theta(x)$ are q distinct irreducible characters of B.

Similarly, let $\epsilon = 1$ and $x \in \mathbb{F}_q^*$. Define a linear character of $X_b U_1$ by

$$x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6) \mapsto \phi(xt_2 + xt_3 + t_5).$$

Inducing this character to B, we obtain ${}_B\theta(x)$ $(x \in \mathbb{F}_q^*)$. Since

$$\left\langle \sum_{x\in\mathbb{F}_q^*} {}_B\theta(x), \sum_{x\in\mathbb{F}_q^*} {}_B\theta(x) \right\rangle_B = q - 1,$$

it follows that ${}_B\theta(x)$ are $q-1$ distinct irreducible characters of B.

Define a linear character of U by

$$x_a(t_1)x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6) \mapsto \phi(t_1 + t_2).$$

Inducing this character to B, we obtain ${}_B\theta_1$.

Next, suppose $\epsilon = 1$ and let ${}_B\theta_2(k, l)$ be the character of B induced from the following linear character of the subgroup $\langle h(\omega, \omega, \omega)\rangle X_b U_1$

$$h_\omega(i, i, i)x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6) \mapsto \tilde{\omega}^{ik}\phi(t_2\gamma^l + t_5),$$

where $k, l = 0, 1, 2$. Let ${}_B\psi(k) = \sum_{l=0}^{2} {}_B\theta_2(k, l)$. Then

$$\langle {}_B\psi(k), {}_B\psi(k)\rangle_B = 3$$

and ${}_B\theta_2(k, l)$ are nine distinct irreducible characters of B.

Let $x \in \mathbb{F}_q^*$, $k \in \{0, 1\}$ and $\delta = \pm 1$. Define a linear character of the subgroup $L = \langle h(-1, -1, 1)\rangle X_b U_1$ of order $2q^5$ by

$$\epsilon_1(k, x) : h(\delta, \delta, 1)x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6) \mapsto \delta^k\phi(xt_2 + t_4).$$

Then we get four irreducible characters of B from the induction

$${}_B\theta_3(k) = {}^B_L\epsilon_1(k, -1) \quad \text{and} \quad {}_B\theta_4(k) = {}^B_L\epsilon_1(k, -\gamma).$$

Define a linear character of the subgroup $L = \langle h(1,-1,-1)\rangle X_b U_1$ of order $2q^5$ by

$$\epsilon_2(k,x) : h(1,\delta,\delta)x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6) \mapsto \delta^k\phi(xt_3+t_5),$$

where $\delta = \pm 1$, $k = 0$ or 1, and $x \in \mathbb{F}_q^*$. Then we get four irreducible characters of B from the induction

$$_B\theta_5(k) = {}_L^B\epsilon_2(k,-1) \quad \text{and} \quad {}_B\theta_6(k) = {}_L^B\epsilon_2(k,-\gamma).$$

If $_B\psi(k) = {}_B\theta_5(k) + {}_B\theta_6(k)$, then

$$\langle {}_B\psi(k), {}_B\psi(k)\rangle_B = 2$$

for $k = 0, 1$, and $_B\theta_i(k)$ $(i = 5, 6)$ are distinct irreducible characters of B.

Finally, consider the subgroup $L = \langle h(-1,1,-1)\rangle X_a X_{2a+b} X_{3a+b} X_{3a+2b}$ of order $2q^4$. Define a linear character of L by

$$\epsilon_3(k,x) : h(\delta,1,\delta)x_a(t_1)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6) \mapsto \delta^k\phi(xt_1+t_6),$$

where $\delta = \pm 1$, $k = 0$ or 1, and $x \in \mathbb{F}_q^*$. Then we get four irreducible characters of B from the induction

$$_B\theta_7(k) = {}_L^B\epsilon_3(k,-1) \quad \text{and} \quad {}_B\theta_8(k) = {}_L^B\epsilon_3(k,-\gamma).$$

Thus we get all $q^2 + 5q + 12 + 4\epsilon$ irreducible characters of B.

4. THE CHARACTER TABLE OF P

In this section, we construct the character table of P. The $q^2 + 3q + 10$ conjugacy classes of P are given in Table A.3, and the class $B_2(i)$ occurs only when $\epsilon = -1$, and B_2, B_3, and $B_4(i)$ occur only when $\epsilon = 1$, while B_0 and B_1 exist in both cases. The character table of P is given in Table A.4 for the case $\epsilon = -1$. In case of $\epsilon = 1$, only the values of the characters at the classes B_0, B_1, B_2, B_3, and $B_4(i)$ are given in Table A.5. The values of the characters at the classes A_{41}, A_{42}, and A_5 are obtained from Table A.4 by setting $\epsilon = 1$ and the values of the characters at the other classes are omitted, since they are identical with those in case $\epsilon = -1$.

Let $U_P = X_b X_{a+b} X_{2a+b} X_{3a+b} X_{3a+2b}$ and $L_P = \langle HX_a, \omega_a\rangle$. Then $P = U_P \rtimes L_P$ and $L_P \cong \mathrm{GL}_2(q)$. First, we lift the irreducible characters of L_P to P and get four families of the irreducible characters of P

$$_P\chi_1(k), \quad k \in R_0, \qquad {}_P\chi_2(k,l), \quad (k,l) \in {}^2R_3, \qquad {}_P\chi_3(k), \quad k \in R_0,$$
$$\text{and} \qquad {}_P\chi_4(k), \quad k \in {}^2T_1$$

(for details, see Huang, 2005a, 3.3.1). Next, we induce certain irreducible characters of B to P and get more irreducible characters of P. By Table A.1, we know the fusions of conjugacy classes of B in P, so we can compute the values of a induced character. The characters

$$_P\chi_5(k) = {}_B^P\chi_3(k), \qquad {}_P\chi_6(k) = {}_B^P\chi_4(k), \qquad {}_P\chi_6(l) = {}_B^P\chi_4(l)$$

are irreducible, where $k \in R_0$ and $l \in {}^2R_1$. If $k = 0$ or 1, then

$$ {}_P\theta_1 = {}^P_B\theta_1 \qquad \text{and} \qquad {}_P\theta_2(k) = {}^P_B\theta_4(k) $$

are irreducible.

The remaining $\frac{1}{2}(q-1)+12$ irreducible characters of P are constructed as follows. Let $L = \langle H, X_{\pm b}, X_{\pm(3a+b)}, X_{\pm(3a+2b)}\rangle$, $H(\omega_a) = \{h(z, z^{q-1}, z^{-q}) \mid z \in \mathbb{F}_q^*\}$,

$$ U_0 = \left\{x_a(t), x_{3a+b}(t^q)x_{3a+2b}(u) \mid t, u \in \mathbb{F}_{q^2}^*, u + u^q = t^{q+1}\right\}, $$

$M = \langle H(\omega_a)U_0, \omega_{3a+2b}\rangle$ and $K = \langle HX_aX_{3a+2b}, \omega_a\rangle$. Then $L \cong \mathrm{SL}_3(q)$, $M \cong \mathrm{SU}_3(q)$ and the Borel subgroups $B_L = HX_bX_{3a+b}X_{3a+2b}$, $B_M = H(\omega_a)U_0$. The character tables of K, B_M, and B_L are given by Tables A.10, A.13, and A.15, respectively.

Suppose $\epsilon = -1$ and let

$$ \begin{aligned} \phi_1 &= -2\,{}^P_B\theta_7(0) + \frac{1}{2}(q-2-\zeta)({}_P\theta_9 - {}_P\theta_{10}) + \frac{1}{2}(q-4+\zeta)\,{}_P\theta_{11} - \frac{1}{2}(q+\zeta)\,{}_P\theta_{12}, \\ \phi_2 &= -2\,{}^P_B\theta_7(1) + \frac{1}{2}(q-2+\zeta)({}_P\theta_9 - {}_P\theta_{10}) + \frac{1}{2}(q-4-\zeta)\,{}_P\theta_{11} - \frac{1}{2}(q-\zeta)\,{}_P\theta_{12}, \\ \psi &= \frac{1}{3}(q-2)\left\{{}^P_B\theta_3(0) + {}^P_B\theta_3(1) + \sum_{x\in\mathbb{F}_q} {}^P_B\theta(x) + ({}_P\theta_2(0) + {}_P\theta_2(1))\right\}. \end{aligned} $$

Then the following $\frac{1}{2}(q-1)+12$ characters of P are irreducible:

$$ \begin{aligned} {}_P\theta_7 &= {}^P_{B_M}\chi_2(1) - \left({}^P_B\theta_7(1) - {}^P_B\theta_8(1)\right) - 2\,{}^P_B\theta_7(0) - {}^P_K\chi_6(1) + {}^P_K\theta_1, \\ {}_P\theta_8 &= {}^P_B\chi_7(0) - {}_P\theta_7, \\ {}_P\theta_{11} &= \frac{1}{2}\left({}^P_B\theta_7(k) - {}^P_B\theta_8(k) + {}^P_B\chi_7\left(\frac{1}{2}(q-1)\right)\right), && k = \begin{cases} 0 & \text{if } \zeta = 1, \\ 1 & \text{if } \zeta = -1, \end{cases} \\ {}_P\theta_{12} &= {}^P_B\chi_7\left(\frac{1}{2}(q-1)\right) - {}_P\theta_{11}, \\ {}_P\theta_{10} &= {}^P_K\theta_4(-1) - {}^P_K\theta_3(-1) + {}^P_K\theta_1 - {}_P\theta_7 + {}_P\theta_{12}, \\ {}_P\theta_9 &= {}^P_B\theta_7(k) - {}^P_B\theta_8(k) + {}_P\theta_{10}, && k = \begin{cases} 1 & \text{if } \zeta = 1, \\ 0 & \text{if } \zeta = -1, \end{cases} \\ {}_P\chi_8(k) &= {}^P_{B_M}\chi_2(k) - {}^P_B\chi_7(0) - (q-3)\,{}^P_B\theta_7(0) \\ &\qquad - (q-3)\,{}^P_B\theta_7(1) - 2\sum_{l\in{}^2R_1} {}_P\chi_7(l) + \begin{cases} \phi_1 & \text{if } k \text{ is odd}, \\ \phi_2 & \text{if } k \text{ is even}, \end{cases} && k \in {}^2S_1, \\ {}_P\theta_4 &= {}^P_{B_M}\theta(1,0) - \sum_{l\in R_0} {}_P\chi_5(l) - (q-1)\,{}_P\theta_1 - \psi, \\ {}_P\theta_3(k) &= {}^P_B\theta_3(k) - {}_P\theta_4, && k = 0, 1, \\ {}_P\theta_5 &= {}^P_{B_M}\theta(0,1) - \sum_{l\in R_0} {}_P\chi_6(l) - \psi, \\ {}_P\theta_6(k) &= {}^P_{B_M}\theta(1,k) - \sum_{l\in R_0} {}_P\chi_6(l) - \psi, && k = 1, 2. \end{aligned} $$

Suppose $\epsilon = 1$ and let

$$\psi = \frac{1}{3}(q+2)\sum_{x\in\mathbb{F}_q^*} {}^P_B\theta(x) + \left({}^P_B\theta_3(0) + {}^P_B\theta_3(1)\right) - \frac{1}{3}(q-1)({}_P\theta_2(0) + {}_P\theta_2(1)).$$

Then the following $\frac{1}{2}(q-1)+12$ characters of P are irreducible.

$$\begin{aligned}
{}_P\theta_7 &= {}^P_K\theta_1 - \left({}^P_B\theta_7(0) + {}^P_B\theta_7(1)\right) - \left({}^P_B\chi_7(0) + {}^P_B\chi_7\left(\frac{1}{2}(q-1)\right)\right) \\
&\quad + \sum_{\substack{l\in {}^2R_1 \\ l\not\equiv 0 \pmod 3}} {}_P\chi_7(l) - 2 \sum_{\substack{l\in {}^2R_1 \\ l\equiv 0 \pmod 3}} {}_P\chi_7(l), \\
{}_P\theta_8 &= {}^P_B\chi_7(0) - {}_P\theta_7, \\
{}_P\theta_{11} &= \frac{1}{2}\left({}^P_B\theta_7(s) - {}^P_B\theta_8(s) + {}^P_B\chi_7\left(\frac{1}{2}(q-1)\right)\right), && s = \begin{cases} 0 & \text{if } \zeta = 1, \\ 1 & \text{if } \zeta = -1, \end{cases} \\
{}_P\theta_{12} &= {}^P_B\chi_7\left(\frac{1}{2}(q-1)\right) - {}_P\theta_{11}, \\
{}_P\theta_{10} &= {}^P_K\theta_4(-\zeta) - {}^P_K\theta_3(\zeta) + 2{}^P_B\chi_7(0) + 3{}_P\theta_{11} + 2{}_P\theta_{12} \\
&\quad + \sum_{\substack{l\in {}^2R_1 \\ l\not\equiv 0 \pmod 3}} {}_P\chi_7(l) + 4 \sum_{\substack{l\in {}^2R_1 \\ l\equiv 0 \pmod 3}} {}_P\chi_7(l), \\
{}_P\theta_9 &= {}^P_B\theta_7(s) - {}^P_B\theta_8(s) + {}_P\theta_{10}, && s = \begin{cases} 1 & \text{if } \zeta = 1, \\ 0 & \text{if } \zeta = -1, \end{cases} \\
{}_P\chi_8(k) &= {}^P_K\chi_6(k) - (q-2){}^P_K\theta_1 + (2q-5){}_P\theta_7 + (q-3){}_P\theta_8 \\
&\quad + \frac{1}{2}(q-1)({}_P\theta_9 - {}_P\theta_{10}) + \frac{1}{2}(3q-7){}_P\theta_{11} + \frac{1}{2}(q-5){}_P\theta_{12} \\
&\quad - q \sum_{\substack{l\in {}^2R_1 \\ l\not\equiv \pmod 3}} {}_P\chi_7(l) + 2(q-3) \sum_{\substack{l\in {}^2R_1 \\ l\equiv 0 \pmod 3}} {}_P\chi_7(l), && k \in {}^2S_1, \\
{}_P\theta_4 &= {}^P_{B_L}\theta(1,0) - (q-1){}_P\theta_1 + \sum_{l\in R_0}(2{}_P\chi_6(l) - {}_P\chi_5(l)) - \psi, \\
{}_P\theta_3(k) &= {}^P_B\theta_3(k) - {}_P\theta_4, && k = 0, 1, \\
{}_P\theta_5 &= {}^P_{B_L}\theta(0,1) + \sum_{l\in R_0} {}_P\chi_6(l) - \psi, \\
{}_P\theta_6(k) &= {}^P_{B_L}\theta(1,k) + \sum_{l\in R_0} {}_P\chi_6(l) - \psi, && k = 1, 2.
\end{aligned}$$

This completes the construction of irreducible characters of P.

5. THE CHARACTER TABLE OF Q

In this section, we construct the character table of Q using the an approach similar to that of P. The $q^2 + 4q + 10 + 4\epsilon$ conjugacy classes of Q are given in Table A.6, and the classes $B_0(i)$, $B_1(i)$, $B_2(i)$, $B_3(i,j)$ occur only when $\epsilon = 1$. The character table of Q is given in Table A.7. Apart from the values of the character

${}_Q\chi_7(k)$ at the class $A_{42}(0)$ are distinct in the two cases, the other values of the characters of Q are independent of ϵ.

Let $L_Q = \langle HX_b, \omega_b \rangle$ and $U_Q = X_a X_{a+b} X_{2a+b} X_{3a+b} X_{3a+2b}$, so that $Q = U_Q \rtimes L_Q$ and $L_Q \cong \mathrm{GL}_2(q)$. Lift the irreducible characters of L_P to Q and get four families of the irreducible characters of Q

$$
{}_Q\chi_1(k),\quad k \in R_0, \qquad {}_Q\chi_2(k,l),\quad (k,l) \in {}^2R_2, \qquad {}_Q\chi_3(k),\quad k \in R_0,
$$
$$
\text{and} \qquad {}_Q\chi_4(k),\quad k \in {}^2T_1.
$$

Induce certain irreducible characters of B to Q and get more irreducible characters of Q. The characters

$$
{}_Q\chi_5(k) = {}^Q_B\chi_2(k), \qquad {}_Q\chi_6(l) = {}^Q_B\chi_5(l), \qquad {}_Q\chi_7(k) = {}^Q_B\chi_6(k), \qquad {}_Q\theta_1 = {}^Q_B\theta_1
$$

are irreducible, where $k \in R_0$ and $l \in {}^2R_1$. If $k, l \in \{0, 1, 2\}$, $x \in \mathbb{F}_q^*$, and $\epsilon = 1$, then the characters

$$
{}_Q\theta_2(k,l) = {}^Q_B\theta_2(k,l) \qquad \text{and} \qquad {}_Q\theta_3(x) = {}^Q_B\theta(x)
$$

are distinct and irreducible. If $\epsilon = -1$ and $x \in \mathbb{F}_q$, then each

$$
{}_Q\theta_4(x) = {}^Q_B\theta(x)
$$

is irreducible.

Let ${}_Q\theta_5(k) = {}^Q_B\theta_5(k)$ and ${}_Q\theta_6(k) = {}^Q_B\theta_6(k)$. Then the values of ${}_Q\psi(k) = {}_Q\theta_5(k) + {}_Q\theta_6(k)$ can be determined and

$$
\langle {}_Q\psi(k), {}_Q\psi(k) \rangle_Q = 2
$$

for $k = 0, 1$. This shows that ${}_Q\theta_i(k)$ $(i = 5, 6)$ are distinct irreducible characters of Q.

To obtain the remaining irreducible characters of Q, we lift irreducible characters of $\langle HX_b X_{2a+b}, \omega_b \rangle$ (see Table A.11) to the subgroup

$$
C = \langle HX_b X_{2a+b} X_{3a+b} X_{3a+2b}, \omega_b \rangle.
$$

The remaining irreducible characters of Q are given as follows.

$$
\begin{aligned}
{}_Q\chi_8(k) &= {}^Q_B\theta_4(0) + {}^Q_B\theta_4(1) - {}^Q_C\chi_6(k), \qquad k \in {}^2S_1,\\
{}_Q\theta_8 &= {}^Q_C\theta_4(-\zeta) - \sum_{k \in {}^2S_1} {}_Q\chi_8(k),\\
{}_Q\theta_7(k) &= {}^Q_B\theta_3(1) - {}^Q_B\theta_4(1) + {}_Q\theta_8,\\
{}_Q\theta_9 &= {}^Q_C\theta_1 - {}_P\theta_7,\\
{}_Q\theta_{12} &= {}^Q_C\theta_3(-\zeta) - \sum_{k \in {}^2S_1} {}_Q\chi_8(k) - {}_Q\theta_7,
\end{aligned}
$$

$$
\begin{aligned}
{}_Q\theta_{10} &= {}^Q_B\chi_5(0) - {}_Q\theta_9,\\
{}_Q\theta_{11} &= {}^Q_B\chi_5\left(\frac{1}{2}(q-1)\right) - {}_Q\theta_{12}.
\end{aligned}
$$

This completes the construction of irreducible characters of Q.

APPENDIX A

Table A.1 The conjugacy classes of B, $\epsilon = \pm 1$. (In this table $\lambda \in \mathbb{F}_q$ is a fixed nonsquare and $\mu \in \mathbb{F}_q$ is a fixed noncube and y_j is defined in Section 2)

Notation	Class representative	Parameter	Number of classes	Order of centralizer	Class in P	Class in Q
A_0	$h(1,1,1)$		1	$q^6(q-1)^2$	A_0	A_0
A_1	$x_{3a+2b}(1)$		1	$q^6(q-1)$	A_1	A_1
A_2	$x_{3a+b}(1)$		1	$q^5(q-1)$	A_2	A_1
A_3	$x_{2a+b}(1)$		1	$q^4(q-1)$	A_3	A_2
A_{41}	$x_{a+b}(1)$		1	$q^4(q-1)$	A_3	A_{31}
A_{42}	$x_{a+b}(1)x_{3a+b}(-1)$		1	$2q^4$	A_{41}	A_{32}
A_{43}	$x_{a+b}(\lambda)x_{3a+b}(-1)$		1	$2q^4$	A_{42}	A_{33}
A_{51}	$x_b(1)$		1	$q^4(q-1)$	A_2	A_{41}
$A_{52}(i)$	$x_b(1)x_{3a+b}(\mu^i)$	$0 \le i \le 1+\epsilon$	$2+\epsilon$	$(2+\epsilon)q^4$	A_{41} $(i=0)$ A_5 $(i \neq 0)$	$A_{42}(i)$
$A_{53}(t)$	$x_b(t)x_{2a+b}(1)x_{3a+b}(1)$	$t \in \mathbb{F}_q^*$	$q-1$	q^4	A_3 $(t=-4)$ A_{41} $(t \in \Omega_3)$ A_{42} $(t \in \Omega_1)$ A_5 $(t \in \Omega_0)$	$A_5(t)$
A_{54}	$x_b(1)x_{2a+b}(1)$		1	$2q^4$	A_{41}	A_{61}
A_{55}	$x_b(1)x_{2a+b}(\lambda)$		1	$2q^4$	A_{42}	A_{62}
A_{61}	$x_a(1)$		1	$q^3(q-1)$	A_{61}	A_{31}
A_{62}	$x_a(1)x_{3a+2b}(1)$		1	$2q^3$	A_{62}	A_{32}
A_{63}	$x_a(1)x_{3a+2b}(\lambda)$		1	$2q^3$	A_{63}	A_{33}
A_7	$x_a(1)x_b(1)$		1	q^2	A_7	A_7
$B_0(i)$	$h_\omega(i,i,i)$	$1 \le i \le 1+\epsilon$	$1+\epsilon$	$q^3(q-1)^2$	B_0	$B_0(i)$
$B_1(i)$	$h_\omega(i,i,i)x_{3a+2b}(1)$	$1 \le i \le 1+\epsilon$	$1+\epsilon$	$q^3(q-1)$	B_1	$B_1(i)$
$B_2(i)$	$h_\omega(i,i,i)x_{3a+b}(1)$	$1 \le i \le 1+\epsilon$	$1+\epsilon$	$q^2(q-1)$	B_2 $(i=1)$ B_3 $(i=2)$	$B_1(i)$
$B_3(i)$	$h_\omega(i,i,i)x_b(1)$	$1 \le i \le 1+\epsilon$	$1+\epsilon$	$q^2(q-1)$	B_3 $(i=1)$ B_2 $(i=2)$	$B_2(i)$
$B_4(i,j)$	$h_\omega(i,i,i)y_j$	$1 \le i \le 1+\epsilon$ $0 \le j \le 1+\epsilon$	$3(1+\epsilon)$	$3q^2$	$B_4(0)$ $(j=0)$ $B_4(1)$ $(i=j)$ $B_4(2)$ $(i \neq j)$	$B_3(i,j)$
B_{11}	$h(1,-1,-1)$		1	$q^2(q-1)^2$	B_{11}	B_{11}
B_{12}	$h(1,-1,-1)x_{3a+b}(1)$		1	$q^2(q-1)$	B_{12}	B_{12}
B_{13}	$h(1,-1,-1)x_{a+b}(1)$		1	$q^2(q-1)$	B_{13}	B_{13}
B_{14}	$h(1,-1,-1)x_{a+b}(1)x_{3a+b}(1)$		1	$2q^2$	B_{14}	B_{14}
B_{15}	$h(1,-1,-1)x_{a+b}(1)x_{3a+b}(\lambda)$		1	$2q^2$	B_{15}	B_{15}
B_{21}	$h(-1,-1,1)$		1	$q^2(q-1)^2$	B_{11}	B_{21}
B_{22}	$h(-1,-1,1)x_b(1)$		1	$q^2(q-1)$	B_{12}	B_{22}
B_{23}	$h(-1,-1,1)x_{2a+b}(1)$		1	$q^2(q-1)$	B_{13}	B_{23}
B_{24}	$h(-1,-1,1)x_b(1)x_{2a+b}(1)$		1	$2q^2$	B_{14}	B_{24}

(continued)

Table A.1 (Continued)

Notation	Class representative	Parameter	Number of classes	Order of centralizer	Class in P	Class in Q
B_{25}	$h(-1,-1,1)x_b(1)x_{2a+b}(\lambda)$		1	$2q^2$	B_{15}	B_{25}
B_{31}	$h(-1,1,-1)$		1	$q^2(q-1)^2$	B_{21}	B_{11}
B_{32}	$h(-1,1,-1)x_{3a+2b}(1)$		1	$q^2(q-1)$	B_{22}	B_{12}
B_{33}	$h(-1,1,-1)x_a(1)$		1	$q^2(q-1)$	B_{23}	B_{13}
B_{34}	$h(-1,1,-1)x_a(1)x_{3a+2b}(1)$		1	$2q^2$	B_{24}	B_{14}
B_{35}	$h(-1,1,-1)x_a(1)x_{3a+2b}(\lambda)$		1	$2q^2$	B_{25}	B_{15}
$C_{11}(i)$	$h_\gamma(i,-2i,i)$	$i \in R_1^*$	$q-4-\epsilon$	$q(q-1)^2$	$C_{11}(i)$	$C_{11}(i)$
$C_{12}(i)$	$h_\gamma(i,-2i,i)x_{3a+2b}(1)$	$i \in R_1^*$	$q-4-\epsilon$	$q(q-1)$	$C_{12}(i)$	$C_{12}(i)$
$C_{21}(i)$	$h_\gamma(-2i,i,i)$	$i \in R_1^*$	$q-4-\epsilon$	$q(q-1)^2$	$C_{21}(i)$	$C_{11}(i)$
$C_{22}(i)$	$h_\gamma(-2i,i,i)x_{3a+b}(1)$	$i \in R_1^*$	$q-4-\epsilon$	$q(q-1)$	$C_{22}(i)$	$C_{12}(i)$
$C_{31}(i)$	$h_\gamma(i,-i,0)$	$i \in R_1$	$q-3$	$q(q-1)^2$	$C_{31}(i)$	$C_{21}(i)$
$C_{32}(i)$	$h_\gamma(i,-i,0)x_{2a+b}(1)$	$i \in R_1$	$q-3$	$q(q-1)$	$C_{32}(i)$	$C_{22}(i)$
$C_{41}(i)$	$h_\gamma(0,i,-i)$	$i \in R_1$	$q-3$	$q(q-1)^2$	$C_{31}(i)$	$C_{31}(i)$
$C_{42}(i)$	$h_\gamma(0,i,-i)x_{a+b}(1)$	$i \in R_1$	$q-3$	$q(q-1)$	$C_{32}(i)$	$C_{32}(i)$
$C_{51}(i)$	$h_\gamma(i,i,-2i)$	$i \in R_1^*$	$q-4-\epsilon$	$q(q-1)^2$	$C_{21}(-i)$	$C_{41}(i)$
$C_{52}(i)$	$h_\gamma(i,i,-2i)x_b(1)$	$i \in R_1^*$	$q-4-\epsilon$	$q(q-1)$	$C_{22}(-i)$	$C_{42}(i)$
$C_{61}(i)$	$h_\gamma(i,0,-i)$	$i \in R_1$	$q-3$	$q(q-1)^2$	$C_{41}(i)$	$C_{31}(i)$
$C_{62}(i)$	$h_\gamma(i,0,-i)x_a(1)$	$i \in R_1$	$q-3$	$q(q-1)$	$C_{42}(i)$	$C_{32}(i)$
$E(i,j)$	$h_\gamma(i,j,-i-j)$	$(i,j) \in R_5$	$q^2-8q+17+2\epsilon$	$(q-1)^2$	$C(i,j)$	$C(i,j)$

Table A.2 The character table of B, $\epsilon = -1$ [respectively $\epsilon = 1$]. (In this table, the parameter $\zeta \in \{1,-1\}$ depends on the residue class of q modulo 4 and is defined in Section 2)

Character Number of characters	${}_B\chi_1(k,l)$ $k,l \in R_0$ $(q-1)^2$	${}_B\chi_2(k)$ $k \in R_0$ $q-1$	${}_B\chi_3(k)$ $k \in R_0$ $q-1$	${}_B\chi_4(k)$ $k \in R_0$ $q-1$	${}_B\chi_5(k)$ $k \in R_0$ $q-1$	${}_B\chi_6(k)$ $k \in R_0$ $q-1$
A_0	1	$q-1$	$q-1$	$q(q-1)$	$q(q-1)$	$q(q-1)$
A_1	1	$q-1$	$q-1$	$q(q-1)$	$q(q-1)$	$q(q-1)$
A_2	1	$q-1$	$q-1$	$q(q-1)$	$q(q-1)$	$-q$
A_3	1	$q-1$	$q-1$	$q(q-1)$	$-q$	·
A_{41}	1	$q-1$	$q-1$	$-q$	·	·
A_{42}	1	$q-1$	$q-1$	$-q$	·	q $(-3 \in S)$ $-q$ $(-3 \in N)$
A_{43}	1	$q-1$	$q-1$	$-q$	·	$-q$ $(-3 \in S)$ q $(-3 \in N)$
A_{51}	1	$q-1$	-1	·	·	·
$A_{52}(i)$	1	$q-1$	-1	·	·	· $[q\omega_i]$
$A_{53}(t)$	1	$q-1$	-1	·	q $(-t \in S)$ $-q$ $(-t \in N)$	$q\delta(t)$
A_{54}	1	$q-1$	-1	·	ζq	$2q$ $(-3 \in S)$ · $(-3 \in N)$
A_{55}	1	$q-1$	-1	·	$-\zeta q$	· $(-3 \in S)$ $2q$ $(-3 \in N)$
A_{61}	1	-1	$q-1$	·	·	·
A_{62}	1	-1	$q-1$	·	·	·
A_{63}	1	-1	$q-1$	·	·	·
A_7	1	-1	-1	·	·	·

(continued)

Table A.2 (Continued)

Character Number of characters	${}_B\chi_1(k, l)$ $k, l \in R_0$ $(q-1)^2$	${}_B\chi_2(k)$ $k \in R_0$ $q-1$	${}_B\chi_3(k)$ $k \in R_0$ $q-1$	${}_B\chi_4(k)$ $k \in R_0$ $q-1$	${}_B\chi_5(k)$ $k \in R_0$ $q-1$	${}_B\chi_6(k)$ $k \in R_0$ $q-1$
$B_0(i)$	$\tilde{\omega}^{i(k+l)}$	·	$(q-1)\tilde{\omega}^{ik}$	·	·	$(q-1)\tilde{\omega}^{ik}$
$B_1(i)$	$\tilde{\omega}^{i(k+l)}$	·	$(q-1)\tilde{\omega}^{ik}$	·	·	$(q-1)\tilde{\omega}^{ik}$
$B_2(i)$	$\tilde{\omega}^{i(k+l)}$	·	$(q-1)\tilde{\omega}^{ik}$	·	·	$-\tilde{\omega}^{ik}$
$B_3(i)$	$\tilde{\omega}^{i(k+l)}$	·	$-\tilde{\omega}^{ik}$	·	·	$(q-1)\tilde{\omega}^{ik}$
$B_4(i, j)$	$\tilde{\omega}^{i(k+l)}$	·	$-\tilde{\omega}^{ik}$	·	·	$-\tilde{\omega}^{ik}$
B_{12}	$(-1)^l$	·	·	$(q-1)(-1)^k$	·	$-(-1)^k$
B_{13}	$(-1)^l$	·	·	$-(-1)^k$	·	$(q-1)(-1)^k$
B_{14}	$(-1)^l$	·	·	$-(-1)^k$	·	$-(-1)^k$
B_{15}	$(-1)^l$	·	·	$-(-1)^k$	·	$-(-1)^k$
B_{21}	$(-1)^{k+l}$	·	$(q-1)(-1)^k$	·	$(q-1)(-1)^k$	·
B_{22}	$(-1)^{k+l}$	·	$-(-1)^k$	·	$(q-1)(-1)^k$	·
B_{23}	$(-1)^{k+l}$	·	$(q-1)(-1)^k$	·	$-(-1)^k$	·
B_{24}	$(-1)^{k+l}$	·	$-(-1)^k$	·	$-(-1)^k$	·
B_{25}	$(-1)^{k+l}$	·	$-(-1)^k$	·	$-(-1)^k$	·
B_{31}	$(-1)^k$	$(q-1)(-1)^k$	·	·	·	·
B_{32}	$(-1)^k$	$(q-1)(-1)^k$	·	·	·	·
B_{33}	$(-1)^k$	$-(-1)^k$	·	·	·	·
B_{34}	$(-1)^k$	$-(-1)^k$	·	·	·	·
B_{35}	$(-1)^k$	$-(-1)^k$	·	·	·	·
$C_{11}(i)$	$\tilde{\gamma}^{i(k-2l)}$	·	·	·	·	·
$C_{12}(i)$	$\tilde{\gamma}^{i(k-2l)}$	·	·	·	·	·
$C_{21}(i)$	$\tilde{\gamma}^{i(l-2k)}$	·	·	·	·	$(q-1)\tilde{\gamma}^{ik}$
$C_{22}(i)$	$\tilde{\gamma}^{i(l-2k)}$	·	·	·	·	$-\tilde{\gamma}^{ik}$
$C_{31}(i)$	$\tilde{\gamma}^{i(k-l)}$	·	·	·	$(q-1)\tilde{\gamma}^{ik}$	·
$C_{32}(i)$	$\tilde{\gamma}^{i(k-l)}$	·	·	·	$-\tilde{\gamma}^{ik}$	·
$C_{41}(i)$	$\tilde{\gamma}^{il}$	·	·	$(q-1)\tilde{\gamma}^{ik}$	·	·
$C_{42}(i)$	$\tilde{\gamma}^{il}$	·	·	$-\tilde{\gamma}^{ik}$	·	·
$C_{51}(i)$	$\tilde{\gamma}^{i(k+l)}$	·	$(q-1)\tilde{\gamma}^{ik}$	·	·	·
$C_{52}(i)$	$\tilde{\gamma}^{i(k+l)}$	·	$-\tilde{\gamma}^{ik}$	·	·	·
$C_{61}(i)$	$\tilde{\gamma}^{ik}$	$(q-1)\tilde{\gamma}^{ik}$	·	·	·	·
$C_{62}(i)$	$\tilde{\gamma}^{ik}$	$-\tilde{\gamma}^{ik}$	·	·	·	·
$E(i, j)$	$\tilde{\gamma}^{ik+jl}$	·	·	·	·	·

Character Number of characters	${}_B\chi_7(k)$ $k \in R_0$ $q-1$	$\left[\sum_{x\in\mathbb{F}_q{}^*} {}_B\theta(x)\right]$ $q-1$	$\sum_{x\in\mathbb{F}_q} {}_B\theta(x)$ q	${}_B\theta_1$ 1	$\left[\sum_{l=0}^{2} {}_B\theta_2(k, l)\right]$ 9
A_0	$q^2(q-1)$	$q(q-1)^3$	$q^2(q-1)^2$	$(q-1)^2$	$q(q-1)^2$
A_1	$-q^2$	$q(q-1)^3$	$q^2(q-1)^2$	$(q-1)^2$	$q(q-1)^2$
A_2	·	$-q(q-1)^2$	$-q^2(q-1)$	$(q-1)^2$	$-q(q-1)$
A_3	·	·	·	$(q-1)^2$	·
A_{41}	·	·	·	$(q-1)^2$	·
A_{42}	·	$-q(q-1)\ (-3 \in S)$ $q(q-1)\ (-3 \in N)$	·	$(q-1)^2$	$q(q-1)\ (-3 \in S)$ $-q(q-1)\ (-3 \in N)$
A_{43}	·	$q(q-1)\ (-3 \in S)$ $-q(q-1)\ (-3 \in N)$	·	$(q-1)^2$	$-q(q-1)\ (-3 \in S)$ $q(q-1)\ (-3 \in N)$
A_{51}	·	$-q(q-1)$	$-q(q-1)$	$-(q-1)$	·
$A_{52}(i)$	· ·	$3q\ (i=0)$ $\cdot\ (i \neq 0)$	q	$-(q-1)$	$-q\omega_i$
$A_{53}(t)$	·	$q(\delta(t)+1)$	q	$-(q-1)$	$-q\delta(t)$

(continued)

Table A.2 (Continued)

Character	${}_B\chi_7(k)$	$\left[\sum_{x\in\mathbb{F}_q{}^*} {}_B\theta(x)\right]$	$\sum_{x\in\mathbb{F}_q} {}_B\theta(x)$	${}_B\theta_1$	$\left[\sum_{l=0}^{2} {}_B\theta_2(k,l)\right]$
Number of	$k \in R_0$				
characters	$q-1$	$q-1$	q	1	9
A_{54}	·	$-q(q-3)$ $(-3\in S)$ $-q(q-1)$ $(-3\in N)$	$-q(q-1)$	$-(q-1)$	$-2q$ $(-3\in S)$ · $(-3\in N)$
A_{55}	·	$-q(q-1)$ $(-3\in S)$ $-q(q-3)$ $(-3\in N)$	$-q(q-1)$	$-(q-1)$	· $(-3\in S)$ $-2q$ $(-3\in N)$
A_{61}	·	·	·	$-(q-1)$	·
A_{62}	q $(-3\in S)$ $-q$ $(-3\in N)$	·	·	$-(q-1)$	·
A_{63}	$-q$ $(-3\in S)$ q $(-3\in N)$	·	·	$-(q-1)$	·
A_7	·	·	·	1	·
$B_0(i)$	$q(q-1)\tilde{\omega}^{ik}$	·	·	·	$(q-1)^2\tilde{\omega}^{ik}$
$B_1(i)$	$-q\tilde{\omega}^{ik}$	·	·	·	$(q-1)^2\tilde{\omega}^{ik}$
$B_2(i)$	·	·	·	·	$-(q-1)\tilde{\omega}^{ik}$
$B_3(i)$	·	·	·	·	$-(q-1)\tilde{\omega}^{ik}$
$B_4(i,j)$	·	·	·	·	$\tilde{\omega}^{ik}$
B_{11}	·	·	·	·	·
B_{12}	·	·	·	·	·
B_{13}	·	·	·	·	·
B_{14}	·	·	·	·	·
B_{15}	·	·	·	·	·
B_{21}	·	·	·	·	·
B_{22}	·	·	·	·	·
B_{23}	·	·	·	·	·
B_{24}	·	·	·	·	·
B_{25}	·	·	·	·	·
B_{31}	$(q-1)(-1)^k$	·	·	·	·
B_{32}	$-(-1)^k$	·	·	·	·
B_{33}	$(q-1)(-1)^k$	·	·	·	·
B_{34}	$-(-1)^k$	·	·	·	·
B_{35}	$-(-1)^k$	·	·	·	·
$C_{11}(i)$	$(q-1)\tilde{\gamma}^{ik}$	·	·	·	·
$C_{12}(i)$	$-\tilde{\gamma}^{ik}$	·	·	·	·
$C_{21}(i)$	·	·	·	·	·
$C_{22}(i)$	·	·	·	·	·
$C_{31}(i)$	·	·	·	·	·
$C_{32}(i)$	·	·	·	·	·
$C_{41}(i)$	·	·	·	·	·
$C_{42}(i)$	·	·	·	·	·
$C_{51}(i)$	·	·	·	·	·
$C_{52}(i)$	·	·	·	·	·
$C_{61}(i)$	·	·	·	·	·
$C_{62}(i)$	·	·	·	·	·
$E(i,j)$	·	·	·	·	·

Table A.2 (Continued)

Character Number of characters	${}_B\theta_3(k)$ $k = 0, 1$ 2	${}_B\theta_4(k)$ $k = 0, 1$ 2	${}_B\theta_5(k) + {}_B\theta_6(k)$ $k = 0, 1$ 4	${}_B\theta_7(k)$ $k = 0, 1$ 2	${}_B\theta_8(k)$ $k = 0, 1$ 2
A_0	$\frac{1}{2}q(q-1)^2$	$\frac{1}{2}q(q-1)^2$	$q(q-1)^2$	$\frac{1}{2}q^2(q-1)^2$	$\frac{1}{2}q^2(q-1)^2$
A_1	$\frac{1}{2}q(q-1)^2$	$\frac{1}{2}q(q-1)^2$	$q(q-1)^2$	$-\frac{1}{2}q^2(q-1)$	$-\frac{1}{2}q^2(q-1)$
A_2	$\frac{1}{2}q(q-1)^2$	$\frac{1}{2}q(q-1)^2$	$-q(q-1)$	$\cdot$	$\cdot$
A_3	$-\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$	$\cdot$	$\cdot$	$\cdot$
A_{41}	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
A_{42}	$\cdot$	$\cdot$	$-q\ (-3 \in S)$ $q\ (-3 \in N)$	$\cdot$	$\cdot$
A_{43}	$\cdot$	$\cdot$	$q\ (-3 \in S)$ $-q\ (-3 \in N)$	$\cdot$	$\cdot$
A_{51}	$\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$	$q(q-1)$	$\cdot$	$\cdot$
$A_{52}(i)$	$\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$	$-(2+\epsilon)q\ (i = 0)$ $\cdot\ (i \neq 0)$	$\cdot$	$\cdot$ $\cdot$
$A_{53}(t)$	$-q\ (-t \in S)$ $\cdot\ (-t \in N)$	$\cdot\ (-t \in S)$ $q\ (-t \in N)$	$-q(\delta(t)+1)$	$\cdot$	$\cdot$
A_{54}	$-\frac{1}{2}(1+\zeta)q$	$\frac{1}{2}(1-\zeta)q$	$q(q-3)\ (-3 \in S)$ $q(q-1)\ (-3 \in N)$	$\cdot$	$\cdot$
A_{55}	$-\frac{1}{2}(1-\zeta)q$	$\frac{1}{2}(1+\zeta)q$	$q(q-1)\ (-3 \in S)$ $q(q-3)\ (-3 \in N)$	$\cdot$	$\cdot$
A_{61}	$\cdot$	$\cdot$	$\cdot$	$\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$
A_{62}	$\cdot$	$\cdot$	$\cdot$	$-\frac{1}{2}(1+\zeta)q$	$\frac{1}{2}(1-\zeta)q$
A_{63}	$\cdot$	$\cdot$	$\cdot$	$-\frac{1}{2}(1-\zeta)q$	$\frac{1}{2}(1+\zeta)q$
A_7	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$B_0(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$B_1(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$B_2(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$B_3(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$B_4(i, j)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
B_{11}	$\cdot$	$\cdot$	$(-1)^k(q-1)^2$	$\cdot$	$\cdot$
B_{12}	$\cdot$	$\cdot$	$-(-1)^k(q-1)$	$\cdot$	$\cdot$
B_{13}	$\cdot$	$\cdot$	$-(-1)^k(q-1)$	$\cdot$	$\cdot$
B_{14}	$\cdot$	$\cdot$	$(-1)^k$	$\cdot$	$\cdot$
B_{15}	$\cdot$	$\cdot$	$(-1)^k$	$\cdot$	$\cdot$
B_{21}	$\frac{1}{2}(q-1)^2(-1)^k$	$\frac{1}{2}(q-1)^2(-1)^k$	$\cdot$	$\cdot$	$\cdot$
B_{22}	$-\frac{1}{2}(q-1)(-1)^k$	$-\frac{1}{2}(q-1)(-1)^k$	$\cdot$	$\cdot$	$\cdot$
B_{23}	$-\frac{1}{2}(q-1)(-1)^k$	$-\frac{1}{2}(q-1)(-1)^k$	$\cdot$	$\cdot$	$\cdot$
B_{24}	$\frac{1}{2}(q+1)(-1)^k$	$-\frac{1}{2}(q-1)(-1)^k$	$\cdot$	$\cdot$	$\cdot$
B_{25}	$-\frac{1}{2}(q-1)(-1)^k$	$\frac{1}{2}(q+1)(-1)^k$	$\cdot$	$\cdot$	$\cdot$

(continued)

Table A.2 (Continued)

Character Number of characters	${}_B\theta_3(k)$ $k = 0, 1$ 2	${}_B\theta_4(k)$ $k = 0, 1$ 2	${}_B\theta_5(k) + {}_B\theta_6(k)$ $k = 0, 1$ 4	${}_B\theta_7(k)$ $k = 0, 1$ 2	${}_B\theta_8(k)$ $k = 0, 1$ 2
B_{31}	·	·	·	$\frac{1}{2}(q-1)^2(-1)^k$	$\frac{1}{2}(q-1)^2(-1)^k$
B_{32}	·	·	·	$-\frac{1}{2}(q-1)(-1)^k$	$-\frac{1}{2}(q-1)(-1)^k$
B_{33}	·	·	·	$-\frac{1}{2}(q-1)(-1)^k$	$-\frac{1}{2}(q-1)(-1)^k$
B_{34}	·	·	·	$\frac{1}{2}(q+1)(-1)^k$	$-\frac{1}{2}(q-1)(-1)^k$
B_{35}	·	·	·	$-\frac{1}{2}(q-1)(-1)^k$	$\frac{1}{2}(q+1)(-1)^k$
$C_{11}(i)$	·	·	·	·	·
$C_{12}(i)$	·	·	·	·	·
$C_{21}(i)$	·	·	·	·	·
$C_{22}(i)$	·	·	·	·	·
$C_{31}(i)$	·	·	·	·	·
$C_{32}(i)$	·	·	·	·	·
$C_{41}(i)$	·	·	·	·	·
$C_{42}(i)$	·	·	·	·	·
$C_{51}(i)$	·	·	·	·	·
$C_{52}(i)$	·	·	·	·	·
$C_{61}(i)$	·	·	·	·	·
$C_{62}(i)$	·	·	·	·	·
$E(i, j)$	·	·	·	·	·

Table A.3 The conjugacy classes of P, $\epsilon = -1$ [respectively $\epsilon = 1$]. (In this table $\lambda \in \mathbb{F}_q$ is a fixed nonsquare, $\mu \in \mathbb{F}_q$ is a fixed noncube, furthermore, υ is a fixed element of $\mathbb{F}_q$ such that the polynomial $x^3 - 3x - \upsilon$ is irreducible over $\mathbb{F}_q$)

Notation	Class representative	Parameter	Number of classes	Order of centralizer
A_0	$h(1, 1, 1)$		1	$q^6(q-1)(q^2-1)$
A_1	$x_{3a+2b}(1)$		1	$q^6(q^2-1)$
A_2	$x_{3a+b}(1)$		1	$q^5(q-1)$
A_3	$x_{2a+b}(1)$		1	$q^4(q-1)$
A_{41}	$x_{a+b}(1)x_{3a+b}(-1)$		1	$(4+2\epsilon)q^4$
A_{42}	$x_{a+b}(\lambda)x_{3a+b}(-1)$		1	$(4-2\epsilon)q^4$
A_5	$x_b(1)x_{3a+b}(\mu)$		1 $(\epsilon = 1)$	$3q^4$
A_5	$x_b(1)x_{2a+b}(-1)x_{3a+b}(\upsilon)$		1 $(\epsilon = -1)$	$3q^4$
A_{61}	$x_a(1)$		1	$q^3(q-1)$
A_{62}	$x_a(1)x_{3a+2b}(1)$		1	$2q^3$
A_{63}	$x_a(1)x_{3a+2b}(\lambda)$		1	$2q^3$
A_7	$x_a(1)x_b(1)$		1	q^2

(continued)

Table A.3 (Continued)

Notation	Class representative	Parameter	Number of classes	Order of centralizer
B_0	$h(\omega, \omega, \omega)$		1	$q^3(q-1)(q-\epsilon)$
B_1	$h(\omega, \omega, \omega)x_{3a+2b}(1)$		1	$q^3(q-\epsilon)$
$[B_2]$	$h(\omega, \omega, \omega)x_{3a+b}(1)$		1	$q^2(q-1)$
$[B_3]$	$h(\omega, \omega, \omega)x_b(1)$		1	$q^2(q-1)$
$B_2(i)$ $[B_4(i)]$	$h(\omega, \omega, \omega)y_i$	$0 \le i \le 2$	3	$3q^2$
B_{11}	$h(1, -1, -1)$		1	$q^2(q-1)^2$
B_{12}	$h(1, -1, -1)x_{3a+b}(1)$		1	$q^2(q-1)$
B_{13}	$h(1, -1, -1)x_{a+b}(1)$		1	$q^2(q-1)$
B_{14}	$h(1, -1, -1)x_{a+b}(1)x_{3a+b}(1)$		1	$2q^2$
B_{15}	$h(1, -1, -1)x_{a+b}(1)x_{3a+b}(\lambda)$		1	$2q^2$
B_{21}	$h(-1, 1, -1)$		1	$q^2(q-1)(q^2-1)$
B_{22}	$h(-1, 1, -1)x_{3a+2b}(1)$		1	$q^2(q^2-1)$
B_{23}	$h(-1, 1, -1)x_a(1)$		1	$q^2(q-1)$
B_{24}	$h(-1, 1, -1)x_a(1)x_{3a+2b}(1)$		1	$2q^2$
B_{25}	$h(-1, 1, -1)x_a(1)x_{3a+2b}(\lambda)$		1	$2q^2$
$C_{11}(i)$	$h_\gamma(i, -2i, i)$	$i \in {}^2R_1^*$	$(q-4-\epsilon)/2$	$q(q-1)^2$
$C_{12}(i)$	$h_\gamma(i, -2i, i)x_{3a+2b}(1)$	$i \in {}^2R_1^*$	$(q-4-\epsilon)/2$	$q(q-1)$
$C_{21}(i)$	$h_\gamma(-2i, i, i)$	$i \in R_1^*$	$q-4-\epsilon$	$q(q-1)^2$
$C_{22}(i)$	$h_\gamma(-2i, i, i)x_{3a+b}(1)$	$i \in R_1^*$	$q-4-\epsilon$	$q(q-1)$
$C_{31}(i)$	$h_\gamma(i, -i, 0)$	$i \in R_1$	$q-3$	$q(q-1)^2$
$C_{32}(i)$	$h_\gamma(i, -i, 0)x_{2a+b}(1)$	$i \in R_1$	$q-3$	$q(q-1)$
$C_{41}(i)$	$h_\gamma(i, 0, -i)$	$i \in R_1$	$q-3$	$q(q-1)(q^2-1)$
$C_{42}(i)$	$h_\gamma(i, 0, -i)x_a(1)$	$i \in R_1$	$q-3$	$q(q-1)$
$C(i, j)$	$h_\gamma(i, j, -i-j)$	$(i, j) \in {}^2R_5$	$(q^2-8q+17+2\epsilon)/2$	$(q-1)^2$
$D_{11}(i)$	$h_\eta(i, -2i, i)$	$i \in {}^2S_1^*$	$(q-2+\epsilon)/2$	$q(q^2-1)$
$D_{12}(i)$	$h_\eta(i, -2i, i)x_{3a+2b}(1)$	$i \in {}^2S_1^*$	$(q-2+\epsilon)/2$	$q(q+1)$
$E(i)$	$h_\theta(i, (q-1)i, -qi)$	$i \in {}^2T_3$	$(q-1)^2/2$	q^2-1

Table A.4 The character table of P, $\epsilon = -1$. (In this table, the parameter $\zeta \in \{1, -1\}$ depends on the residue class of q modulo 4 and is defined in Section 2)

Character	${}_P\chi_1(k)$	${}_P\chi_2(k,l)$	${}_P\chi_3(k)$	${}_P\chi_4(k)$	${}_P\chi_5(k)$	${}_P\chi_6(k)$	${}_P\chi_7(k)$	${}_P\chi_8(k)$	${}_P\theta_1$	${}_P\theta_2(k)$
Number of	$k \in R_0$	$(k,l) \in {}^2R_3$	$k \in R_0$	$k \in {}^2T_1$	$k \in R_0$	$k \in R_0$	$k \in {}^2R_1$	$k \in {}^2S_1$		$k = 0, 1$
characters	$q-1$	$(q-1)(q-2)/2$	$q-1$	$q(q-1)/2$	$q-1$	$q-1$	$(q-3)/2$	$(q-1)/2$	1	2
A_0	1	$q+1$	q	$q-1$	q^2-1	$q(q^2-1)$	$q^2(q^2-1)$	$q^2(q-1)^2$	$(q-1)(q^2-1)$	$\frac{1}{2}q(q-1)(q^2-1)$
A_1	1	$q+1$	q	$q-1$	q^2-1	$q(q^2-1)$	$-q^2(q+1)$	$-q^2(q-1)$	$(q-1)(q^2-1)$	$\frac{1}{2}q(q-1)(q^2-1)$
A_2	1	$q+1$	q	$q-1$	-1	$q(q-1)$	.	.	$-(q-1)$	$-\frac{1}{2}q(q-1)$
A_3	1	$q+1$	q	$q-1$	$q-1$	$q(q-2)$	.	.	$(q-1)^2$	$-\frac{1}{2}q(q-1)$
A_{41}	1	$q+1$	q	$q-1$	$(1+\epsilon)q-1$	$-(2+\epsilon)q$	.	.	$(q-1)((1+\epsilon)q-1)$	$-\frac{1}{2}q(q-1)$
A_{42}	1	$q+1$	q	$q-1$	$(1-\epsilon)q-1$	$-(2-\epsilon)q$	.	.	$(q-1)((1-\epsilon)q-1)$	$\frac{1}{2}q(q+1)$
A_5	1	$q+1$	q	$q-1$	$-(q+1)$	.	.	.	$-(q^2-1)$	$-\frac{1}{2}q(\epsilon q-1)$
A_{61}	1	1	.	-1	$q-1$	.	.	.	$-(q-1)$	.
A_{62}	1	1	.	-1	$q-1$	.	ζq	$-\zeta q$	$-(q-1)$	.
A_{63}	1	1	.	-1	$q-1$	.	$-\zeta q$	ζq	$-(q-1)$	.
A_7	1	1	.	-1	-1	.	.	.	1	.
B_0	1	.	-1	$-\omega_k$	.	.	.	$q(q-1)\omega_k$	.	.
B_1	1	.	-1	$-\omega_k$	.	.	.	$-q\omega_k$	.	.
$B_2(0)$	1	.	-1	$-\omega_k$	.	.	.	.	.	.
$B_2(i)$	1	.	-1	$-\omega_k$	.	.	.	.	.	.
B_{11}	$(-1)^k$	$(-1)^l+(-1)^{k+l}$	$(-1)^k$	.	$(q-1)(-1)^k$	$(q-1)(-1)^k$	.	.	.	$\frac{1}{2}(q-1)^2(-1)^k$
B_{12}	$(-1)^k$	$(-1)^l+(-1)^{k+l}$	$(-1)^k$	.	$-(-1)^k$	$(q-1)(-1)^k$	.	.	.	$-\frac{1}{2}(q-1)(-1)^k$
B_{13}	$(-1)^k$	$(-1)^l+(-1)^{k+l}$	$(-1)^k$	.	$(q-1)(-1)^k$	$-(-1)^k$	.	.	.	$-\frac{1}{2}(q-1)(-1)^k$
B_{14}	$(-1)^k$	$(-1)^l+(-1)^{k+l}$	$(-1)^k$	.	$-(-1)^k$	$-(-1)^k$	.	.	.	$-\frac{1}{2}(q-1)(-1)^k$
B_{15}	$(-1)^k$	$(-1)^l+(-1)^{k+l}$	$(-1)^k$	.	$-(-1)^k$	$-(-1)^k$	.	.	.	$\frac{1}{2}(q+1)(-1)^k$
B_{21}	1	$(q+1)(-1)^k$	q	$(q-1)(-1)^k$	.	.	$(q^2-1)(-1)^k$	$(q-1)^2(-1)^k$	.	.
B_{22}	1	$(q+1)(-1)^k$	q	$(q-1)(-1)^k$	.	.	$-(q+1)(-1)^k$	$-(q-1)(-1)^k$	.	.
B_{23}	1	$(-1)^k$	.	$-(-1)^k$	.	.	$(q-1)(-1)^k$	$-(q-1)(-1)^k$	.	.
B_{24}	1	$(-1)^k$	.	$-(-1)^k$	.	.	$-(-1)^k$	$(-1)^k$	.	.
B_{25}	1	$(-1)^k$	.	$-(-1)^k$	.	.	$-(-1)^k$	$(-1)^k$	.	.

$C_{11}(i)$	1	$\tilde{\gamma}^{i(k-2l)} + \tilde{\gamma}^{-i(k-2l)}$	1	.	.	.	$(q-1)\alpha_{ik}$	.	.	.
$C_{12}(i)$	1	$\tilde{\gamma}^{i(k-2l)} + \tilde{\gamma}^{-i(k-2l)}$	1	.	.	.	$-\alpha_{ik}$	.	.	.
$C_{21}(i)$	$\tilde{\gamma}^{-3ik}$	$\tilde{\gamma}^{i(l-2k)} + \tilde{\gamma}^{-i(k+l)}$	$\tilde{\gamma}^{-3ik}$	.	$(q-1)\tilde{\gamma}^{-ik}$	.	.	.	.	.
$C_{22}(i)$	$\tilde{\gamma}^{-3ik}$	$\tilde{\gamma}^{i(l-2k)} + \tilde{\gamma}^{-i(k+l)}$	$\tilde{\gamma}^{-3ik}$	.	$-\tilde{\gamma}^{-ik}$	.	.	.	.	.
$C_{31}(i)$	$\tilde{\gamma}^{ik}$	$\tilde{\gamma}^{i(k-l)} + \tilde{\gamma}^{il}$	$\tilde{\gamma}^{ik}$	.	.	$(q-1)\tilde{\gamma}^{ik}$	.	.	.	.
$C_{32}(i)$	$\tilde{\gamma}^{ik}$	$\tilde{\gamma}^{i(k-l)} + \tilde{\gamma}^{il}$	$\tilde{\gamma}^{ik}$	.	.	$-\tilde{\gamma}^{ik}$	.	.	.	.
$C_{41}(i)$	$\tilde{\gamma}^{2ik}$	$(q+1)\tilde{\gamma}^{ik}$	$q\tilde{\gamma}^{2ik}$	$(q-1)\tilde{\gamma}^{ik}$	.	.	.	.	.	.
$C_{42}(i)$	$\tilde{\gamma}^{2ik}$	$\tilde{\gamma}^{ik}$	.	$-\tilde{\gamma}^{ik}$	.	.	.	.	.	.
$C(i,j)$	$\tilde{\gamma}^{(2i+j)k}$	$\tilde{\gamma}^{ik+jl} + \tilde{\gamma}^{ik+j(k-l)}$	$\tilde{\gamma}^{(2i+j)k}$	.	.	.	.	.	.	.
$D_{11}(i)$	1	.	-1	$-\beta_{ik}$	.	.	.	$-(q-1)\beta_{ik}$	.	.
$D_{12}(i)$	1	.	-1	$-\beta_{ik}$	.	.	.	β_{ik}	.	.
$E(i)$	$\tilde{\gamma}^{ik}$	.	$-\tilde{\gamma}^{ik}$	$-(\tilde{\theta}^{ik} + \tilde{\theta}^{qik})$	.	.	.	.	.	.
Character	${}_P\theta_3(k)$	${}_P\theta_4$	${}_P\theta_5$	${}_P\theta_6(k)$	${}_P\theta_7$	${}_P\theta_8$	${}_P\theta_9$	${}_P\theta_{10}$	${}_P\theta_{11}$	${}_P\theta_{12}$
Number of	$k = 0, 1$			$k = 1, 2$						
characters	2	1	1	2	1	1	1	1	1	1
A_0	$\frac{1}{6}q(q-1)(q^2-1)$	$\frac{1}{3}q(q-1)(q^2-1)$	$\frac{1}{3}q(q-1)(q^2-1)$	$\frac{1}{3}q(q-1)(q^2-1)$	$q^2(q-1)$	$q^3(q-1)$	$\frac{1}{2}q^2(q-1)^2$	$\frac{1}{2}q^2(q-1)^2$	$\frac{1}{2}q^2(q^2-1)$	$\frac{1}{2}q^2(q^2-1)$
A_1	$\frac{1}{6}q(q-1)(q^2-1)$	$\frac{1}{3}q(q-1)(q^2-1)$	$\frac{1}{3}q(q-1)(q^2-1)$	$\frac{1}{3}q(q-1)(q^2-1)$	$-q^2$	$-q^3$	$-\frac{1}{2}q^2(q-1)$	$-\frac{1}{2}q^2(q-1)$	$-\frac{1}{2}q^2(q+1)$	$-\frac{1}{2}q^2(q+1)$
A_2	$\frac{1}{6}q(q-1)(2q-1)$	$\frac{1}{3}q(q-1)(2q-1)$	$-\frac{1}{3}q(q^2-1)$	$-\frac{1}{3}q(q^2-1)$	.	.	.	.	.	.
A_3	$-\frac{1}{2}q(q-1)$	$-q(q-1)$	.	.	.	.	.	.	.	.
A_{41}	$\frac{1}{6}q(q+3+2\epsilon)$	$\frac{1}{3}q(q+3+2\epsilon)$	$\frac{1}{3}q(q-\epsilon)$	$\frac{1}{3}q(q-\epsilon)$	.	.	.	.	.	.
A_{42}	$\frac{1}{6}q(-q+3-2\epsilon)$	$\frac{1}{3}q(-q+3-2\epsilon)$	$-\frac{1}{3}q(q-\epsilon)$	$-\frac{1}{3}q(q-\epsilon)$	.	.	.	.	.	.
A_5	$\frac{1}{6}q(\epsilon q-1)$	$\frac{1}{3}q(\epsilon q-1)$	$\frac{1}{3}q(\epsilon q+2)$	$\frac{1}{3}q(\epsilon q+2)$	.	.	.	.	.	.
A_{61}	.	.	.	.	.	.	$\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$
A_{62}	.	.	.	.	ζq	.	$-\frac{1}{2}(1+\zeta)q$	$\frac{1}{2}(1-\zeta)q$	$-\frac{1}{2}(1-\zeta)q$	$\frac{1}{2}(1+\zeta)q$
A_{63}	.	.	.	.	$-\zeta q$	.	$-\frac{1}{2}(1-\zeta)q$	$\frac{1}{2}(1+\zeta)q$	$-\frac{1}{2}(1+\zeta)q$	$\frac{1}{2}(1-\zeta)q$
A_7	.	.	.	.	.	.	.	.	.	.
B_0	$\frac{1}{3}(q^2-1)$	$-\frac{1}{3}(q^2-1)$	$\frac{2}{3}(q^2-1)$	$-\frac{1}{3}(q^2-1)$	$-q(q-1)$	$q(q-1)$	$q(q-1)$	$q(q-1)$	.	.
B_1	$\frac{1}{3}(q^2-1)$	$-\frac{1}{3}(q^2-1)$	$\frac{2}{3}(q^2-1)$	$-\frac{1}{3}(q^2-1)$	q	$-q$	$-q$	$-q$	.	.
$B_2(0)$	$\frac{1}{3}(2q-1)$	$-\frac{1}{3}(2q-1)$	$-\frac{2}{3}(q+1)$	$\frac{1}{3}(q+1)$	.	.	.	.	.	.

(continued)

Table A.4 (Continued)

Character Number of characters	${}_P\theta_3(k)$ $k=0,1$ 2	${}_P\theta_4$ 1	${}_P\theta_5$ 1	${}_P\theta_6(k)$ $k=1,2$ 2	${}_P\theta_7$ 1	${}_P\theta_8$ 1	${}_P\theta_9$ 1	${}_P\theta_{10}$ 1	${}_P\theta_{11}$ 1	${}_P\theta_{12}$ 1
$B_2(i)$	$-\frac{1}{3}(q+1)$	$\frac{1}{3}(q+1)$	$\frac{1}{3}(q-2)$	$\frac{1}{3}(q+1)+q\omega^{ik}$	.	.	.	.	.	.
B_{11}	$\frac{1}{2}(q-1)^2(-1)^k$	.	.	.	.	.	.	.	.	.
B_{12}	$-\frac{1}{2}(q-1)(-1)^k$	.	.	.	.	.	.	.	.	.
B_{13}	$-\frac{1}{2}(q-1)(-1)^k$	.	.	.	.	.	.	.	.	.
B_{14}	$\frac{1}{2}(q+1)(-1)^k$	.	.	.	.	.	.	.	.	.
B_{15}	$-\frac{1}{2}(q-1)(-1)^k$	.	.	.	.	.	.	.	.	.
B_{21}	.	.	.	.	$q-1$	$q(q-1)$	$-\frac{1}{2}(q-1)^2\zeta$	$-\frac{1}{2}(q-1)^2\zeta$	$\frac{1}{2}(q^2-1)\zeta$	$\frac{1}{2}(q^2-1)\zeta$
B_{22}	.	.	.	.	-1	$-q$	$\frac{1}{2}(q-1)\zeta$	$\frac{1}{2}(q-1)\zeta$	$-\frac{1}{2}(q+1)\zeta$	$-\frac{1}{2}(q+1)\zeta$
B_{23}	.	.	.	.	$q-1$	.	$\frac{1}{2}(q-1)\zeta$	$\frac{1}{2}(q-1)\zeta$	$\frac{1}{2}(q-1)\zeta$	$\frac{1}{2}(q-1)\zeta$
B_{24}	.	.	.	.	-1	.	$-\frac{1}{2}(q+1)\zeta$	$\frac{1}{2}(q-1)\zeta$	$\frac{1}{2}(q-1)\zeta$	$-\frac{1}{2}(q+1)\zeta$
B_{25}	.	.	.	.	-1	.	$\frac{1}{2}(q-1)\zeta$	$-\frac{1}{2}(q+1)\zeta$	$-\frac{1}{2}(q+1)\zeta$	$\frac{1}{2}(q-1)\zeta$
$C_{11}(i)$	.	.	.	.	$q-1$	$q-1$	.	.	$(q-1)(-i)^i$	$(q-1)(-i)^i$
$C_{12}(i)$	.	.	.	.	-1	-1	.	.	$-(-1)^i$	$-(-1)^i$
$C_{21}(i)$	.	.	.	.	.	.	.	.	.	.
$C_{22}(i)$	.	.	.	.	.	.	.	.	.	.
$C_{31}(i)$	.	.	.	.	.	.	.	.	.	.
$C_{32}(i)$	.	.	.	.	.	.	.	.	.	.
$C_{41}(i)$	.	.	.	.	.	.	.	.	.	.
$C_{42}(i)$	.	.	.	.	.	.	.	.	.	.
$C(i,j)$	.	.	.	.	.	.	.	.	.	.
$D_{11}(i)$	.	.	.	.	$q-1$	$-(q-1)$	$-(q-1)(-1)^i$	$-(q-1)(-1)^i$	.	.
$D_{12}(i)$	.	.	.	.	-1	1	$(-1)^i$	$(-1)^i$	.	.
$E(i)$	.	.	.	.	.	.	.	.	.	.

Table A.5 The character table of P, $\epsilon = 1$

	${}_P\chi_1(k)$	${}_P\chi_2(k,l)$	${}_P\chi_3(k)$	${}_P\chi_4(k)$	${}_P\chi_5(k)$	${}_P\chi_6(k)$	${}_P\chi_7(k)$	${}_P\chi_8(k)$	${}_P\theta_1$	${}_P\theta_2(k)$
B_0	1	ω_{k+l}	1	·	$(q-1)\omega_k$	·	$q(q-1)\omega_k$	·	·	·
B_1	1	ω_{k+l}	1	·	$(q-1)\omega_k$	·	$-q\omega_k$	·	·	·
B_2	1	ω_{k+l}	1	·	$(q-1)\tilde{\omega}^k - \tilde{\omega}^{-k}$	·	·	·	·	·
B_3	1	ω_{k+l}	1	·	$-\tilde{\omega}^k + (q-1)\tilde{\omega}^{-k}$	·	·	·	·	·
$B_4(0)$	1	ω_{k+l}	1	·	$-\omega_k$	·	·	·	·	·
$B_4(1)$	1	ω_{k+l}	1	·	$-\omega_k$	·	·	·	·	·
$B_4(2)$	1	ω_{k+l}	1	·	$-\omega_k$	·	·	·	·	·
	${}_P\theta_3(k)$	${}_P\theta_4$	${}_P\theta_5$	${}_P\theta_6(k)$	${}_P\theta_7$	${}_P\theta_8$	${}_P\theta_9$	${}_P\theta_{10}$	${}_P\theta_{11}$	${}_P\theta_{12}$
B_0	$\frac{1}{3}(q-1)^2$	$-\frac{1}{3}(q-1)^2$	$\frac{2}{3}(q-1)^2$	$-\frac{1}{3}(q-1)^2$	$q(q-1)$	$q(q-1)$	·	·	$q(q-1)$	$q(q-1)$
B_1	$\frac{1}{3}(q-1)^2$	$-\frac{1}{3}(q-1)^2$	$\frac{2}{3}(q-1)^2$	$-\frac{1}{3}(q-1)^2$	$-q$	$-q$	·	·	$-q$	$-q$
B_2	$-\frac{1}{3}(q-1)$	$\frac{1}{3}(q-1)$	$-\frac{2}{3}(q-1)$	$\frac{1}{3}(q-1)$	·	·	·	·	·	·
B_3	$-\frac{1}{3}(q-1)$	$\frac{1}{3}(q-1)$	$-\frac{2}{3}(q-1)$	$\frac{1}{3}(q-1)$	·	·	·	·	·	·
$B_4(0)$	$\frac{1}{3}(2q+1)$	$-\frac{1}{3}(2q+1)$	$-\frac{2}{3}(q-1)$	$\frac{1}{3}(q-1)$	·	·	·	·	·	·
$B_4(1)$	$-\frac{1}{3}(q-1)$	$\frac{1}{3}(q-1)$	$\frac{1}{3}(q+2)$	$\frac{1}{3}(q-1) + q\tilde{\omega}^k$	·	·	·	·	·	·
$B_4(2)$	$-\frac{1}{3}(q-1)$	$\frac{1}{3}(q-1)$	$\frac{1}{3}(q+2)$	$\frac{1}{3}(q-1) + q\tilde{\omega}^{-k}$	·	·	·	·	·	·

Table A.6 The conjugacy classes of Q, $\epsilon = \pm 1$. (In this table $\lambda \in \mathbb{F}_q$ is a fixed nonsquare, $\mu \in \mathbb{F}_q$ is a fixed noncube and y_j is defined in Section 2)

Notation	Class representative	Parameter	Number of classes	Order of centralizer
A_0	$h(1, 1, 1)$		1	$q^6(q-1)(q^2-1)$
A_1	$x_{3a+2b}(1)$		1	$q^6(q-1)$
A_2	$x_{2a+b}(1)$		1	$q^4(q^2-1)$
A_{31}	$x_{a+b}(1)$		1	$q^4(q-1)$
A_{32}	$x_{a+b}(1)x_{3a+b}(-1)$		1	$2q^4$
A_{33}	$x_{a+b}(\lambda)x_{3a+b}(-1)$		1	$2q^4$
A_{41}	$x_b(1)$		1	$q^4(q-1)$
$A_{42}(i)$	$x_b(1)x_{3a+b}(\mu^i)$	$0 \leq i \leq 1+\epsilon$	$2+\epsilon$	$(2+\epsilon)q^4$
$A_5(t)$	$x_b(t)x_{2a+b}(1)x_{3a+b}(1)$	$t \in \mathbb{F}_q^*$	$q-1$	q^4
A_{61}	$x_b(1)x_{2a+b}(1)$		1	$2q^4$
A_{62}	$x_b(1)x_{2a+b}(\lambda)$		1	$2q^4$
A_7	$x_a(1)x_b(1)$		1	q^2
$B_0(i)$	$h_\omega(i, i, i)$	$1 \leq i \leq 1+\epsilon$	$1+\epsilon$	$q^3(q-1)(q^2-1)$
$B_1(i)$	$h_\omega(i, i, i)x_{3a+2b}(1)$	$1 \leq i \leq 1+\epsilon$	$1+\epsilon$	$q^3(q-1)$
$B_2(i)$	$h_\omega(i, i, i)x_b(1)$	$1 \leq i \leq 1+\epsilon$	$1+\epsilon$	$q^2(q-1)$
$B_3(i, j)$	$h_\omega(i, i, i)y_j$	$1 \leq i \leq 1+\epsilon$ $0 \leq j \leq 1+\epsilon$	$3(1+\epsilon)$	$3q^2$
B_{11}	$h(1, -1, -1)$		1	$q^2(q-1)^2$
B_{12}	$h(1, -1, -1)x_{3a+b}(1)$		1	$q^2(q-1)$
B_{13}	$h(1, -1, -1)x_{a+b}(1)$		1	$q^2(q-1)$
B_{14}	$h(1, -1, -1)x_{a+b}(1)x_{3a+b}(1)$		1	$2q^2$
B_{15}	$h(1, -1, -1)x_{a+b}(1)x_{3a+b}(\lambda)$		1	$2q^2$
B_{21}	$h(-1, -1, 1)$		1	$q^2(q-1)(q^2-1)$
B_{22}	$h(-1, -1, 1)x_b(1)$		1	$q^2(q-1)$
B_{23}	$h(-1, -1, 1)x_{2a+b}(1)$		1	$q^2(q^2-1)$
B_{24}	$h(-1, -1, 1)x_b(1)x_{2a+b}(1)$		1	$2q^2$
B_{25}	$h(-1, -1, 1)x_b(1)x_{2a+b}(\lambda)$		1	$2q^2$
$C_{11}(i)$	$h_\gamma(i, -2i, i)$	$i \in R_1^*$	$q-4-\epsilon$	$q(q-1)^2$
$C_{12}(i)$	$h_\gamma(i, -2i, i)x_{3a+2b}(1)$	$i \in R_1^*$	$q-4-\epsilon$	$q(q-1)$
$C_{21}(i)$	$h_\gamma(i, -i, 0)$	$i \in {}^2R_1$	$(q-3)/2$	$q(q-1)^2$
$C_{22}(i)$	$h_\gamma(i, -i, 0)x_{2a+b}(1)$	$i \in {}^2R_1$	$(q-3)/2$	$q(q-1)$
$C_{31}(i)$	$h_\gamma(0, i, -i)$	$i \in R_1$	$q-3$	$q(q-1)^2$
$C_{32}(i)$	$h_\gamma(0, i, -i)x_{a+b}(1)$	$i \in R_1$	$q-3$	$q(q-1)$
$C_{41}(i)$	$h_\gamma(i, i, -2i)$	$i \in R_1^*$	$q-4-\epsilon$	$q(q-1)(q^2-1)$
$C_{42}(i)$	$h_\gamma(i, i, -2i)x_b(1)$	$i \in R_1^*$	$q-4-\epsilon$	$q(q-1)$
$C(i, j)$	$h_\gamma(i, j, -i-j)$	$(i, j) \in {}^2R_5'$	$(q^2-8q+17+2\epsilon)/2$	$(q-1)^2$
$D_{11}(i)$	$h_\eta(i, -i, 0)$	$i \in {}^2S_1$	$(q-1)/2$	$q(q^2-1)$
$D_{12}(i)$	$h_\eta(i, -i, 0)x_{2a+b}(1)$	$i \in {}^2S_1$	$(q-1)/2$	$q(q+1)$
$E(i)$	$h_\theta(i, qi, -(q+1)i)$	$i \in {}^2T_3$	$(q-1)^2/2$	q^2-1

Table A.7 The character table of Q, $\epsilon = -1$ [respectively $\epsilon = 1$]. (In this table, the parameter $\zeta \in \{1, -1\}$ depends on the residue class of q modulo 4 and is defined in Section 2)

Character Number of characters	${}_Q\chi_1(k)$ $k \in R_0$ $q-1$	${}_Q\chi_2(k,l)$ $(k,l) \in {}^2R_2$ $(q-1)(q-2)/2$	${}_Q\chi_3(k)$ $k \in R_0$ $q-1$	${}_Q\chi_4(k)$ $k \in {}^2T_1$ $q(q-1)/2$	${}_Q\chi_5(k)$ $k \in R_0$ $q-1$	${}_Q\chi_6(k)$ $k \in {}^2R_1$ $(q-3)/2$	${}_Q\chi_7(k)$ $k \in R_0$ $q-1$	${}_Q\chi_8(k)$ $k \in {}^2S_1$ $(q-1)/2$	${}_Q\theta_1$ 1	$\left[\sum_{l=0}^{2} {}_Q\theta_2(k,l)\right]$ $\epsilon = 1$ 9
A_0	1	$q+1$	q	$q-1$	q^2-1	$q(q^2-1)$	$q(q^2-1)$	$q(q-1)^2$	$(q-1)(q^2-1)$	$q(q-1)(q^2-1)$
A_1	1	$q+1$	q	$q-1$	q^2-1	$q(q^2-1)$	$-q$	$q(q-1)^2$	$(q-1)(q^2-1)$	$-q(q-1)$
A_2	1	$q+1$	q	$q-1$	q^2-1	$-q(q+1)$	.	$-q(q-1)$	$(q-1)(q^2-1)$	.
A_{31}	1	$q+1$	q	$q-1$	-1	.	.	.	$-(q-1)$	.
A_{32}	1	$q+1$	q	$q-1$	-1	.	$q\ (-3 \in S)$ $-q\ (-3 \in N)$	.	$-(q-1)$	$q(q-1)\ (-3 \in S)$ $-q(q-1)\ (-3 \in N)$
A_{33}	1	$q+1$	q	$q-1$	-1	.	$-q\ (-3 \in S)$	.	$-(q-1)$	$-q(q-1)\ (-3 \in S)$
							$-q\ (-3 \in S)$	.	$-(q-1)$	$-q(q-1)\ (-3 \in S)$
A_{41}	1	1	.	-1	$q-1$	.	.	.	$-(q-1)$	.
$A_{42}(i)$	1	1	.	-1	$q-1$	.	$\cdot\,[q\omega_i]$	.	$-(q-1)$	$-q\omega_i$
$A_5(t)$	1	1	.	-1	$q-1$	$q\ (-t \in S)$ $-q\ (-t \in N)$	$q\delta(t)$	$-q\ (-t \in S)$ $q\ (-t \in N)$	$-(q-1)$	$-q\delta(t)$
A_{61}	1	1	.	-1	$q-1$	ζq	$2q\ (-3 \in S)$ $\cdot\ (-3 \in N)$	$-\zeta q$	$-(q-1)$	$-2q\ (-3 \in S)$ $-q(q-1)\ (-3 \in N)$
A_{62}	1	1	.	-1	$q-1$	$-\zeta q$	$\cdot\ (-3 \in S)$ $2q\ (-3 \in N)$	ζq	$-(q-1)$	$\cdot\ (-3 \in S)$ $-q(q-3)\ (-3 \in N)$
A_7	1	1	.	-1	-1	.	.	.	1	.
$B_0(i)$	$\tilde{\omega}^{-ik}$	$(q+1)\tilde{\omega}^{i(k+l)}$	$q\tilde{\omega}^{-ik}$	$(q-1)\tilde{\omega}^{ik}$	.	.	$(q^2-1)\tilde{\omega}^{ik}$	.	.	$(q-1)(q^2-1)\tilde{\omega}^{ik}$
$B_1(i)$	$\tilde{\omega}^{-ik}$	$(q+1)\tilde{\omega}^{i(k+l)}$	$q\tilde{\omega}^{-ik}$	$(q-1)\tilde{\omega}^{ik}$	.	.	$-\tilde{\omega}^{ik}$	.	.	$-(q-1)\tilde{\omega}^{ik}$
$B_2(i)$	$\tilde{\omega}^{-ik}$	$\tilde{\omega}^{i(k+l)}$	.	$-\tilde{\omega}^{ik}$	.	.	$(q-1)\tilde{\omega}^{ik}$	.	.	$-(q-1)\tilde{\omega}^{ik}$
$B_3(i,j)$	$\tilde{\omega}^{-ik}$	$\tilde{\omega}^{i(k+l)}$	.	$-\tilde{\omega}^{ik}$	.	.	$-\tilde{\omega}^{ik}$	.	.	$\tilde{\omega}^{ik}$
B_{11}	$(-1)^k$	$(-1)^k+(-1)^l$	$(-1)^k$	.	$(q-1)(-1)^k$	.	$(q-1)(-1)^k$	.	.	.
B_{12}	$(-1)^k$	$(-1)^k+(-1)^l$	$(-1)^k$	.	$(q-1)(-1)^k$	.	$-(-1)^k$	.	.	.
B_{13}	$(-1)^k$	$(-1)^k+(-1)^l$	$(-1)^k$	.	$-(-1)^k$	.	$(q-1)(-1)^k$	.	.	.
B_{14}	$(-1)^k$	$(-1)^k+(-1)^l$	$(-1)^k$	.	$-(-1)^k$	.	$-(-1)^k$	.	.	.
B_{15}	$(-1)^k$	$(-1)^k+(-1)^l$	$(-1)^k$	.	$-(-1)^k$	.	$-(-1)^k$	.	.	.
B_{21}	1	$(q+1)(-1)^{k+l}$	q	$(q-1)(-1)^k$	.	$(q^2-1)(-1)^k$	.	$-(q-1)^2(-1)^k$	.	.
B_{22}	1	$(-1)^{k+l}$	.	$-(-1)^k$	.	$(q-1)(-1)^k$	.	$(q-1)(-1)^k$	.	.

(continued)

Table A.7 (Continued)

Character Number of characters	${}_Q\chi_1(k)$ $k \in R_0$ $q-1$	${}_Q\chi_2(k,l)$ $(k,l) \in {}^2R_2$ $(q-1)(q-2)/2$	${}_Q\chi_3(k)$ $k \in R_0$ $q-1$	${}_Q\chi_4(k)$ $k \in {}^2T_1$ $q(q-1)/2$	${}_Q\chi_5(k)$ $k \in R_0$ $q-1$	${}_Q\chi_6(k)$ $k \in {}^2R_1$ $(q-3)/2$	${}_Q\chi_7(k)$ $k \in R_0$ $q-1$	${}_Q\chi_8(k)$ $k \in {}^2S_1$ $(q-1)/2$	${}_Q\theta_1$ 1	$\left[\sum_{l=0}^{2} {}_Q\theta_2(k,l)\right]$ $\epsilon = 1$ 9
B_{23}	1	$(q+1)(-1)^{k+l}$	q	$(q-1)(-1)^k$	·	$-(q+1)(-1)^k$	·	$(q-1)(-1)^k$	·	·
B_{24}	1	$(-1)^{k+l}$	·	$-(-1)^k$	·	$-(-1)^k$	·	$-(-1)^k$	·	·
B_{25}	1	$(-1)^{k+l}$	·	$-(-1)^k$	·	$-(-1)^k$	·	$-(-1)^k$	·	·
$C_{11}(i)$	$\tilde\gamma^{-ik}$	$\tilde\gamma^{i(k-2l)} + \tilde\gamma^{i(l-2k)}$	$\tilde\gamma^{-ik}$	·	·	·	$(q-1)\tilde\gamma^{ik}$	·	·	·
$C_{12}(i)$	$\tilde\gamma^{-ik}$	$\tilde\gamma^{i(k-2l)} + \tilde\gamma^{i(l-2k)}$	$\tilde\gamma^{-ik}$	·	·	·	$-\tilde\gamma^{ik}$	·	·	·
$C_{21}(i)$	1	$\tilde\gamma^{i(k-l)} + \tilde\gamma^{i(l-k)}$	1	·	·	$(q-1)\alpha_{ik}$	·	·	·	·
$C_{22}(i)$	1	$\tilde\gamma^{i(k-l)} + \tilde\gamma^{i(l-k)}$	1	·	·	$-\alpha_{ik}$	·	·	·	·
$C_{31}(i)$	$\tilde\gamma^{ik}$	$\tilde\gamma^{ik} + \tilde\gamma^{il}$	$\tilde\gamma^{ik}$	·	$(q-1)\tilde\gamma^{ik}$	·	·	·	·	·
$C_{32}(i)$	$\tilde\gamma^{ik}$	$\tilde\gamma^{ik} + \tilde\gamma^{il}$	$\tilde\gamma^{ik}$	·	$-\tilde\gamma^{ik}$	·	·	·	·	·
$C_{41}(i)$	$\tilde\gamma^{2ik}$	$(q+1)\tilde\gamma^{i(k+l)}$	$q\tilde\gamma^{2ik}$	$(q-1)\tilde\gamma^{ik}$	·	·	·	·	·	·
$C_{42}(i)$	$\tilde\gamma^{2ik}$	$\tilde\gamma^{i(k+l)}$	·	$-\tilde\gamma^{ik}$	·	·	·	·	·	·
$C(i,j)$	$\tilde\gamma^{(i+j)k}$	$\tilde\gamma^{ik+jl} + \tilde\gamma^{il+jk}$	$\tilde\gamma^{(i+j)k}$	·	·	·	·	·	·	·
$D_{11}(i)$	1	·	-1	$-\beta_{ik}$	·	·	·	$(q-1)\beta_{ik}$	·	·
$D_{12}(i)$	1	·	-1	$-\beta_{ik}$	·	·	·	$-\beta_{ik}$	·	·
$E(i)$	$\tilde\gamma^{ik}$	·	$-\tilde\gamma^{ik}$	$-(\tilde\theta^{ik} + \tilde\theta^{qik})$	·	·	·	·	·	·

Charater Number of characters	$\left[\sum_{x\in\mathbb{F}_q^*} {}_Q\theta_3(x)\right]$ $q-1$	$\sum_{x\in\mathbb{F}_q} {}_Q\theta_4(x)$ q	${}_Q\theta_5(k) + {}_Q\theta_6(k)$ $k = 0, 1$ 4	${}_Q\theta_7$ 1	${}_Q\theta_8$ 1	${}_Q\theta_9$ 1	${}_Q\theta_{10}$ 1	${}_Q\theta_{11}$ 1	${}_Q\theta_{12}$ 1
A_0	$q(q-1)^2(q^2-1)$	$q^2(q-1)(q^2-1)$	$q(q-1)(q^2-1)$	$\frac{1}{2}q(q-1)^2$	$\frac{1}{2}q(q-1)^2$	$\frac{1}{2}q(q^2-1)$	$\frac{1}{2}q(q^2-1)$	$q(q-1)$	$q^2(q-1)$
A_1	$-q(q-1)^2$	$-q^2(q-1)$	$-q(q-1)$	$\frac{1}{2}q(q-1)^2$	$\frac{1}{2}q(q-1)^2$	$\frac{1}{2}q(q^2-1)$	$\frac{1}{2}q(q^2-1)$	$q(q-1)$	$q^2(q-1)$
A_2	·	·	·	$-\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q+1)$	$-\frac{1}{2}q(q+1)$	$-q$	$-q^2$
A_{31}	·	·	·	·	·	·	·	·	·
A_{32}	$-q(q-1)$ $(-3 \in S)$ $q(q-1)$ $(-3 \in N)$	·	$-q$ $(-3 \in S)$ q $(-3 \in N)$	·	·	·	·	·	·
A_{33}	$q(q-1)$ $(-3 \in S)$ $-q(q-1)$ $(-3 \in N)$	·	q $(-3 \in S)$ $q(q-1)$ $(-3 \in N)$	·	·	·	·	·	·
A_{41}	$-q(q-1)$	$-q(q-1)$	$q(q-1)$	$\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$	·	·
$A_{42}(i)$	$3q$ $(i=0)$ $\cdot$ $(i \neq 0)$	q	$-q[-3q]$ $(i=0)$ $\cdot$ $(i \neq 0)$	$\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{2}q(q-1)$	$-\frac{1}{2}q(q-1)$	·	·

$A_5(t)$	$q(\delta(t)+1)$	q	$-q(\delta(t)+1)$	$-q\ (-t \in S)$ $\cdot\ (-t \in N)$	$\cdot\ (-t \in S)$ $q\ (-t \in N)$	$\cdot\ (-t \in S)$ $-q\ (-t \in N)$	$q\ (-t \in S)$ $\cdot\ (-t \in N)$	$q\ (-t \in S)$ $-q\ (-t \in N)$	$\cdot$
A_{61}	$-q(q-3)\ (-3 \in S)$ $-q(q-1)\ (-3 \in N)$	$-q(q-1)$	$q(q-3)\ (-3 \in S)$ $q(q-1)\ (-3 \in N)$	$-\frac{1}{2}(1+\zeta)q$	$\frac{1}{2}(1-\zeta)q$	$-\frac{1}{2}(1-\zeta)q$	$\frac{1}{2}(1+\zeta)q$	ζq	$\cdot$
A_{62}	$-q(q-1)\ (-3 \in S)$ $-q(q-3)\ (-3 \in N)$	$-q(q-1)$	$q(q-1)\ (-3 \in S)$ $q(q-3)\ (-3 \in N)$	$-\frac{1}{2}(1-\zeta)q$	$\frac{1}{2}(1+\zeta)q$	$-\frac{1}{2}(1+\zeta)q$	$\frac{1}{2}(1-\zeta)q$	$-\zeta q$	$\cdot$
A_7	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$B_0(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$B_1(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$B_2(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$B_3(i,j)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
B_{11}	$\cdot$	$\cdot$	$(-1)^k(q-1)^2$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
B_{12}	$\cdot$	$\cdot$	$-(-1)^k(q-1)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
B_{13}	$\cdot$	$\cdot$	$-(-1)^k(q-1)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
B_{14}	$\cdot$	$\cdot$	$(-1)^k$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
B_{15}	$\cdot$	$\cdot$	$(-1)^k$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
B_{21}	$\cdot$	$\cdot$	$\cdot$	$-\frac{1}{2}(q-1)^2$	$-\frac{1}{2}(q-1)^2$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\zeta(q-1)$	$\zeta q(q-1)$
B_{22}	$\cdot$	$\cdot$	$\cdot$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	$\zeta(q-1)$	$\cdot$
B_{23}	$\cdot$	$\cdot$	$\cdot$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(q+1)$	$-\frac{1}{2}(q+1)$	$-\zeta$	$-\zeta q$
B_{24}	$\cdot$	$\cdot$	$\cdot$	$-\frac{1}{2}(q+1)$	$\frac{1}{2}(q-1)$	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(q+1)$	$-\zeta$	$\cdot$
B_{25}	$\cdot$	$\cdot$	$\cdot$	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(q+1)$	$-\frac{1}{2}(q+1)$	$\frac{1}{2}(q-1)$	$-\zeta$	$\cdot$
$C_{11}(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$C_{12}(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$C_{21}(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$q-1$	$q-1$	$(q-1)(-1)^i$	$(q-1)(-1)^i$
$C_{22}(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	-1	-1	$-(-1)^i$	$-(-1)^i$
$C_{31}(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$C_{32}(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$C_{41}(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$C_{42}(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$C(i,j)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$D_{11}(i)$	$\cdot$	$\cdot$	$\cdot$	$q-1$	$q-1$	$\cdot$	$\cdot$	$-(q-1)(-1)^i$	$(q-1)(-1)^i$
$D_{12}(i)$	$\cdot$	$\cdot$	$\cdot$	-1	-1	$\cdot$	$\cdot$	$(-1)^i$	$-(-1)^i$
$E(i)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Table A.8 The conjugacy classes of $\langle HX_aX_{3a+2b}, \omega_a\rangle$. (In this table $\lambda \in \mathbb{F}_q$ is a fixed nonsquare)

Notation	Class representative	Parameter	Number of classes	Order of centralizer
A_0	$h(1,1,1)$		1	$q^2(q-1)(q^2-1)$
A_1	$x_{3a+2b}(1)$		1	$q^2(q^2-1)$
A_{21}	$x_a(1)$		1	$q^2(q-1)$
A_{22}	$x_a(1)x_{3a+2b}(1)$		1	$2q^2$
A_{23}	$x_a(1)x_{3a+2b}(\lambda)$		1	$2q^2$
B_0	$h(\omega,\omega,\omega)$		1	$q(q-1)(q-\epsilon)$
B_1	$h(\omega,\omega,\omega)x_{3a+2b}(1)$		1	$q(q-\epsilon)$
B_{11}	$h(-1,1,-1)$		1	$q^2(q-1)(q^2-1)$
B_{12}	$h(-1,1,-1)x_{3a+2b}(1)$		1	$q^2(q^2-1)$
B_{13}	$h(-1,1,-1)x_a(1)$		1	$q^2(q-1)$
B_{14}	$h(-1,1,-1)x_a(1)x_{3a+2b}(1)$		1	$2q^2$
B_{15}	$h(-1,1,-1)x_a(1)x_{3a+2b}(\lambda)$		1	$2q^2$
B_2	$h(1,-1,-1)$		1	$(q-1)^2$
$C_{11}(i)$	$h_\gamma(i,-2i,i)$	$i \in {}^2R_1^*$	$(q-4-\epsilon)/2$	$q(q-1)^2$
$C_{12}(i)$	$h_\gamma(i,-2i,i)x_{3a+2b}(1)$	$i \in {}^2R_1^*$	$(q-4-\epsilon)/2$	$q(q-1)$
$C_{21}(i)$	$h_\gamma(i,0,-i)$	$i \in R_1$	$q-3$	$q(q-1)(q^2-1)$
$C_{22}(i)$	$h_\gamma(i,0,-i)x_a(1)$	$i \in R_1$	$q-3$	$q(q-1)$
$C_3(i)$	$h_\gamma(-2i,i,i)$	$i \in R_1^*$	$q-4-\epsilon$	$(q-1)^2$
$C_4(i)$	$h_\gamma(i,-i,0)$	$i \in R_1$	$q-3$	$(q-1)^2$
$C(i,j)$	$h_\gamma(i,j,-i-j)$	$(i,j) \in {}^2R_5$	$(q^2-8q+17+2\epsilon)/2$	$(q-1)^2$
$D_{11}(i)$	$h_\eta(i,-2i,i)$	$i \in {}^2S_1^*$	$(q-2+\epsilon)/2$	$q(q^2-1)$
$D_{12}(i)$	$h_\eta(i,-2i,i)x_{3a+2b}(1)$	$i \in {}^2S_1^*$	$(q-2+\epsilon)/2$	$q(q+1)$
$E(i)$	$h_\theta(i,(q-1)i,-qi)$	$i \in {}^2T_3$	$(q-1)^2/2$	q^2-1

Table A.9 The conjugacy classes of $\langle HX_bX_{2a+b}, \omega_b\rangle$. (In this table $\lambda \in \mathbb{F}_q$ is a fixed nonsquare)

Notation	Class representative	Parameter	Number of classes	Order of centralizer
A_0	$h(1,1,1)$		1	$q^2(q-1)(q^2-1)$
A_1	$x_{2a+b}(1)$		1	$q^2(q^2-1)$
A_{21}	$x_b(1)$		1	$q^2(q-1)$
A_{22}	$x_b(1)x_{2a+b}(1)$		1	$2q^2$
A_{23}	$x_b(1)x_{2a+b}(\lambda)$		1	$2q^2$
$B_0(i)$	$h_\omega(i,i,i)$	$1 \le i \le 1+\epsilon$	$1+\epsilon$	$q(q-1)(q^2-1)$
$B_1(i)$	$h_\omega(i,i,i)x_b(1)$	$1 \le i \le 1+\epsilon$	$1+\epsilon$	$q(q-1)$
B_{11}	$h(-1,-1,1)$		1	$q^2(q-1)(q^2-1)$
B_{12}	$h(-1,-1,1)x_b(1)$		1	$q^2(q-1)$
B_{13}	$h(-1,-1,1)x_{2a+b}(1)$		1	$q^2(q^2-1)$
B_{14}	$h(-1,-1,1)x_b(1)x_{2a+b}(1)$		1	$2q^2$
B_{15}	$h(-1,-1,1)x_b(1)x_{2a+b}(\lambda)$		1	$2q^2$
B_2	$h(1,-1,-1)$		1	$(q-1)^2$
$C_{11}(i)$	$h_\gamma(i,-i,0)$	$i \in {}^2R_1$	$(q-3)/2$	$q(q-1)^2$
$C_{12}(i)$	$h_\gamma(i,-i,0)x_{2a+b}(1)$	$i \in {}^2R_1$	$(q-3)/2$	$q(q-1)$
$C_{21}(i)$	$h_\gamma(i,i,-2i)$	$i \in R_1^*$	$q-4-\epsilon$	$q(q-1)(q^2-1)$
$C_{22}(i)$	$h_\gamma(i,i,-2i)x_b(1)$	$i \in R_1^*$	$q-4-\epsilon$	$q(q-1)$
$C_3(i)$	$h_\gamma(0,i,-i)$	$i \in R_1$	$q-3$	$(q-1)^2$
$C_4(i)$	$h_\gamma(i,-2i,i)$	$i \in R_1^*$	$q-4-\epsilon$	$(q-1)^2$
$C(i,j)$	$h_\gamma(i,j,-i-j)$	$(i,j) \in {}^2R_5'$	$(q^2-8q+17+2\epsilon)/2$	$(q-1)^2$
$D_{11}(i)$	$h_\eta(i,-i,0)$	$i \in {}^2S_1$	$(q-1)/2$	$q(q^2-1)$
$D_{12}(i)$	$h_\eta(i,-i,0)x_{2a+b}(1)$	$i \in {}^2S_1$	$(q-1)/2$	$q(q+1)$
$E(i)$	$h_\theta(i,qi,-(q+1)i)$	$i \in {}^2T_3$	$(q-1)^2/2$	q^2-1

Table A.10 The character table of $\langle HX_a X_{3a+2b}, \omega_a \rangle$, $\epsilon = -1$ [respectively $\epsilon = 1$]. (In this table, the parameter $\zeta \in \{1, -1\}$ depends on the residue class of q modulo 4 and is defined in Section 2)

Character Number of characters	$\chi_1(k)$ $k \in R_0$ $q-1$	$\chi_2(k,l)$ $(k,l) \in {}^2R_3$ $(q-1)(q-2)/2$	$\chi_3(k)$ $k \in R_0$ $q-1$	$\chi_4(k)$ $k \in {}^2T_1$ $q(q-1)/2$	$\chi_5(k)$ $k \in {}^2R_1$ $(q-3)/2$	$\chi_6(k)$ $k \in {}^2S_1$ $(q-1)/2$	θ_1 1	θ_2 1	$\theta_3(\delta)$ $\delta = \pm 1$ 2	$\theta_4(\delta)$ $\delta = \pm 1$ 2
A_0	1	$q+1$	q	$q-1$	q^2-1	$(q-1)^2$	$q-1$	$q(q-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q-1)^2$
A_1	1	$q+1$	q	$q-1$	$-(q+1)$	$-(q-1)$	-1	$-q$	$-\frac{1}{2}(q+1)$	$-\frac{1}{2}(q-1)$
A_{21}	1	1	.	-1	$q-1$	$-(q-1)$	$q-1$	.	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(q-1)$
A_{22}	1	1	.	-1	-1	1	-1	.	$\frac{1}{2}(q\delta-1)$	$\frac{1}{2}(q\delta+1)$
A_{23}	1	1	.	-1	-1	1	-1	.	$-\frac{1}{2}(q\delta+1)$	$-\frac{1}{2}(q\delta-1)$
B_0	1	$\cdot[\omega_{k+l}]$	$-1[1]$	$-\omega_k[\cdot]$	$\cdot[-(q-1)]$	$-(q-1)\omega_k[\cdot]$	$q-1$	$-(q-1)[q-1]$	$\cdot[q-1]$	$-(q-1)[\cdot]$
B_1	1	$\cdot[\omega_{k+l}]$	$-1[1]$	$-\omega_k[\cdot]$	$\cdot[1]$	$\omega_k[\cdot]$	-1	$1[-1]$	$\cdot[-1]$	$1[\cdot]$
B_{11}	1	$(q+1)(-1)^k$	q	$(q-1)(-1)^k$	$(q^2-1)(-1)^k$	$(q-1)^2(-1)^k$	$q-1$	$q(q-1)$	$\frac{1}{2}(q^2-1)\zeta$	$-\frac{1}{2}(q-1)^2\zeta$
B_{12}	1	$(q+1)(-1)^k$	q	$(q-1)(-1)^k$	$-(q+1)(-1)^k$	$-(q-1)(-1)^k$	-1	$-q$	$-\frac{1}{2}(q+1)\zeta$	$\frac{1}{2}(q-1)\zeta$
B_{13}	1	$(-1)^k$	.	$-(-1)^k$	$(q-1)(-1)^k$	$-(q-1)(-1)^k$	$q-1$	.	$\frac{1}{2}(q-1)\zeta$	$\frac{1}{2}(q-1)\zeta$
B_{14}	1	$(-1)^k$	.	$-(-1)^k$	$-(-1)^k$	$(-1)^k$	-1	.	$\frac{1}{2}(q\delta-1)\zeta$	$-\frac{1}{2}(q\delta+1)\zeta$
B_{15}	1	$(-1)^k$	.	$-(-1)^k$	$-(-1)^k$	$(-1)^k$	-1	.	$-\frac{1}{2}(q\delta+1)\zeta$	$\frac{1}{2}(q\delta-1)\zeta$
B_2	$(-1)^k$	$(-1)^{k+l}+(-1)^l$	$(-1)^k$	.	.	.	.	.	.	.
$C_{11}(i)$	1	$\alpha_{i(k-2l)}$	1	.	$(q-1)\alpha_{ik}$	.	$q-1$	$q-1$	$(q-1)(-1)^i$	.
$C_{12}(i)$	1	$\alpha_{i(k-2l)}$	1	.	$-\alpha_{ik}$	.	-1	-1	$-(-1)^i$	.
$C_{21}(i)$	$\tilde{\gamma}^{2ik}$	$(q+1)\tilde{\gamma}^{ik}$	$q\tilde{\gamma}^{2ik}$	$(q-1)\tilde{\gamma}^{ik}$	.	.	.	.	.	.
$C_{22}(i)$	$\tilde{\gamma}^{2ik}$	$\tilde{\gamma}^{ik}$	.	$-\tilde{\gamma}^{ik}$	.	.	.	.	.	.
$C_3(i)$	$\tilde{\gamma}^{-3ik}$	$\tilde{\gamma}^{i(l-2k)}+\tilde{\gamma}^{-i(k+l)}$	$\tilde{\gamma}^{-3ik}$	.	.	.	.	.	.	.
$C_4(i)$	$\tilde{\gamma}^{ik}$	$\tilde{\gamma}^{i(k-l)}+\tilde{\gamma}^{il}$	$\tilde{\gamma}^{ik}$	.	.	.	.	.	.	.
$C(i,j)$	$\tilde{\gamma}^{(2i+j)k}$	$\tilde{\gamma}^{ik+jl}+\tilde{\gamma}^{ik+j(k-l)}$	$\tilde{\gamma}^{(2i+j)k}$	.	.	.	.	.	.	.
$D11(i)$	1	.	-1	$-\beta_{ik}$	.	$-(q-1)\beta ik$	$q-1$	$-(q-1)$	.	$-(q-1)(-1)^i$
$D12(i)$	1	.	-1	$-\beta_{ik}$	.	βik	-1	1	.	$(-1)^i$
$E(i)$	$\tilde{\gamma}^{ik}$	.	$-\tilde{\gamma}^{ik}$	$-(\tilde{\theta}^{ik}+\tilde{\theta}^{qik})$	.	.	.	.	.	.

Table A.11 The character table of $\langle HX_bX_{2a+b}, \omega_b \rangle$. (In this table, the parameter $\zeta \in \{1, -1\}$ depends on the residue class of q modulo 4 and is defined in Section 2)

Character Number of characters	$\chi_1(k)$ $k \in R_0$ $q-1$	$\chi_2(k,l)$ $(k,l) \in {}^2R_2$ $(q-1)(q-2)/2$	$\chi_3(k)$ $k \in R_0$ $q-1$	$\chi_4(k)$ $k \in {}^2T_1$ $q(q-1)/2$	$\chi_5(k)$ $k \in {}^2R_1$ $(q-3)/2$	$\chi_6(k)$ $k \in {}^2S_1$ $(q-1)/2$	θ_1 1	θ_2 1	$\theta_3(\delta)$ $\delta = \pm 1$ 2	$\theta_4(\delta)$ $\delta = \pm 1$ 2
A_0	1	$q+1$	q	$q-1$	q^2-1	$(q-1)^2$	$q-1$	$q(q-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q-1)^2$
A_1	1	$q+1$	q	$q-1$	$-(q+1)$	$-(q-1)$	-1	$-q$	$-\frac{1}{2}(q+1)$	$-\frac{1}{2}(q-1)$
A_{21}	1	1	.	-1	$q-1$	$-(q-1)$	$q-1$	.	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(q-1)$
A_{22}	1	1	.	-1	-1	1	-1	.	$\frac{1}{2}(q\delta-1)$	$\frac{1}{2}(q\delta+1)$
A_{23}	1	1	.	-1	-1	1	-1	.	$-\frac{1}{2}(q\delta+1)$	$-\frac{1}{2}(q\delta-1)$
$B_0(i)$	$\tilde{\omega}^{-ik}$	$(q+1)\tilde{\omega}^{i(k+l)}$	$q\tilde{\omega}^{-ik}$	$(q-1)\tilde{\omega}^{ik}$	.	.	.	.	.	.
$B_1(i)$	$\tilde{\omega}^{-ik}$	$\tilde{\omega}^{i(k+l)}$	.	$-\tilde{\omega}^{ik}$	.	.	.	.	.	.
B_{11}	1	$(q+1)(-1)^{k+l}$	q	$(q-1)(-1)^k$	$(q^2-1)(-1)^k$	$(q-1)^2(-1)^k$	$q-1$	$q(q-1)$	$\frac{1}{2}(q^2-1)\zeta$	$-\frac{1}{2}(q-1)^2\zeta$
B_{12}	1	$(-1)^{k+l}$	.	$-(-1)^k$	$(q-1)(-1)^k$	$-(q-1)(-1)^k$	$q-1$	.	$\frac{1}{2}(q-1)\zeta$	$\frac{1}{2}(q-1)\zeta$
B_{13}	1	$(q+1)(-1)^{k+l}$	q	$(q-1)(-1)^k$	$-(q+1)(-1)^k$	$-(q-1)(-1)^k$	-1	$-q$	$-\frac{1}{2}(q+1)\zeta$	$\frac{1}{2}(q-1)\zeta$
B_{14}	1	$(-1)^{k+l}$	.	$-(-1)^k$	$-(-1)^k$	$(-1)^k$	-1	.	$\frac{1}{2}(q\delta-1)\zeta$	$-\frac{1}{2}(q\delta+1)\zeta$
B_{15}	1	$(-1)^{k+l}$	.	$-(-1)^k$	$-(-1)^k$	$(-1)^k$	-1	.	$-\frac{1}{2}(q\delta+1)\zeta$	$\frac{1}{2}(q\delta-1)\zeta$
B_2	$(-1)^k$	$(-1)^k+(-1)^l$	$(-1)^k$	.	.	.	.	.	.	.
$C_{11}(i)$	1	$\tilde{\gamma}^{i(k-l)}+\tilde{\gamma}^{i(l-k)}$	1	.	$(q-1)\alpha_{ik}$	.	$q-1$	$q-1$	$(q-1)(-1)^i$	.
$C_{12}(i)$	1	$\tilde{\gamma}^{i(k-l)}+\tilde{\gamma}^{i(l-k)}$	1	.	$-\alpha_{ik}$	.	-1	-1	$-(-1)^i$	.
$C_{21}(i)$	$\tilde{\gamma}^{2ik}$	$(q+1)\tilde{\gamma}^{i(k+l)}$	$q\tilde{\gamma}^{2ik}$	$(q-1)\tilde{\gamma}^{ik}$	.	.	.	.	.	.
$C_{22}(i)$	$\tilde{\gamma}^{2ik}$	$\tilde{\gamma}^{i(k+l)}$	.	$-\tilde{\gamma}^{ik}$	.	.	.	.	.	.
$C_3(i)$	$\tilde{\gamma}^{ik}$	$\tilde{\gamma}^{ik}+\tilde{\gamma}^{il}$	$\tilde{\gamma}^{ik}$	.	.	.	.	.	.	.
$C_4(i)$	$\tilde{\gamma}^{-ik}$	$\tilde{\gamma}^{i(k-2l)}+\tilde{\gamma}^{i(l-2k)}$	$\tilde{\gamma}^{-ik}$	.	.	.	.	.	.	.
$C(i,j)$	$\tilde{\gamma}^{(i+j)k}$	$\tilde{\gamma}^{ik+jl}+\tilde{\gamma}^{il+jk}$	$\tilde{\gamma}^{(i+j)k}$	.	.	.	.	.	.	.
$D11(i)$	1	.	-1	$-\beta_{ik}$	.	$-(q-1)\beta ik$	$q-1$	$-(q-1)$	.	$-(q-1)(-1)^i$
$D12(i)$	1	.	-1	$-\beta_{ik}$	.	βik	-1	1	.	$(-1)^i$
$E(i)$	$\tilde{\gamma}^{ik}$	.	$-\tilde{\gamma}^{ik}$	$-(\tilde{\theta}^{ik}+\tilde{\theta}^{qik})$	.	.	.	.	.	.

Table A.12 The conjugacy classes of $B_M = H(\omega_a)U_0$, $\epsilon = \pm 1$. (In this table v denotes an element of $\mathbb{F}_{q^2}$ such that $v + v^q = 1$)

Notation	Class representative	Parameter	Number of classes	Order of centralizer
$A_0(i)$	$h_\omega(i, i, i)$	$0 \le i \le 1 - \epsilon$	$2 - \epsilon$	$q^3(q^2 - 1)$
$A_1(i)$	$h_\omega(i, i, i)x_{3a+2b}(1)$	$0 \le i \le 1 - \epsilon$	$2 - \epsilon$	$q^3(q + 1)$
$A_2(i, 0)$	$h_\omega(i, i, i)x_b(1)x_{3a+b}(1)x_{3a+2b}(v)$	$0 \le i \le 1 - \epsilon$	$2 - \epsilon$	$q^2(2 - \epsilon)$
$A_2(i, j)$	$h_\omega(i, i, i)x_b(\eta^j)x_{3a+b}(\eta^{-j})x_{3a+2b}(\omega)$	$0 \le i \le 1 - \epsilon$ $1 \le j \le 1 - \epsilon$	$3(1 - \epsilon)$	$3q^2$
B_1	$h(-1, 1, -1)$		1	$q(q^2 - 1)$
B_2	$h(-1, 1, -1)x_{3a+2b}(1)$		1	$q(q + 1)$
$C(i)$	$h_\gamma(i, 0, -i)$	$i \in R_1$	$q - 3$	$q^2 - 1$
$D_1(i)$	$h_\eta(i, -2i, i)$	$i \in S_1^*$	$q - 2 + \epsilon$	$q(q^2 - 1)$
$D_2(i)$	$h_\eta(i, -2i, i)x_{3a+2b}(1)$	$i \in S_1^*$	$q - 2 + \epsilon$	$q(q + 1)$
$E(i)$	$h_\theta(i, (q - 1)i, -qi)$	$i \in T_3$	$(q - 1)^2$	$q^2 - 1$

Table A.13 The character table of $B_M = H(\omega_a)U_0$, $\epsilon = -1$ [respectively $\epsilon = 1$]

Character Number of Characters	$\chi_1(k)$ $k \in T_0$ $q^2 - 1$	$\chi_2(k)$ $k \in S_0$ $q + 1$	$\theta(k, l)$ $k, l = 0, 1, 2$ 9	$[\theta]$ 1
$A_0(i)$	$\tilde{\omega}^{ik}$	$q(q - 1)\tilde{\omega}^{ik}$	$\frac{1}{3}(q^2 - 1)\tilde{\omega}^{ik}$	$q^2 - 1$
$A_1(i)$	$\tilde{\omega}^{ik}$	$-q\tilde{\omega}^{ik}$	$\frac{1}{3}(q^2 - 1)\tilde{\omega}^{ik}$	$q^2 - 1$
$A_2(i, j)$	$\tilde{\omega}^{ik}$	$\cdot$	$\frac{1}{3}(q\omega_{j-l} - 1)\tilde{\omega}^{ik}$	-1
B_1	$(-1)^k$	$-(q - 1)(-1)^k$	$\cdot$	$\cdot$
B_2	$(-1)^k$	$(-1)^k$	$\cdot$	$\cdot$
$C(i)$	$\tilde{\gamma}^{ik}$	$\cdot$	$\cdot$	$\cdot$
$D_1(i)$	$\tilde{\eta}^{ik}$	$-(q - 1)\tilde{\eta}^{ik}$	$\cdot$	$\cdot$
$D_2(i)$	$\tilde{\eta}^{ik}$	$\tilde{\eta}^{ik}$	$\cdot$	$\cdot$
$E(i)$	$\tilde{\theta}^{ik}$	$\cdot$	$\cdot$	$\cdot$

Table A.14 The conjugacy classes of $B_L = HX_bX_{3a+b}X_{3a+2b}$, $\epsilon = \pm 1$

Notation	Class representative	Parameter	Number of classes	Order of centralizer
$A_0(i)$	$h_\omega(i, i, i)$	$0 \le i \le 1 + \epsilon$	$2 + \epsilon$	$q^3(q - 1)^2$
$A_1(i)$	$h_\omega(i, i, i)x_{3a+2b}(1)$	$0 \le i \le 1 + \epsilon$	$2 + \epsilon$	$q^3(q - 1)$
$A_2(i)$	$h_\omega(i, i, i)x_{3a+b}(1)$	$0 \le i \le 1 + \epsilon$	$2 + \epsilon$	$q^2(q - 1)$
$A_3(i)$	$h_\omega(i, i, i)x_b(1)$	$0 \le i \le 1 + \epsilon$	$2 + \epsilon$	$q^2(q - 1)$
$A_4(i, j)$	$h_\omega(i, i, i)x_b(1)x_{3a+b}(\gamma^j)$	$0 \le i, j \le 1 + \epsilon$	$(2 + \epsilon)^2$	$q^2(2 + \epsilon)$
B_{11}	$h(-1, 1, -1)$		1	$q(q - 1)^2$
B_{12}	$h(-1, 1, -1)x_{3a+2b}(1)$		1	$q(q - 1)$
B_{21}	$h(1, -1, -1)$		1	$q(q - 1)^2$
B_{22}	$h(1, -1, -1)x_{3a+b}(1)$		1	$q(q - 1)$
B_{31}	$h(-1, -1, 1)$		1	$q(q - 1)^2$
B_{32}	$h(-1, -1, 1)x_b(1)$		1	$q(q - 1)$
$C_{11}(i)$	$h_\gamma(i, -2i, i)$	$i \in R_1^*$	$q - 4 - \epsilon$	$q(q - 1)^2$
$C_{12}(i)$	$h_\gamma(i, -2i, i)x_{3a+2b}(1)$	$i \in R_1^*$	$q - 4 - \epsilon$	$q(q - 1)$
$C_{21}(i)$	$h_\gamma(-2i, i, i)$	$i \in R_1^*$	$q - 4 - \epsilon$	$q(q - 1)^2$
$C_{22}(i)$	$h_\gamma(-2i, i, i)x_{3a+b}(1)$	$i \in R_1^*$	$q - 4 - \epsilon$	$q(q - 1)$
$C_{31}(i)$	$h_\gamma(i, i, -2i)$	$i \in R_1^*$	$q - 4 - \epsilon$	$q(q - 1)^2$
$C_{32}(i)$	$h_\gamma(i, i, -2i)x_b(1)$	$i \in R_1^*$	$q - 4 - \epsilon$	$q(q - 1)$
$C(i, j)$	$h_\gamma(i, j, -i - j)$	$(i, j) \in R_7$	$q^2 - 5q + 8 + 2\epsilon$	$(q - 1)^2$

Table A.15 The character table of $B_L = HX_bX_{3a+b}X_{3a+2b}$, $\epsilon = -1$ [respectively $\epsilon = 1$]

Character	$\chi_1(k,l)$	$\chi_2(k)$	$\chi_3(k)$	$\chi_4(k)$	θ	$[\theta(k,l)]$
Number of	$k,l \in R_0$	$k \in R_0$	$k \in R_0$	$k \in R_0$		$k,l = 0,1,2$
characters	$(q-1)^2$	$q-1$	$q-1$	$q-1$	1	9
$A_0(i)$	$\tilde{\omega}^{i(k+l)}$	$(q-1)\tilde{\omega}^{ik}$	$(q-1)\tilde{\omega}^{ik}$	$q(q-1)\tilde{\omega}^{ik}$	$(q-1)^2$	$\frac{1}{3}(q-1)^2\tilde{\omega}^{ik}$
$A_1(i)$	$\tilde{\omega}^{i(k+l)}$	$(q-1)\tilde{\omega}^{ik}$	$(q-1)\tilde{\omega}^{ik}$	$-q\tilde{\omega}^{ik}$	$(q-1)^2$	$\frac{1}{3}(q-1)^2\tilde{\omega}^{ik}$
$A_2(i)$	$\tilde{\omega}^{i(k+l)}$	$(q-1)\tilde{\omega}^{ik}$	$-\tilde{\omega}^{ik}$	·	$-(q-1)$	$-\frac{1}{3}(q-1)\tilde{\omega}^{ik}$
$A_3(i)$	$\tilde{\omega}^{i(k+l)}$	$-\tilde{\omega}^{ik}$	$(q-1)\tilde{\omega}^{ik}$	·	$-(q-1)$	$-\frac{1}{3}(q-1)\tilde{\omega}^{ik}$
$A_4(i,j)$	$\tilde{\omega}^{i(k+l)}$	$-\tilde{\omega}^{ik}$	$-\tilde{\omega}^{ik}$	·	1	$\frac{1}{3}(q\omega_{j-l}+1)\tilde{\omega}^{ik}$
B_{11}	$(-1)^{l-2k}$	·	·	$(q-1)(-1)^k$	·	·
B_{12}	$(-1)^{l-2k}$	·	·	$-(-1)^k$	·	·
B_{21}	$(-1)^{k+l}$	·	$(q-1)(-1)^k$	·	·	·
B_{22}	$(-1)^{k+l}$	·	$-(-1)^k$	·	·	·
B_{31}	$(-1)^{k-2l}$	$(q-1)(-1)^k$	·	·	·	·
B_{32}	$(-1)^{k-2l}$	$-(-1)^k$	·	·	·	·
$C_{11}(i)$	$\tilde{\gamma}^{i(k-2l)}$	·	·	$(q-1)\tilde{\gamma}^{ik}$	·	·
$C_{12}(i)$	$\tilde{\gamma}^{i(k-2l)}$	·	·	$-\tilde{\gamma}^{ik}$	·	·
$C_{21}(i)$	$\tilde{\gamma}^{i(l-2k)}$	·	$(q-1)\tilde{\gamma}^{ik}$	·	·	·
$C_{22}(i)$	$\tilde{\gamma}^{i(l-2k)}$	·	$-\tilde{\gamma}^{ik}$	·	·	·
$C_{31}(i)$	$\tilde{\gamma}^{i(k+l)}$	$(q-1)\tilde{\gamma}^{ik}$	·	·	·	·
$C_{32}(i)$	$\tilde{\gamma}^{i(k+l)}$	$-\tilde{\gamma}^{ik}$	·	·	·	·
$C(i,j)$	$\tilde{\gamma}^{ik+jl}$	·	·	·	·	·

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