

# Separate Continuity, Joint Continuity, the Lindelöf Property and $p$ -spaces

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**Abstract.** In this paper we prove a theorem more general than the following. Suppose that  $X$  is Čech-complete and  $Y$  is a closed subset of a product of a separable metric space with a compact Hausdorff space. Then for each separately continuous function  $f : X \times Y \rightarrow \mathbb{R}$  there exists a residual set  $R$  in  $X$  such that  $f$  is jointly continuous at each point of  $R \times Y$ . This confirms the suspicions of S. Mercourakis and S. Negrepontis from 1991.

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## 1 Introduction

If  $X$ ,  $Y$  and  $Z$  are topological spaces and  $f : X \times Y \rightarrow Z$  is a function then we say that  $f$  is *jointly continuous at*  $(x_0, y_0) \in X \times Y$  if for each neighbourhood  $W$  of  $f(x_0, y_0)$  there exists a product of open sets  $U \times V \subseteq X \times Y$  containing  $(x_0, y_0)$  such that  $f(U \times V) \subseteq W$  and we say that  $f$  is *separately continuous on*  $X \times Y$  if for each  $x_0 \in X$  and  $y_0 \in Y$  the functions  $y \mapsto f(x_0, y)$  and  $x \mapsto f(x, y_0)$  are both continuous on  $Y$  and  $X$  respectively. If the range space  $Z$  is a metric space, with metric  $d$ , and  $\varepsilon$  is a positive number then we say that  $f$  is  $\varepsilon$ -*jointly continuous at*  $(x_0, y_0) \in X \times Y$  if there exists a product of open sets  $U \times V \subseteq X \times Y$  containing  $(x_0, y_0) \in X \times Y$  such that  $d\text{-diam}f(U \times V) \leq \varepsilon$ .

Since the paper [2] of Baire first appeared there has been continued interest in the question of when a separately continuous function defined on a product of “nice” spaces admit a point (or many points) of joint continuity and over the years there have been many contributions to this area. Most of these results can be classified into one of two types. (I) The existence problem, i.e., if  $f : X \times Y \rightarrow \mathbb{R}$  is separately continuous find conditions on either  $X$  or  $Y$  (or both) such that  $f$  has at least one point of joint continuity. (II) The fibre problem, i.e., if  $f : X \times Y \rightarrow \mathbb{R}$  is separately continuous find conditions on either  $X$  or  $Y$  (or both) such that there exists a “big” subset  $R$  of  $X$  such that  $f$  is jointly continuous at the points of  $R \times Y$ . Our interest in this paper is in the fibre problem. Specifically, we are interested in providing an extension of the following result of M. Talagrand, [7, p. 503].

“Let  $f : X \times Y \rightarrow \mathbb{R}$  be a separately continuous function defined on the product of Čech-complete spaces  $X$  and  $Y$ . If  $Y$  is Lindelöf then there exists a dense  $G_\delta$  subset  $R$  of  $X$  such that  $f$  is jointly continuous at each point of  $R \times Y$ .”

This result of Talagrand is distinctive within the literature because it does not require the space  $Y$  to be either compact ( $\sigma$ -compact) or second countable. What we shall do is show that the conclusion of Talagrand’s theorem remains valid when one: (i) weakens the hypothesis on  $X$  from being Čech-complete to being conditionally  $\alpha$ -favourable and (ii) reduces the hypotheses on  $Y$  from being Lindelöf and Čech-complete to being a Lindelöf  $p$ -space (Recall that a completely regular

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space  $X$  is a  $p$ -space if there exists a sequence  $(\mathcal{G}_n : n \in \mathbb{N})$  of open covers of  $X$  such that if  $x \in X$  and  $x \in G_n \in \mathcal{G}_n$  for each  $n \in \mathbb{N}$ , then  $\bigcap_{n \in \mathbb{N}} \overline{G_n}$  is a compact set for which the sequence  $(\bigcap_{1 \leq k \leq n} \overline{G_k} : n \in \mathbb{N})$  is an outer network, i.e., if  $\bigcap_{n \in \mathbb{N}} \overline{G_n} \subseteq U$  for some open set  $U$  then there exists an  $n \in \mathbb{N}$  such that  $\bigcap_{1 \leq k \leq n} \overline{G_k} \subseteq U$ , [4, Theorem 3.21], or see [1] for the original definition.). In the special case when  $X$  is Čech-complete this confirms the suspicions of the authors in [7, p. 503] and fills, what is probably, a much needed gap in the literature. For more information on problem (II) see [7, p. 495-536].

Some form of our first lemma may be found in many of the papers written on separate and joint continuity.

**Lemma 1** *Let  $X$  and  $Y$  be topological spaces,  $\varepsilon$  be a positive number and  $(Z, d)$  be a metric space. If  $f : X \times Y \rightarrow Z$ ,  $y \mapsto f(x_0, y)$  is continuous on  $Y$  and there exists a pair of open neighbourhoods  $U$  of  $x_0 \in X$  and  $V$  of  $y_0 \in Y$  such that  $d(f(x, y), f(x', y)) \leq \varepsilon/3$  for all  $x$  and  $x'$  in  $U$  and  $y$  in  $V$  then  $f$  is  $\varepsilon$ -jointly continuous at  $(x_0, y_0) \in X \times Y$ .*

**Proof:** Let  $U$  be an open neighbourhood of  $x_0$  and  $V$  be an open neighbourhood of  $y_0$  such that  $d(f(x, y), f(x', y)) \leq \varepsilon/3$  for all  $x$  and  $x'$  in  $U$  and  $y$  in  $V$ . Since  $y \mapsto f(x_0, y)$  is continuous on  $Y$  we can assume, by possibly making  $V$  smaller, that  $d(f(x_0, y), f(x_0, y_0)) < \varepsilon/6$  for all  $y \in V$ . Therefore, for any  $(x, y)$  and  $(x', y')$  in  $U \times V$ ,

$$\begin{aligned} d(f(x, y), f(x', y')) &\leq d(f(x, y), f(x_0, y)) + d(f(x_0, y), f(x_0, y')) + d(f(x_0, y'), f(x', y')) \\ &< d(f(x, y), f(x_0, y)) + d(f(x_0, y'), f(x', y')) + \varepsilon/3 \leq \varepsilon. \end{aligned}$$

Hence,  $f$  is  $\varepsilon$ -jointly continuous at  $(x_0, y_0) \in X \times Y$ .  $\odot$

For a topological space  $Y$  we shall denote by  $C(Y)$  the set of all real-valued continuous functions defined on  $Y$  and by  $C_p(Y)$  the set  $C(Y)$  endowed with the topology of pointwise convergence on  $Y$ . Further, if  $X$  is a topological space and  $f : X \rightarrow C(Y)$  then the mapping  $\tilde{f} : X \times Y \rightarrow \mathbb{R}$  defined by,  $\tilde{f}(x, y) := f(x)(y)$  is separately continuous on  $X \times Y$  if, and only if,  $f : X \rightarrow C_p(Y)$  is continuous. Hence there is a natural correspondence between the study of real-valued separately continuous functions on  $X \times Y$  and the study of continuous mappings from  $X$  into  $C_p(Y)$ . With this in mind, we introduce the following definitions. We say that a mapping  $f : X \rightarrow C(Y)$  is *jointly continuous* at  $(x_0, y_0) \in X \times Y$  if the function  $\tilde{f}$  is jointly continuous at  $(x_0, y_0)$  and for each  $\varepsilon > 0$ , we will say that  $f$  is  $\varepsilon$ -*jointly continuous* at  $(x_0, y_0)$  if the function  $\tilde{f}$  is  $\varepsilon$ -jointly continuous at  $(x_0, y_0)$ .

With these definitions under our belt we can rephrase Lemma 1 as follows.

**Lemma 2** *Let  $X$  and  $Y$  be topological spaces and let  $f : X \rightarrow C(Y)$ . If for some  $\varepsilon > 0$  there exists a pair of open neighbourhoods  $U$  of  $x_0 \in X$  and  $V$  of  $y_0 \in Y$  such that  $|f(x)(y) - f(x')(y)| \leq \varepsilon/3$  for all  $x$  and  $x'$  in  $U$  and  $y$  in  $V$ , then  $f$  is  $\varepsilon$ -jointly continuous at  $(x_0, y_0) \in X \times Y$ .*

## 2 Main Result

To formulate the statement of our main theorem we will need to consider the following topological game.

Let  $(X, \tau)$  be a topological space. The  $\mathcal{G}_X$ -game played on  $X$  is played by two players  $\alpha$  and  $\beta$ . Player  $\beta$  starts by choosing a non-empty open subset  $B_1$  of  $X$ . Player  $\alpha$  then selects a non-empty

open subset  $A_1 \subseteq B_1$  and an element  $a_1 \in X$ .  $\beta$  follows by choosing a non-empty open subset  $B_2 \subseteq A_1 \subseteq B_1$  and  $\alpha$  responds by selecting a non-empty open subset  $A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1$  and a point  $a_2 \in X$ . The players continue this procedure indefinitely to produce a *play* of the  $\mathcal{G}_X$ -game. We say that  $\alpha$  *wins* a play of the  $\mathcal{G}_X$ -game if either  $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$  or  $\bigcap_{n \in \mathbb{N}} A_n \cap \{a_n : n \in \mathbb{N}\} \neq \emptyset$ ; otherwise  $\beta$  wins. A strategy  $s$  for the player  $\alpha$  is a “rule” that tells him/her how to play. More precisely, a strategy  $s$  for  $\alpha$  is a sequence of mappings  $s := (s_n : n \in \mathbb{N})$  defined inductively as follows: The domain of  $s_1$  is  $\tau \setminus \{\emptyset\}$  and to every element  $B_1$  of  $\tau \setminus \{\emptyset\}$

$$s_1(B_1) := (a_{(B_1)}, A_{(B_1)}) \in X \times (\tau \setminus \{\emptyset\})$$

where  $A_{(B_1)} \subseteq B_1$ . In general, the domain of  $s_{n+1}$  consists of all finite sequences  $(B_1, B_2, \dots, B_{n+1})$  in  $(\tau \setminus \{\emptyset\})^{n+1}$  such that

$$B_1 \supseteq A_2 \supseteq B_2 \supseteq A_2 \supseteq \dots \supseteq B_n \supseteq A_n \supseteq B_{n+1}$$

where for every  $1 \leq k \leq n$ ,  $(B_1, B_2, \dots, B_k)$  is from the domain of  $s_k$ ,  $(a_k, A_k) := s_k(B_1, B_2, \dots, B_k)$  and  $B_{n+1}$  is an arbitrary non-empty open subset of  $A_n$ . For any such sequence  $(B_1, B_2, \dots, B_{n+1})$ ,

$$s_{n+1}(B_1, B_2, \dots, B_{n+1}) := (a_{(B_1, B_2, \dots, B_{n+1})}, A_{(B_1, B_2, \dots, B_{n+1})}) \in X \times (\tau \setminus \{\emptyset\})$$

where  $A_{(B_1, B_2, \dots, B_{n+1})} \subseteq B_{n+1}$ .

Any finite [infinite] sequence  $(B_1, B_2, \dots, B_{n+1}) [(B_n : n \in \mathbb{N})]$  such that  $B_{k+1} \subseteq A_k$ , for all  $1 \leq k \leq n$  [for all  $k \in \mathbb{N}$ ] where  $(a_k, A_k) := s_k(B_1, B_2, \dots, B_k)$  is called an *s-sequence*. We shall call a strategy  $s := (s_n : n \in \mathbb{N})$  for  $\alpha$  a *winning strategy* if each infinite *s-sequence* is won by  $\alpha$ . We shall call a topological space  $(X, \tau)$  *conditionally  $\alpha$ -favourable* if  $\alpha$  has a winning strategy in the  $\mathcal{G}_X$ -game played on  $X$ .

It is easy to see that all metric spaces are conditionally  $\alpha$ -favourable, as indeed are all  $p$ -space. However, there are many other examples such as Čech-analytic spaces, or more generally, spaces with countable separation (see, [6, p. 213] for the definition of countable separation). The class  $\chi$  of spaces considered in [3], which includes arbitrary products of  $p$ -spaces, are also conditionally  $\alpha$ -favourable. Finally, let us also mention that all separable spaces are conditionally  $\alpha$ -favourable. In the other direction, all conditionally  $\alpha$ -favourable Baire spaces are  $\sigma$ - $\beta$ -unfavourable, as defined in [9].

In addition to the  $\mathcal{G}_X$ -game we also need to consider the *Banach-Mazur game*. Let  $(X, \tau)$  be a topological space and let  $R$  be a subset of  $X$ . On  $X$  we consider the *BM(R)-game* played between two players  $\alpha$  and  $\beta$ . A *play* of the *BM(R)-game* is a decreasing sequence of non-empty open subsets  $A_n \subseteq B_n \subseteq A_{n-1} \subseteq \dots \subseteq A_1 \subseteq B_1$  which have been chosen alternately; the  $A_n$ 's by  $\alpha$  and the  $B_n$ 's by  $\beta$ . The player  $\alpha$  is said to have *won* a play of the *BM(R)-game* if  $\bigcap_{n \in \mathbb{N}} A_n \subseteq R$ ; otherwise  $\beta$  is said to have won. A strategy  $s$  for the player  $\alpha$  is a “rule” that tells him/her how to play. More precisely, a strategy  $s$  for  $\alpha$  is a sequence of mappings  $s := (s_n : n \in \mathbb{N})$  defined inductively as follows: The domain of  $s_1$  is  $\tau \setminus \{\emptyset\}$  and to every element  $B_1$  of  $\tau \setminus \{\emptyset\}$ ,  $s_1(B_1)$  is a non-empty open subset of  $B_1$ . In general, the domain of  $s_{n+1}$  consists of all finite sequences  $(B_1, B_2, \dots, B_{n+1}) \in (\tau \setminus \{\emptyset\})^{n+1}$  such that

$$B_1 \supseteq A_2 \supseteq B_2 \supseteq A_2 \supseteq \dots \supseteq B_n \supseteq A_n \supseteq B_{n+1}$$

where for every  $1 \leq k \leq n$ ,  $(B_1, B_2, \dots, B_k)$  is from the domain of  $s_k$ ,  $A_k := s_k(B_1, B_2, \dots, B_k)$  and  $B_{n+1}$  is an arbitrary non-empty open subset of  $A_n$ . For any such sequence  $(B_1, B_2, \dots, B_{n+1})$ ,  $s_{n+1}(B_1, B_2, \dots, B_{n+1})$  is a non-empty open subset of  $B_{n+1}$ .

Any finite [infinite] sequence  $(B_1, B_2, \dots, B_{n+1})$  [ $(B_n : n \in \mathbb{N})$ ] such that  $B_{k+1} \subseteq A_k$ , for all  $1 \leq k \leq n$  [for all  $k \in \mathbb{N}$ ] where  $A_k := s_k(B_1, B_2, \dots, B_k)$  is called an *s-sequence*. We shall call a strategy  $s := (s_n : n \in \mathbb{N})$  for  $\alpha$  a *winning strategy* if each infinite *s-sequence* is won by  $\alpha$ .

The following theorem reveals our interest in the Banach-Mazur game.

**Theorem 1** [8] *Let  $R$  be a subset of a topological space  $X$ . Then  $R$  is residual in  $X$  (i.e., contains, as a subset, a countable intersection of dense open subsets of  $X$ ) if, and only if, the player  $\alpha$  has a winning strategy in the  $BM(R)$ -game played on  $X$ .*

The proof of our main result (i.e., Theorem 2) requires two elementary facts from general topology.

**Lemma 3** *Let  $f : X \rightarrow Y$  be continuous mapping acting between topological spaces  $X$  and  $Y$ . If  $\{C_k : 1 \leq k \leq n\}$  is a family of closed subsets of  $Y$  and  $U$  is a non-empty open subset of  $X$  such that  $f(U) \subseteq \bigcup_{k=1}^n C_k$  then there exists a  $k_0 \in \{1, 2, \dots, n\}$  and a non-empty open subset  $W$  of  $U$  such that  $f(W) \subseteq C_{k_0}$ .*

**Proof:** For each  $k \in \{1, 2, \dots, n\}$ , let  $U_k := \{u \in U : f(u) \in C_k\}$ . Then  $\{U_k : 1 \leq k \leq n\}$  is a closed cover of  $U$ . Hence, by a simple induction (on  $n$ ), there is some  $k_0 \in \{1, 2, \dots, n\}$  such that  $W := \text{int } U_{k_0} \neq \emptyset$ . This completes the proof.  $\odot$

The following lemma is contained in the proof of Stone's well-known "lattice formulation" of the Stone-Weierstrass Theorem, see [5, p. 244] or [10].

**Lemma 4** *Let  $K$  be a compact subset of a topological space  $Y$  and let  $\tau_p$  denote the topology on  $C(Y)$  of pointwise convergence on  $Y$ . If  $L$  is a sub-lattice of  $C(Y)$  and  $f \in \overline{L}^{\tau_p}$  then for each  $\varepsilon > 0$  there exists an element  $l_\varepsilon \in L$  and an open subset  $U_\varepsilon$  of  $Y$ , containing  $K$ , such that  $|f(u) - l_\varepsilon(u)| < \varepsilon$  for all  $u \in U_\varepsilon$ .*

**Theorem 2** *Suppose that  $X$  is a conditionally  $\alpha$ -favourable space and  $Y$  is a closed subset of the product of a separable metric space  $M$  with a compact Hausdorff space  $K$ . If  $f : X \rightarrow C_p(Y)$  is a continuous mapping then there exists a residual subset  $R$  of  $X$  such that  $f$  is jointly continuous at each point of  $R \times Y$ .*

**Proof:** We begin with some preliminary definitions. Let  $P : Y \rightarrow M$  be the natural projection of  $Y$  onto  $M$  defined by,  $P(m, k) := m$ . Note that by possibly making  $M$  smaller we may assume that  $P$  maps  $Y$  onto  $M$ . Moreover, it is not difficult to check that  $P$  is a perfect mapping (i.e., continuous, maps closed sets to closed sets and has compact fibres). Let  $\{U_n : n \in \mathbb{N}\}$  be a countable base for the topology on  $M$ . For each  $n \in \mathbb{N}$  (and  $\varepsilon > 0$ ) define  $p_n : C(Y) \rightarrow [0, \infty]$  by,

$$p_n(f) := \sup\{|f(y)| : y \in P^{-1}(U_n)\}$$

and  $B_n(\varepsilon)$  by,  $B_n(\varepsilon) := \{f \in C(Y) : p_n(f) \leq \varepsilon\}$ . Note: each  $B_n(\varepsilon)$  is  $\tau_p$ -closed and convex. We shall also denote by  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  a mapping from  $\mathbb{N}$  onto  $\mathbb{N}$  such that for each  $n \in \mathbb{N}$ ,  $\pi^{-1}(n)$  is cofinal in  $\mathbb{N}$ . Finally, for each  $\varepsilon > 0$  we shall consider the set

$$R_\varepsilon := \{x \in X : f \text{ is } \varepsilon\text{-jointly continuous at each point of } \{x\} \times Y\}.$$

Clearly,  $f$  is jointly continuous at each point of  $(\bigcap_{n \in \mathbb{N}} R_{1/n}) \times Y$ . Therefore, it will be sufficient to show that for each  $\varepsilon > 0$ ,  $R_\varepsilon$  is residual in  $X$ . To this end, fix  $\varepsilon > 0$ . Let  $s := (s_n : n \in \mathbb{N})$  be a winning strategy for the player  $\alpha$  in the  $\mathcal{G}_X$ -game played on  $X$ . We will use this strategy to

inductively define a winning strategy  $\sigma := (\sigma_n : n \in \mathbb{N})$  for the player  $\alpha$  in the Banach-Mazur game  $BM(R_\varepsilon)$  played on  $X$ ; thus showing, via Theorem 1, that  $R_\varepsilon$  is indeed residual in  $X$ .

*Step 1.* Let  $U_1$  be a non-empty open subset of  $X$  ( $U_1$  may be considered as the first move of the player  $\beta$  in the  $BM(R_\varepsilon)$ -game played on  $X$ ) and let  $x_{(U_1)}$ ,  $V_{(U_1)}$  and  $L_{(U_1)}$  be defined by,

$$(x_{(U_1)}, V_{(U_1)}) := s_1(U_1) \quad \text{and} \quad L_{(U_1)} := \langle f(x_{(U_1)}) \rangle \quad [\text{i.e., the lattice generated by } f(x_{(U_1)})].$$

If  $f(V_{(U_1)}) \subseteq L_{(U_1)} + B_{\pi(1)}(\varepsilon/6)$  let  $l_{(U_1)} := f(x_{(U_1)})$  and define  $\sigma_1(U_1) := V_{(U_1)}$ . Otherwise, there exists a non-empty open subset  $W'$  of  $V_{(U_1)}$  such that  $f(W') \cap [L_{(U_1)} + B_{\pi(1)}(\varepsilon/6)] = \emptyset$ . In this case we let  $l_{(U_1)} := f(x_{(U_1)})$  and define  $\sigma_1(U_1) := W'$ . [Note that in either case,  $\sigma_1(U_1) \subseteq V_{(U_1)} \subseteq U_1$ ].

Now suppose that the point  $x_{(U_1, U_2, \dots, U_j)} \in X$ , the finite sub-lattice  $L_{(U_1, U_2, \dots, U_j)}$  of  $C(Y)$ , the element  $l_{(U_1, U_2, \dots, U_j)} \in L_{(U_1, U_2, \dots, U_j)}$ , the non-empty open set  $V_{(U_1, U_2, \dots, U_j)}$  of  $X$  and the strategy  $\sigma_j$  have been defined for each  $\sigma$ -sequence  $(U_1, U_2, \dots, U_j)$  of length  $j$ , with  $1 \leq j \leq n$  so that:

- (i)  $(U_1, U_2, \dots, U_j)$  is an  $s$ -sequence i.e.,  $U_k \subseteq V_{(U_1, U_2, \dots, U_{k-1})}$  for each  $2 \leq k \leq j$ ;
- (ii)  $(x_{(U_1, U_2, \dots, U_j)}, V_{(U_1, U_2, \dots, U_j)}) := s_j(U_1, U_2, \dots, U_j)$ ;
- (iii)  $L_{(U_1, U_2, \dots, U_j)} := \langle f(x_{(U_1)}), f(x_{(U_1, U_2)}), \dots, f(x_{(U_1, U_2, \dots, U_j)}) \rangle$ ;
- (iv) either  $f(\sigma_j(U_1, U_2, \dots, U_j)) \subseteq l_{(U_1, U_2, \dots, U_j)} + B_{\pi(j)}(\varepsilon/6)$  for some  $l_{(U_1, U_2, \dots, U_j)} \in L_{(U_1, U_2, \dots, U_j)}$  or else  $f(\sigma_j(U_1, U_2, \dots, U_j)) \cap [L_{(U_1, U_2, \dots, U_j)} + B_{\pi(j)}(\varepsilon/6)] = \emptyset$ ;
- (v)  $\sigma_j(U_1, U_2, \dots, U_j) \subseteq V_{(U_1, U_2, \dots, U_j)}$ .

*Step  $n + 1$ .* Let  $(U_1, U_2, \dots, U_{n+1})$  be a  $\sigma$ -sequence of length  $n + 1$ . Then,

$$\begin{aligned} U_{n+1} \subseteq \sigma_n(U_1, U_2, \dots, U_n) \subseteq V_{(U_1, U_2, \dots, U_n)} \subseteq U_n &\subseteq \sigma_{n-1}(U_1, U_2, \dots, U_{n-1}) \\ &\subseteq V_{(U_1, U_2, \dots, U_{n-1})} \subseteq \dots \subseteq U_2 \subseteq \sigma_1(U_1) \subseteq V_{(U_1)} \subseteq U_1 \end{aligned}$$

and so

$$U_{n+1} \subseteq V_{(U_1, U_2, \dots, U_n)} \subseteq U_n \subseteq V_{(U_1, U_2, \dots, U_{n-1})} \subseteq U_{n-1} \subseteq \dots \subseteq U_2 \subseteq V_{(U_1)} \subseteq U_1.$$

Therefore,  $(U_1, U_2, \dots, U_{n+1})$  is also an  $s$ -sequence. Let  $x_{(U_1, U_2, \dots, U_{n+1})}$  and  $V_{(U_1, U_2, \dots, U_{n+1})}$  be defined by,

$$(x_{(U_1, U_2, \dots, U_{n+1})}, V_{(U_1, U_2, \dots, U_{n+1})}) := s_j(U_1, U_2, \dots, U_{n+1})$$

and

$$L_{(U_1, U_2, \dots, U_{n+1})} := \langle f(x_{(U_1)}), f(x_{(U_1, U_2)}), \dots, f(x_{(U_1, U_2, \dots, U_{n+1})}) \rangle.$$

If  $f(V_{(U_1, U_2, \dots, U_{n+1})}) \subseteq L_{(U_1, U_2, \dots, U_{n+1})} + B_{\pi(n+1)}(\varepsilon/6)$  then by Lemma 3 there exists a non-empty open subset  $W$  of  $V_{(U_1, U_2, \dots, U_{n+1})}$  and an element  $l_{(U_1, U_2, \dots, U_{n+1})} \in L_{(U_1, U_2, \dots, U_{n+1})}$  such that

$$f(W) \subseteq l_{(U_1, U_2, \dots, U_{n+1})} + B_{\pi(n+1)}(\varepsilon/6).$$

In this case we define  $\sigma_{n+1}(U_1, U_2, \dots, U_{n+1}) := W$ . Otherwise there exists a non-empty open subset  $W'$  of  $V_{(U_1, U_2, \dots, U_{n+1})}$  such that

$$f(W') \cap [L_{(U_1, U_2, \dots, U_{n+1})} + B_{\pi(n+1)}(\varepsilon/2)] = \emptyset.$$

In this situation we let  $l_{(U_1, U_2, \dots, U_{n+1})} := l_{(U_1, U_2, \dots, U_n)}$  and define  $\sigma_{n+1}(U_1, U_2, \dots, U_{n+1}) := f(W')$ . [Note that in either case,  $\sigma_{n+1}(U_1, U_2, \dots, U_{n+1}) \subseteq V_{(U_1, U_2, \dots, U_{n+1})} \subseteq U_{n+1}$ ].

This completes the definition of  $\sigma := (\sigma_n : n \in \mathbb{N})$ . Next, we show that  $\sigma$  is indeed a winning strategy for the player  $\alpha$  in the  $BM(R_\varepsilon)$ -game played on  $X$ . To this end, let  $(U_n : n \in \mathbb{N})$  be a  $\sigma$ -sequence such that  $\bigcap_{n \in \mathbb{N}} U_n \neq \emptyset$  and suppose that  $x_0 \in \bigcap_{n \in \mathbb{N}} U_n$ . We need to show that  $x_0 \in R_\varepsilon$ . So consider an arbitrary element  $y_0 \in Y$  and let  $x_\infty \in \bigcap_{n \in \mathbb{N}} U_n \cap \overline{\{x_{(U_1, U_2, \dots, U_n)} : n \in \mathbb{N}\}} \neq \emptyset$  (this choice is possible since  $(U_n : n \in \mathbb{N})$  is a  $s$ -sequence). Then  $f(x_\infty) \in \overline{\{f(x_{(U_1, U_2, \dots, U_n)}) : n \in \mathbb{N}\}}^{T_p} \subseteq \overline{L_\infty}^{T_p}$ , where  $L_\infty := \bigcup_{n \in \mathbb{N}} L_{(U_1, U_2, \dots, U_n)}$ . Now since  $P^{-1}(P(y_0))$  is compact and  $L_\infty$  is a lattice we have by Lemma 4 that there exists a  $l_\infty \in L_\infty$  and an open set  $U$  containing  $P^{-1}(P(y_0))$  such that  $|f(x_\infty)(u) - l_\infty(u)| < \varepsilon/6$  for all  $u \in U$ . Since  $P$  is a closed mapping (i.e., maps closed sets to closed sets) there exists an  $n_0 \in \mathbb{N}$  such that  $P(y_0) \in U_{n_0}$  and  $p_{n_0}(f(x_\infty), l_\infty) \leq \varepsilon/6$ . Next, if we choose  $j \in \mathbb{N}$  sufficiently large so that  $l_\infty \in L_{(U_1, U_2, \dots, U_j)}$  and  $\pi(j) = n_0$  then at the  $j^{\text{th}}$ -stage of the definition of  $\sigma$  we must have had that

$$f(\sigma_j(U_1, U_2, \dots, U_j)) \subseteq l_{(U_1, U_2, \dots, U_j)} + B_{\pi(j)}(\varepsilon/6) = l_{(U_1, U_2, \dots, U_j)} + B_{n_0}(\varepsilon/6)$$

Thus, for any  $x$  and  $x'$  in  $\sigma_j(U_1, U_2, \dots, U_j)$  and  $y \in P^{-1}(U_{n_0})$ ,  $|f(x)(y) - f(x')(y)| \leq \varepsilon/3$ . Thus, by Lemma 2,  $f$  is  $\varepsilon$ -jointly continuous at  $(x_0, y_0)$  since  $x_0 \in \bigcap_{n \in \mathbb{N}} U_n \subseteq \sigma_j(U_1, U_2, \dots, U_j)$  and  $y_0 \in P^{-1}(U_{n_0})$ ; which is open in  $Y$ .  $\odot$

Let us end this paper by mentioning that for a completely regular space  $Y$  the following are equivalent:

- (i)  $Y$  is a Lindelöf  $p$ -space;
- (ii)  $Y$  is the pre-image of a separable metric space under a perfect mapping;
- (iii)  $Y$  is a closed subset of a product of a separable metric space with a compact Hausdorff space.

For the justification of this see, [4, p. 441] and [4, Corollary 3.20]);

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