IRREGULARITY

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Abstract. Regular and irregular pretopologies are studied. In particular, for every ordinal there exists a topology such that the series of its partial (pretopological) regularizations has length of that ordinal. Regularity and topologicity of standard pretopologies on cascades can be characterized in terms of their states, so that their study for such spaces reduces to that of a combinatorics of states. For example, if an iterated partial regularization $r^k\pi$ is topological for $k > 0$ then $r\pi$ is a regular topology. Irregularity of pretopologies of countable character can be characterized in terms of sequential cascades with standard irregular pretopologies.

Contents

1. Introduction and some preliminaries

In convergence theory there are several generalizations of the topological notion of regularity. In fact, the two regularity concepts, of Fischer [6] and of Grimeisen [7][8], coincide for pseudotopological spaces.[1] By a *convergence* we understand a relation $x \in \lim \mathcal{F}$, between filters $\mathcal{F}$ and points x, such that $\mathcal{F} \subset \mathcal{G}$ implies $\lim \mathcal{F} \subset \lim \mathcal{G}$, and for which the principal ultrafilter of x converges to x for every point x. If ξ and ζ are convergences on the same underlying set, then ζ is *finer* than ξ ($\xi \leq \zeta$) whenever $\lim_\zeta \mathcal{F} \subset \lim_\xi \mathcal{F}$ for every filter $\mathcal{F}$. A map f between pretopological spaces is *continuous* if $f(\lim \mathcal{F}) \subset \lim f(\mathcal{F})$ for every filter $\mathcal{F}$, where $f(\mathcal{F})$ stands for the filter generated by $\{f(F) : F \in \mathcal{F}\}$. A convergence is *Hausdorff* if $\lim \mathcal{F}$ is at most a singleton.

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[1]A convergence is a *pseudotopology* if $\lim \mathcal{F} = \bigcap_{\mathcal{U} \in \beta\mathcal{F}} \lim \mathcal{U}$, where $\beta\mathcal{F}$ stands for the set of ultrafilters that are finer than $\mathcal{F}$.

The *adherence* of a filter $\mathcal{H}$ with respect to a convergence ξ is defined by $\operatorname{adh}_\xi \mathcal{H} = \bigcup_{\mathcal{F}\#\mathcal{H}} \lim_\xi \mathcal{F}$ where $\mathcal{F}\#\mathcal{H}$ means that $\mathcal{F}$ *meshes* with $\mathcal{G}$, that is, $F \cap H \neq \varnothing$ for every $F \in \mathcal{F}$ and each $H \in \mathcal{H}$. In particular, $\operatorname{adh}_\xi H$ denotes the adherence of the principal filter of H. A convergence is a *pretopology* if $\lim \mathcal{F} \supset \bigcap_{H\#\mathcal{F}} \operatorname{adh} H$.

If $\mathcal{F}$ is a filter on the underlying set $|\xi|$ of a convergence ξ, then the symbol $\operatorname{adh}_\xi^\natural \mathcal{F}$ denotes the filter generated by $\{\operatorname{adh}_\xi F : F \in \mathcal{F}\}$. A convergence ξ is *regular* (in the sense of Fischer) if

$$\lim_\xi \mathcal{F} \subset \lim_\xi(\operatorname{adh}_\xi^\natural \mathcal{F}) \tag{1.1}$$

for every filter $\mathcal{F}$.[2] In this sense a regular convergence need not be Hausdorff.

A set A is closed with respect to a convergence ξ if $\operatorname{adh}_\xi A \subset A$. The *closure* $\operatorname{cl}_\xi A$ of A is the least ξ-closed set that includes A. A convergence is a *topology* if $\lim \mathcal{F} \supset \bigcap_{H\#\mathcal{F}} \operatorname{cl} H$. If $\mathcal{F}$ is a filter on the underlying set of a convergence ξ, then the symbol $\operatorname{cl}_\xi^\natural \mathcal{F}$ denotes the filter generated by $\{\operatorname{cl}_\xi F : F \in \mathcal{F}\}$. A convergence ξ is said to be *topologically regular* whenever $\lim_\xi \mathcal{F} \subset \lim_\xi \operatorname{cl}_\xi^\natural \mathcal{F}$ for every filter $\mathcal{F}$.

In this paper we observe (Proposition 3.7) a peculiar but simple fact concerning regular pretopologies, which seems to have passed unnoticed so far. A regular pretopology is topologically regular. This property does not hold for general convergences (Example 3.8).

The classes of pretopologies, topologies and of regular convergences are concretely reflective subcategories of the category of convergences. The corresponding reflectors are denoted by P, T and R respectively. In other words, for every convergence ξ there exists, respectively, the finest pretopology, topology and a regular convergence among those that are coarser than ξ. They are denoted by $P\xi$, $T\xi$ and $R\xi$ and called the *pretopologization, topologization* and *regularization* of ξ.

We notice that the topologization of a Hausdorff regular pretopology on a countable set is normal, hence regular (Theorem 3.9). This fact slightly improves [11, Theorem 2.4] by Nyikos and Vaughan who attribute it to Foged, although these authors do not mention pretopologies, but formulate the result in terms of weak bases of topologies.

If ξ is a convergence then we define its *partial regularization* $r\xi$ as follows: $x \in \lim_{r\xi} \mathcal{F}$ if there exists a filter $\mathcal{G}$ such that $x \in \lim_\xi \mathcal{G}$ and $\mathcal{F} \geq \operatorname{adh}_\xi^\natural \mathcal{G}$. It is clear that $\xi \geq r\xi$ and that ξ is regular if and only if $\xi \leq r\xi$. The partial regularization can be iterated: for each ordinal $\beta > 1$, set

$$r^\beta \xi = r(\bigwedge_{\alpha<\beta} r^\alpha \xi),$$

where $\bigwedge$ stands for the infimum in the complete lattice of convergences on a fixed (underlying) set. Sometime it is useful to define $r^{<\beta}\xi = \bigwedge_{\alpha<\beta} r^\alpha \xi$ for limit β. Of course, for every convergence ξ there is a least ordinal β (called the *irregularity* of ξ) such that $r^{\beta+1}\xi = r^\beta \xi$ and thus $r^\beta \xi = R\xi$. We show that for every ordinal β, there exists a pretopology ξ whose irregularity is β. Kent and Richardson [9][10] introduced another functor of partial regularization, which in our terminology is equal to r^ω. They proved that for every ordinal β there exists a pretopology ξ such

[2] In case where ξ is a topology, that is, $x \in \lim_\xi \mathcal{F}$ whenever $\mathcal{F}$ is finer the neighborhood filter of x, (1.1) means that every neighborhood of x includes a closed neighborhood of x.

that β is the least ordinal for which $(r^{\omega})^{\beta}\xi = R\xi$. Due to the adopted definitions, our result is more precise. On the other hand, our construction is much simpler.

If every filter converging to x in $r\xi$ converges to x also in ξ then x is *regular* with respect to ξ. More precisely, the irregularity spectrum of an element x of a convergence space ξ is the set of ordinals β such that for each $\alpha < \beta$ there is a filter that converges to x in $r^{\beta}\xi$ but not in $r^{\beta}\xi$. An irregularity spectrum need not be an interval of Ord; in particular, the irregularity spectrum of a regular point need not be empty.

We observe that the partial regularization of a pretopology is a pretopology, but the partial regularization of a topology is not necessarily a topology. Nevertheless infinite iterations of the partial regularization of a pretopology need not a pretopology, because an infimum (in the lattice of convergences on a fixed set) of infinitely many pretopologies need not be a pretopology. As the set of pretopologies (on a fixed set) is a complete lattice (actually the pretopological infimum $\bigwedge^{P}$ is equal to $P\bigwedge$), we consider sometimes iterated pretopological partial regularizations.

An important special class is that of convergences of countable character. The partial regularization of a convergence of countable character is of countable character. Moreover, an element x is irregular for a pretopology of countable character if and only if there exists a bisequence $\{x_{n,k} : n, k < \omega\}$ and a sequence $(x_n)_n$ such that $x_n \in \lim(x_{n,k})_k$ for each n, $x \notin \operatorname{adh}(x_n)_n$ and the filter generated by $\{x_{n,k} : n \leq m, k < \omega\}_{m<\omega}$ converges to x. This characterization led us to a study of regularity of special pretopologies of countable character on some sequential trees. It turned out that, at least for finite rank, such a study can be reduced to a combinatorics of finite subintervals of ω. This makes it easier to test general hypotheses or to construct counter-examples.

We prove that if n is an irregularity of an element x of a pretopology of countable character, then there exists a homeomorphically embedded sequential cascade T of rank $n+1$ with a maximally irregular pretopology of countable character such that $\varnothing_T = x$. The converse is not true but for $n = 1$.

2. Partial regularizations

In Section 1 we have defined the partial regularization $r\xi$ of ξ. Namely, $x \in \lim_{r\xi}\mathcal{F}$ if there exists a filter $\mathcal{G}$ such that $x \in \lim_{\xi}\mathcal{G}$ and $\mathcal{F} \geq \operatorname{adh}^{\natural}_{\xi}\mathcal{G}$. The *irregularity* $\rho(\xi)$ of a convergence ξ is the least ordinal β such that $r^{\beta}\xi = r^{\beta+1}\xi$. Thus $R\xi = r^{\rho(\xi)}\xi$.

An element x is *regular* with respect to a convergence ξ if $x \in \lim_{r\xi}\mathcal{F}$ implies that $x \in \lim_{\xi}\mathcal{F}$; otherwise it is *irregular*. An ordinal β is an *irregularity* of x with respect to ξ whenever for every $\alpha < \beta$ there exists a filter $\mathcal{F}$ such that $x \in \lim_{r^{\beta}\xi}\mathcal{F}$ and $x \notin \lim_{r^{\alpha}\xi}\mathcal{F}$. If such a β does not exist, then we say that x is *intrinsically regular*. The set of irregularities of x (with respect to ξ) is called the *irregularity spectrum* of x and is denoted by $\operatorname{spect}_{\xi}(x)$, and $\rho_{\xi}(x) = \sup \operatorname{spect}_{\xi}(x)$ is the *irregularity bound* of x. Therefore an element is regular if and only if 1 does not belong to its spectrum. Of course, $\rho(\xi) = \sup_{x\in|\xi|}\rho_{\xi}(x)$ where $|\xi|$ stands for the underlying set of ξ.

Example 2.1. *Let $T = \{\varnothing\} \cup \{x_n : n < \omega\} \cup \{x_{n,k} : n, k < \omega\}$ be an (extended) bisequence equipped with the following topology ξ: the elements $x_{n,k}$ (of level 2) are isolated, the free part of the neighborhood filter $\mathcal{N}_{\xi}(x_n)$ is the cofinite filter of*

$\{x_{n,k} : k < \omega\}$ *for every* $n < \omega$, *and the free part of the neighborhood filter* $\mathcal{N}_\xi(\varnothing)$ *is* $\mathcal{N}_\xi(\mathcal{H})$, *the contour of* $\mathcal{N}_\xi$ *along the cofinite filter* $\mathcal{H}$ *of* $\{x_n : n < \omega\}$. *We notice that* $\operatorname{adh}^\natural_\xi \mathcal{N}(\mathcal{H}) = \{\varnothing\}^\uparrow \wedge \mathcal{H} \wedge \mathcal{N}_\xi(\mathcal{H})$, *hence* ξ *is not regular, because* $\mathcal{H}$ *is not finer than* $\mathcal{N}_\xi(\mathcal{H})$ *(actually,* $\mathcal{H}$ *does not mesh with* $\mathcal{N}_\xi(\mathcal{H})$*). Notice that* $\mathcal{N}_{r\xi}(\varnothing) = \operatorname{adh}^\natural_\xi \mathcal{N}_\xi(\mathcal{H}), \mathcal{N}_{r\xi}(x_n) = \mathcal{N}_\xi(x_n)$ *and* $x_{n,k}$ *is isolated for* $r\xi$ *for each natural* n *and* k. *We notice that* $r\xi$ *is a regular topology, so that* $\rho(\xi) = 1$.

We notice that in Example 2.1 the filters convergent to $x_{n,k}$ and to x_n are the same for ξ and for $r\xi$, while there are more filters convergent to $\varnothing$ in $r\xi$ than in ξ. Therefore the elements of levels 1 and 2 in Example 2.1 are intrinsically regular, while the irregularity spectrum of $\varnothing$ is $\{1\}$.

If ξ is a convergence on a set Y and $X \subset Y$ then $\xi|_X$ stands for the restriction of ξ to X. We notice that

$$r(\xi|_X) \geq (r\xi)|_X,$$

hence $\rho(\xi) \leq \rho(\xi|_X)$. Indeed, if $x \in \lim_{r(\xi|_X)} \mathcal{F}$ then there is a filter $\mathcal{G}$ containing X such that $x \in \lim_\xi \mathcal{G}$ and $\operatorname{adh}^\natural_{\xi|_X} \mathcal{G} \leq \mathcal{F}$. As $\operatorname{adh}^\natural_{\xi|_X} \mathcal{G} = \operatorname{adh}^\natural_\xi \mathcal{G} \vee X$, we infer that $X \in \mathcal{F}$ and $x \in \lim_{r\xi} \mathcal{F}$, hence $x \in \lim_{(r\xi)|_X} \mathcal{F}$.

The converse inequality does not hold in general; it does if X is an open subset of Y. For instance, take the pretopology ξ of Example 2.1 and let $X = \{0\} \cup \{x_n : n < \omega\}$. Then $\xi|_X$ is discrete, hence regular so that $r(\xi|_X) = \xi|_X$, while $r\xi$ is the natural topology of T, thus $(r\xi)|_X$ is the natural topology of $\{0\} \cup \{x_n : n < \omega\}$, which is strictly coarser than the discrete topology $\xi|_X$.

If $\mathcal{W}(y)$ is a family of subsets of X for every $y \in Y$, and if $\mathcal{A}$ is a family of subsets of Y, then the *contour* of $\mathcal{W}$ along $\mathcal{A}$ is defined by

$$\int_{\mathcal{A}} \mathcal{W} = \mathcal{W}(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} \bigcap_{y \in A} \mathcal{W}(y). \tag{2.1}$$

The infimum $\mathcal{V}_\xi(x)$ of all filters, which converge to x, is called the *vicinity filter* of x with respect to ξ. A convergence ξ is a pretopology if and only if $x \in \lim_\xi \mathcal{V}_\xi(x)$ for every $x \in |\xi|$, where $|\xi|$ stands for the underlying set of ξ. It is straightforward that

$$\mathcal{A} \# \mathcal{V}_\xi(\mathcal{B}) \iff (\operatorname{adh}^\natural_\xi \mathcal{A}) \# \mathcal{B} \tag{2.2}$$

for each families of sets $\mathcal{A}$ and $\mathcal{B}$. It is known [9] that a pseudotopology ξ is regular if and only if

$$\operatorname{adh}_\xi \mathcal{V}_\xi(\mathcal{H}) \subset \operatorname{adh}_\xi \mathcal{H} \tag{2.3}$$

for every filter $\mathcal{H}$.

By definition, $A \sqcup B$ is defined and equal to $A \cup B$ whenever $A \cap B = \varnothing$; similarly, $\bigsqcup_{A \in \mathcal{A}} A$ is defined and equal to $\bigcup_{A \in \mathcal{A}} A$ whenever $A_0 \cap A_1 = \varnothing$ for every two distinct elements A_0, A_1 of $\mathcal{A}$.

Proposition 2.2. *For every ordinal* β, *there exists a Hausdorff pretopology of irregularity* β *(of cardinality* $|\beta| \vee \aleph_0$*).*

Proof. Actually we will show that this irregularity is attained at an element, for which it is strong. The irregularity of each regular pretopology is 0. Example 2.1 describes a Hausdorff topology of irregularity 1 and of cardinality $\aleph_0$. Suppose that $\beta > 1$ and that for each $\alpha < \beta$, there exists a set X_α (of cardinality $|\alpha| \vee \aleph_0$), a Hausdorff pretopology π_α on X_α, an element x_α of X_α, and a free filter $\mathcal{F}_\alpha$ on X_α

such that $x_\alpha \in \lim_{r^\alpha \pi_\alpha} \mathcal{F}_\alpha \setminus \lim_{r^\gamma \pi_\alpha} \mathcal{F}_\alpha$ for each $\gamma < \alpha$. If β is limit, then consider the simple sum $\bigoplus_{\alpha<\beta} \pi_\alpha$ on $\bigsqcup_{\alpha<\beta} X_\alpha$ and let $\mathcal{F}$ be the image on $\{x_\alpha : \alpha < \beta\}$ of the coarsest filter on $\beta = \{\alpha : \alpha < \beta\}$ that converges to β in the natural topology. Define a pretopology π on $\bigsqcup_{\alpha<\beta} X_\alpha \sqcup \{\varnothing\}$ (which is of cardinality $|\beta| \vee \aleph_0$) of by setting $\{\varnothing\} = \lim_\pi \int_{\mathcal{F}} (\mathcal{F}\ \alpha)_{\alpha<\beta}$.

This is a Hausdorff pretopology of cardinality $|\beta|$, and $\varnothing \notin \operatorname{adh}_{r^\gamma \pi} \mathcal{F}$ for each $\gamma < \beta$ but $\varnothing \in \lim_{r^\beta \pi} \mathcal{F}$, because $\operatorname{adh}^\natural_{r^\beta \pi} \int_{\mathcal{F}} (\mathcal{F}\ \alpha)_{\alpha<\beta} \leq \mathcal{F}$. If β is isolated, then mimic the construction above, on replacing $\{\pi_\alpha : \alpha < \beta\}$ by countable infinite simple sum of copies of $\beta - 1$, and $\mathcal{F}$ by the cofinite filter of a countable infinite set of copies of $x_{\beta-1}$. ∎

Let us observe that if an irregularity β of an element x is isolated, then there is a filter $\mathcal{F}$ such that $x \in \lim_{r^\beta \xi} \mathcal{F}$ and $x \notin \lim_{r^\alpha \xi} \mathcal{F}$ for all $\alpha < \beta$. This inversion of quantifiers leads to a slightly stronger property in case of limit ordinals; we can call such an irregularity *strong*. The construction in the proof of Proposition 2.2 shows the existence of pretopologies of arbitrary irregularity attained at point as strong irregularity. Example 2.3 illustrates a case of irregularity that is not strong. The construction in the proof of Proposition 2.2 uses elements the irregularity of which is strong. Here is an example of an element whose irregularity is ω_0 and is not strong.

Example 2.3. *Let π_n be a Hausdorff pretopology on X_n of cardinality $\aleph_0$ of irregularity n attained at x_n. Let $\mathcal{F}_n$ be such a filter that $x_n \in \lim_{r^n \pi_n} \mathcal{F}_n \setminus \lim_{r^{n-1} \pi_n} \mathcal{F}_n$. Take the simple sum $\bigoplus_{n<\omega} \pi_n$ on $\bigsqcup_{n<\omega} X_n$ and take the pretopological quotient π by identifying all x_n in $\varnothing$. Then $\varnothing \in \lim_{r^n \pi} \mathcal{F}_k$ exactly for $k \leq n$, $r^\omega \pi = r^{<\omega} \pi$, and $\varnothing \in \lim_{r^\omega \pi} \mathcal{F}_n \setminus \lim_{r^{n-1} \pi_n} \mathcal{F}_n$ for every $n < \omega$, while there is no filter which converges to $\varnothing$ in $r^\omega \pi$ but does not converge in $r^n \pi$ for every $n < \omega$, that is, does not converge in $r^{<\omega} \pi$. If we pretopologize $r^\omega \pi$ then we get a regular pretopology, which vicinity filter at $\varnothing$ is equal to $\bigwedge_{n<\omega} \mathcal{F}_n \wedge \{\varnothing\}$ where $\{\varnothing\}$ stands for the principal ultrafilter of $\varnothing$.*

3. Regularity for pretopologies

We shall concentrate here on regularity in the case of pretopologies. This level of generality, on one hand, enables one to notice several interesting phenomena (like the propagation of irregularities) that are not visible in the realm of topologies, and on the other to avoid certain complexity, which can be qualified as technical, and which is not essential for the phenomena mentioned above.

Proposition 3.1. *If $\mathcal{H}$ is a filter, then*

$$\operatorname{adh}_{r\xi} \mathcal{H} = \operatorname{adh}_\xi \mathcal{V}_\xi(\mathcal{H}). \tag{3.1}$$

Proof. By definition, $x \in \operatorname{adh}_{r\xi} \mathcal{H}$ if there exists a filter $\mathcal{F} \geq \mathcal{H}$ such that $x \in \lim_{r\xi} \mathcal{F}$, hence there is a filter $\mathcal{G}$ such that $x \in \lim_\xi \mathcal{G}$ and $\operatorname{adh}^\natural_\xi \mathcal{G} \leq \mathcal{F}$, thus $\operatorname{adh}^\natural_\xi \mathcal{G}$ meshes with $\mathcal{H}$, equivalently $\mathcal{G}$ meshes with $\mathcal{V}_\xi(\mathcal{H})$, which means that $x \in \operatorname{adh}_\xi \mathcal{V}_\xi(\mathcal{H})$. Conversely, if $x \in \operatorname{adh}_\xi(\mathcal{V}_\xi(\mathcal{H}))$ then there is a filter $\mathcal{G}$ such that $\mathcal{G} \# \mathcal{V}_\xi(\mathcal{H})$ and $x \in \lim_\xi \mathcal{G}$, hence $\operatorname{adh}^\natural_\xi \mathcal{G}$ meshes with $\mathcal{H}$ and $\operatorname{adh}^\natural_\xi \mathcal{G}$ converges to x in $r\xi$, so that $x \in \operatorname{adh}_{r\xi} \mathcal{H}$. ∎

Corollary 3.2. *For every set H,*

$$\operatorname{adh}_{r\xi} H = \operatorname{adh}_{\xi}(\mathcal{V}_{\xi}(H)).$$

Therefore,

$$\operatorname{adh}_{r\xi}^{\natural} \mathcal{H} \approx \{\operatorname{adh}_{\xi}(\mathcal{V}_{\xi}(H)) : H \in \mathcal{H}\},$$

where $\mathcal{A} \approx \mathcal{B}$ means that $\mathcal{A}$ is *finer* than $\mathcal{B}$ (for every $B \in \mathcal{B}$ there is $A \in \mathcal{A}$ with $A \subset B$) and $\mathcal{B}$ is *finer* than $\mathcal{A}$. In the particular case above this signifies that the family on the right-hand side is a base of the filter $\operatorname{adh}_{r\xi}^{\natural} \mathcal{H}$.

Proposition 3.3. *The filter $\operatorname{adh}_{r\xi}^{\natural} \mathcal{H}$ is finer than $\operatorname{adh}_{\xi}^{\natural} \mathcal{V}(\mathcal{H})$ for each family $\mathcal{H}$.*

Proof. Let $A \in \operatorname{adh}_{\xi}^{\natural} \mathcal{V}_{\xi}(\mathcal{H})$; this means that there is $H \in \mathcal{H}$ and $V \in \mathcal{V}_{\xi}(H)$ such that $\operatorname{adh}_{\xi} V \subset A$. In general, $\operatorname{adh}_{\xi}(\mathcal{V}_{\xi}(H)) \subset \operatorname{adh}_{\xi} V$ for every $V \in \mathcal{V}_{\xi}(H)$. Therefore $A \in \operatorname{adh}_{r\xi}^{\natural} \mathcal{H}$. ∎

In general, $\operatorname{adh}_{r\xi}^{\natural} \mathcal{H} \neq \operatorname{adh}_{\xi}^{\natural} \mathcal{V}_{\xi}(\mathcal{H})$. Indeed, the irregular bisequence is an example where the equality does not hold.

Example 3.4. *Consider the pretopology of Example 2.1. For every $H \in \mathcal{H}$ we have $\operatorname{adh}_{\xi} H = \operatorname{adh}_{\xi} \mathcal{V}_{\xi}(H) = \bigcap_{V \in \mathcal{V}(H)} \operatorname{adh}_{\xi} V = \bigcap_{V \in \mathcal{V}(H)} V = H$, so that $\operatorname{adh}_{r\xi}^{\natural} \mathcal{H} = \mathcal{H}$. On the other hand, $\operatorname{adh}_{\xi}^{\natural} \mathcal{V}_{\xi}(\mathcal{H}) = \{0\}^{\uparrow} \wedge \mathcal{H} \wedge \mathcal{V}_{\xi}(\mathcal{H})$, which is strictly coarser than $\mathcal{H} = \operatorname{adh}_{r\xi}^{\natural} \mathcal{H}$.*

Proposition 3.5. *If ξ is a pretopology, then $r\xi$ is a pretopology, and*

$$\mathcal{V}_{r\xi}(x) = \operatorname{adh}_{\xi}^{\natural} \mathcal{V}_{\xi}(x). \tag{3.2}$$

Proof. If ξ is a pretopology, then by (3.1) for every set A,

$$\operatorname{adh}_{r\xi} A = \bigcap_{V \in \mathcal{V}_{\xi}(A)} \operatorname{adh}_{\xi} V.$$

By definition, a set A meshes with $\mathcal{V}_{r\xi}(x)$ if and only if $x \in \operatorname{adh}_{r\xi} A$, so when ξ is a pretopology, if and only if $x \in \operatorname{adh}_{\xi} V$ for every $V \in \mathcal{V}_{\xi}(A)$, equivalently if $V \in \mathcal{V}_{\xi}(A)$ then $\mathcal{V}_{\xi}(x) \# V$, that is, $\mathcal{V}_{\xi}(A) \# \mathcal{V}_{\xi}(x)$, which amounts to $A \# \operatorname{adh}_{\xi}^{\natural} \mathcal{V}_{\xi}(x)$. Therefore (3.2) holds. Now if ξ is a pretopology $x \in \lim_{r\xi} \mathcal{F}$ whenever $\mathcal{F} \geq \operatorname{adh}_{\xi}^{\natural} \mathcal{V}_{\xi}(x)$, which proves that $r\xi$ is a pretopology. ∎

Corollary 3.6. *If ξ is a pretopology, then $r^n\xi$ is a pretopology for every n.*

Let $\mathcal{W}_{\xi}(x) = \operatorname{adh}_{\xi}^{\natural} \mathcal{V}_{\xi}(x)$, and if $\mathcal{W}_{\xi}^{n}(x)$ is defined for $n \geq 1$, let

$$\mathcal{W}_{\xi}^{n+1}(x) = \operatorname{adh}_{\xi}^{\natural} \mathcal{V}_{\xi}(\mathcal{W}_{\xi}^{n}(x)).$$

It is important to stress that $\mathcal{W}_{\xi}^{n}(x)$ is not (in general) an iterated contour of the system of vicinities $\operatorname{adh}_{\xi}^{\natural} \mathcal{V}_{\xi}$. The following diagonal iteration of a pretopology is a special case of diagonal product introduced in [3] by G. H. Greco and one of the present authors. By definition, $\operatorname{adh}_{\xi^0} A = A$ and

$$\operatorname{adh}_{\xi^{\beta}} A = \operatorname{adh}_{\xi}\left(\bigcup_{\alpha<\beta} \operatorname{adh}_{\xi^{\alpha}} A\right)$$

for $\beta > 0$. The adherence above defines a pretopology ξ^{β} for every ordinal β. For every pretopology ξ there is a least ordinal β such that $\xi^{\beta} = T\xi$; this is the least ordinal for which $\operatorname{adh}_{\xi^{\beta}} A = \operatorname{adh}_{\xi^{\beta+1}} A$ (equivalently, $\operatorname{adh}_{\xi^{\beta}} A = \operatorname{cl}_{\xi} A$) for every set A.

Proposition 3.7. *Each regular pretopology is topologically regular.*

Proof. Let ξ be a regular pretopology, that is, $\mathcal{V}_\xi(x) \subset \operatorname{adh}^\natural_\xi \mathcal{V}_\xi(x)$. By applying, $\operatorname{adh}^\natural_\xi$ to this inclusion, we get $\operatorname{adh}^\natural_\xi \mathcal{V}_\xi(x) \subset \operatorname{adh}^\natural_{\xi^2} \mathcal{V}_\xi(x)$ hence $\mathcal{V}_\xi(x) \subset \operatorname{adh}^\natural_{\xi^2} \mathcal{V}_\xi(x)$. Therefore $\mathcal{V}_\xi(x) \subset \operatorname{adh}^\natural_{\xi^n} \mathcal{V}_\xi(x)$ for every $n < \omega$ and thus $\mathcal{V}_\xi(x) \subset \operatorname{adh}^\natural_{\xi^\beta} \mathcal{V}_\xi(x)$ for every ordinal β, so that $\mathcal{V}_\xi(x) \subset \operatorname{cl}^\natural_\xi \mathcal{V}_\xi(x)$. ∎

The property above does not hold for general convergences.[3] For each $n, m < \omega$ let $A_{n,m}$ be a countably infinite set such that $A_{n,m+1}$ is a partition of $A_{n,m}$. Let $A_m = \bigsqcup_{n<\omega} A_{n,m}$ and $A = \bigsqcup_{m<\omega} A_m$ will be called a *sink* (of countable character). The map $\pi^m_{m+1} : A_m \to A_{m+1}$ is the quotient defined by A_{m+1} on A_m. The *natural convergence* of a sink is defined by the fact that for each n, m and $p \in A_{n,m+1}$ the cofinite filter $\mathcal{N}(p)$ of $(\pi^m_{m+1})^-(p)$ converges to p. Of course, the natural convergence of a sink is sequential. Let $\mathcal{F}_m$ be the filter generated by $\{\bigcup_{n\geq l} A_{n,m} : l < \omega\}$.

Example 3.8. *Let $A = \bigsqcup_{n,m<\omega} A_{n,m}$ be a sink endowed with its natural convergence. We extend the convergence of A to $X = \{\infty\} \cup A$ so that $\bigwedge_{m\leq k} \mathcal{F}_m$ converges to ∞ for every $k < \omega$. This is a Hausdorff pseudotopology of countable character. It is regular, because $\operatorname{adh}^\natural \mathcal{F}_m = \mathcal{F}_{m+1} \wedge \mathcal{F}_m$, hence $\operatorname{adh}^\natural \left(\bigwedge_{m\leq k} \mathcal{F}_m\right) \geq \bigwedge_{m\leq k+1} \mathcal{F}_m$, and $\operatorname{adh}^\natural \mathcal{N}(p) = \mathcal{N}(p)$ for each n, m and $p \in A_{n,m+1}$. But it is not topologically regular, because $\operatorname{cl}^\natural \mathcal{F}_0 = \bigwedge_{m<\omega} \mathcal{F}_k$, and the latter filter does not converge to ∞.*

Actually, much more can be said if the underlying set is countable. If a convergence is a pretopology then it is Hausdorff if $\mathcal{V}(x_0)$ does not mesh with $\mathcal{V}(x_1)$ when $x_0 \neq x_1$. It is straightforward that each point of a Hausdorff pretopology is closed, in other words, the pretopology is T_1.

Theorem 3.9. *The topologization of a Hausdorff regular pretopology on a countable set is normal, hence regular.*

Theorem 3.9 slightly improves [11, Theorem 2.4] by Nyikos and Vaughan who attribute it to Foged. Actually the authors do not mention pretopologies, but talk about weak bases of a topology. A *weak base* of a topology τ on X is a union of filter bases $\mathcal{B}(x)$ where $x \in X$ such that $x \in B$ if $B \in \mathcal{B}(x)$, and O is open whenever $x \in O$ implies the existence of $B \in \mathcal{B}(x)$ such that $B \subset O$. If we define a pretopology π by declaring $\mathcal{B}(x)$ to be a base of the vicinity filter $\mathcal{V}_\pi(x)$, then it is clear that $\tau = T\pi$. In these terms, τ is *weakly* T_2 means that π is Hausdorff, and τ is *weakly* T_3 means that π is topologically regular. Thus by virtue of Proposition 3.7, we could relax the original assumption of topological regularity of [11, Theorem 2.4].

4. Regularity of special pretopologies on cascades

We found it useful to study regularity problems first for some special pretopologies on trees, which are well-founded with respect to the inverse order. Such pretopologies are akin to the construction of Proposition 2.2.

It turns out that in case of finite rank, regularity and topologicity properties of such pretopologies can be reduced to some combinatorial properties of finite subintervals on ω. This enables one to test hypotheses in an easier way.

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[3] Not even for pseudotopologies.

Cascades are discussed in detail in [4] and [2]. A *cascade* is a tree T with a single origin $\varnothing_T$, every non-empty subset of which admits a maximal element, and such that for each non-maximal $t \in T$ the set $T^+(t)$ (of immediate successors of t) is infinite. A cascade is *sequential* if $T^+(t)$ is countable infinite for every $t \in T \setminus \max T$. A subset S of a sequential cascade T is a *subcascade* if $\varnothing_T \in S$, if for every $t \in S \setminus \max T$, the set $S^+(t)$ is an infinite subset of $T^+(t)$, and if S is *closed downwards*, that is, $t \in S$ implies that $\{s \in S : s \leq t\} \subset S$. It follows that S is a cascade and $\max S \subset \max T$.

The *rank* $r_T(t)$ of an element t of T is 0 if $t \in \max T$, and otherwise

$$r_T(t) = \sup\{r_T(s) + 1 : t < s\};$$

hence, for every $t \in T \setminus \max T$,

$$r_T(t) = \sup\{r_T(s) + 1 : s \in T^+(t)\}. \tag{4.1}$$

The *rank* $r(T)$ of T is the rank of $\varnothing_T$, that is, $r(T) = r_T(\varnothing_T)$. The coincidence of the notation of the rank r with the partial regularizer r should not hopefully lead to confusion. For every sequential cascade T there exists an order embedding h of T in $\Sigma = \omega^{<\omega}$ ordered by inclusion, which is *full*, that is, such that $h(\varnothing) = \varnothing$ and h restricted to $T^+(t)$ is a bijection onto $\Sigma^+(h(t))$ for every $t \in T \setminus \max T$; therefore $\Sigma^+(h(t)) = \{(t, n) : n < \omega\}$ induces on $T^+(t)$ the natural order of the type ω. From now on we use the term *sequential cascade* for a sequential cascade fully embedded in $\omega^{<\omega}$. A sequential cascade is *monotone* if r_T is non-decreasing on $T^+(t)$ (with respect to the order induced from $\omega^{<\omega}$ by a fixed full embedding) for every $t \in T \setminus \max T$.

The *natural pretopology* of a cascade T is the finest pretopology such that for every non-maximal element t of T, the coarsest free filter that converges to t is the cofinite filter of $T^+(t)$. Consequently the maximal elements of a cascade are isolated for the natural pretopology. The *natural topology* is the topologization of the natural pretopology. It is straightforward that the natural topology is Hausdorff.

The trace of the neighborhood filter of the natural topology of T on $\max T$ is denoted by $\int T$, the reason being that it is equal to the iterated contour defined by the following well-founded induction: for $t \in \max T$ we take $\int T^{\uparrow}(t)$ to be the principal ultrafilter of t; and if $\int T^{\uparrow}(s)$ is defined for each $s \in T^+(t)$, then $\int T^{\uparrow}(t) = \int_{s \in T^+(t)} \int T^{\uparrow}(s)$, that is, the contour of $\{\int T^{\uparrow}(s) : s \in T^+(t)\}$ along the cofinite filter of $T^+(t)$.

If T is a sequential cascade then a map $f : \max T \to X$ is called a *multisequence* on X, and a map $g : T \to X$, an extended *multisequence*. If X is a pretopological space, then a multisequence $f : \max T \to X$ *converges* to x_∞ whenever there exists a continuous map $\hat{f} : T \to X$ (with T equipped with the natural pretopology) such that $\hat{f}|_{\max T} = f$ and $\hat{f}(\varnothing_T) = x_\infty$.

If $\mathcal{N}(t)$ stands for the neighborhood filter of t for the natural topology of a (monotone) sequential cascade, then denote by $\mathcal{N}_{(k)}^{(l)}(t)$ the restriction of the neighborhood filter of t , of level k, to the level $T^{(l)}$ of T. This notation is redundant, but spares the necessity of repeating that $l_T(t) = k$. The natural topology has the following property: for $k < l < m$,

$$\mathcal{N}_{(k)}^{(m)}(t) = \mathcal{N}_{(l)}^{(m)}\left(\mathcal{N}_{(k)}^{(l)}(t)\right), \tag{4.2}$$

where the formula denotes the contour of $\mathcal{N}_{(l)}^{(m)}$ along $\mathcal{N}_{(k)}^{(l)}(t)$ in the sense of (2.1). By definition, the closure (from the level l to the level k) is defined by

$$t \in \mathrm{cl}_{(k)}^{(l)} A \Leftrightarrow A \in \left(\mathcal{N}_{(k)}^{(l)}(t)\right)^{\#}.$$

In other words, $\mathrm{cl}_{(k)}^{(l)} A = T^{(k)} \cap \mathrm{cl}(A \cap T^{(l)})$. Hence, If T is of finite rank, then we can decompose the closure and the neighborhood filters, namely,

$$\mathrm{cl}\, A = \bigcup_{k \le l \le r(T)} \mathrm{cl}_{(k)}^{(l)} A,$$

$$\mathcal{N}(t) = \bigwedge_{k \le l \le r(T)} \mathcal{N}_{(k)}^{(l)}(t).$$

Lemma 4.1. *For the natural topology of a monotone cascade of finite rank,*

$$\left(\mathrm{cl}_{(l)}^{(m)}\right)^{\natural} \mathcal{N}_{(k)}^{(m)}(t) = \mathcal{N}_{(k)}^{(l)}(t)$$

for $k < l < m$.

Proof. If $W \in \mathcal{N}_{(k)}^{(l)}(t)$ and if $V_w \in \mathcal{N}_{(l)}^{(m)}(w)$ and $V_w \in T^{\uparrow}(w)$ for each $w \in W$, then

$$\mathrm{cl}_{(l)}^{(m)} \left(\bigcup_{w \in W} V_w \right) = W.$$

In fact, by definition, $x \in \mathrm{cl}_{(l)}^{(m)} \left(\bigcup_{w \in W} V_w\right)$ whenever $\bigcup_{w \in W} V_w$ meshes with $\mathcal{N}_{(l)}^{(m)}(x)$, that is whenever there is $w \in W$ such that V_w meshes with $\mathcal{N}_{(l)}^{(m)}(x)$, which means that $x = w \in W$. By (4.2) $V \in \mathcal{N}_{(k)}^{(m)}(t)$ if and only if there exists $W \in \mathcal{N}_{(k)}^{(l)}(t)$ such that for every $w \in W$ there is $V_w \in \mathcal{N}_{(l)}^{(m)}(w)$ such that $V \supset \bigcup_{w \in W} V_w$, and this can be done so that $V_w \in T^{\uparrow}(w)$. ∎

Let $\mathcal{V}(t)$ stand for a vicinity filter at $t \in T$ of a pretopology on a sequential cascade T. If t is of level k and $l > k$, then $\mathcal{V}_{(k)}^{(l)}(t)$ denotes the restriction of $\mathcal{V}(t)$ to the level l of T. A pretopology on a sequential cascade is *almost standard* if for every $t \in T$ and for each $k > l_T(t)$ either $\mathcal{V}_{(l_T(t))}^{(k)}(t) = \mathcal{N}_{(l_T(t))}^{(k)}(t)$ or $\mathcal{V}_{(l_T(t))}^{(k)}(t)$ is degenerate. A pretopology on T is said to be *standard* if for every $0 \le k < l \le r(T)$, either $\mathcal{V}_{(k)}^{(l)}(t) = \mathcal{N}_{(k)}^{(l)}(t)$ or $\mathcal{V}_{(k)}^{(l)}(t)$ is degenerate for every t of level k.

Proposition 4.2. *For each almost standard pretopology π on a sequential cascade T of finite rank there exists a monotone sequential subcascade S of T such that the restriction of π to S is standard.*

Proof. The claim is obvious for cascades of rank 1. Let us proceed by induction on the rank. If the claim holds for the rank n and $r(T) = n + 1$ then $n = \sup_{t \in T^{+}(\varnothing_T)} r_T(t)$, thus by inductive assumption, for every $t \in T^{+}(\varnothing_T)$, there is a monotone subcascade S_t of $T^{\uparrow}(t)$ such that the restriction of the pretopology to $T^{\uparrow}(t)$ is standard. As there exist (up to homeomorphism) only finitely many standard pretopologies on a monotone cascade finite rank, there exists a standard pretopology ξ on a monotone cascade of rank n and infinitely many $t \in T^{+}(\varnothing)$ such the pretopology on S_t is homeomorphic to ξ. By setting $S^{+}(\varnothing)$ to be the set of such t, we define a required subcascade of T. ∎

Proposition 4.3. *The irregularity of a standard pretopology on a (monotone) sequential cascade of rank $n+1$ is at most n.*

Proof. If $r(T)=1$, then the only standard pretopology π on T is such that the elements of $\max T$ are isolated and there is on $\max T$ a least filter (which is free) converging to $\varnothing_T$. This defines a regular topology, hence the maximal irregularity of standard pretopologies is 0. If $r(T)=n+1>1$ then $n=\sup_{t\in T^+(\varnothing_T)}(r_T(t)+1)$, and if π is a standard pretopology on T, then by inductive assumption $r^{n-1}\pi_t$ is regular for every $t\in T^+(\varnothing_T)$. It follows that $\operatorname{adh}^{\natural}_{r^{n-1}\pi}\mathcal{F}\wedge\{\varnothing_T\}\le\operatorname{adh}^{\natural}_{r^n\pi}\mathcal{F}$ for every filter $\mathcal{F}$ such that $\varnothing_T\in\lim_\pi\mathcal{F}$. Therefore the irregularity of π is at most n. ∎

Because in the proof of Proposition 2.2 in case of countable ordinals, we have used standard pretopologies on sequential cascades, the upper bounds in Proposition 4.3 are actually attained.

5. States of standard pretopologies

We denote the interval $\{k,k+1,\ldots,l\}$ of natural numbers by $[k,l]$. A *state* on $[0,n]$ is a collection of intervals $[k,l]$ of $[0,n]$ such that $k<l$. Two intervals $[i,j]$ and $[k,l]$ are called *consecutive* if $j=k$. If $i<k$, we define

$$[i,l]\sim[k,l]=[i,k].$$

Therefore there are $2^{\frac{n(n+1)}{2}}$ states on $[0,n]$. A state $\mathcal{S}$ is *regular* if $k<l<m$ and $[k,m],[l,m]\in\mathcal{S}$ implies that $[k,l]\in\mathcal{S}$. For a state $\mathcal{T}$ on $[0,n]$, an element k of $[0,n]$ is *regular* (with respect to $\mathcal{S}$) if $[k,m],[l,m]\in\mathcal{T}$ with $k<l<m$ implies that $[k,l]\in\mathcal{T}$; otherwise, we say that k is *irregular.* Of course, a state is regular if and only if every point is regular with respect to it. The least regular state that includes $\mathcal{S}$ is called the *regularization* of $\mathcal{S}$ and is denoted by $R\mathcal{S}$.

A state is *topological* if $[k,l],[l,m]\in\mathcal{S}$, then $[k,m]\in\mathcal{S}$. For a given state $\mathcal{T}$, a point k is *topological* if $[k,l],l,m]\in\mathcal{T}$ implies that $[k,m]\in\mathcal{T}$. Sure enough, a state is topological if and only if every point is topological with respect to it. If $\mathcal{S}$ is a state, then $T\,\mathcal{S}$ denotes the *topologization* of $\mathcal{S}$, that is, the least topological state that includes $\mathcal{S}$. It is straightforward that $T\mathcal{S}$ consists of all the finite unions of consecutive intervals from $\mathcal{S}$.

There is a one-to-one correspondence between standard pretopologies on a cascade of rank n and states on $[0,n]$, namely if $\mathcal{V}$ denotes the vicinity system of such a pretopology, then the corresponding state $\mathcal{S}$ is defined by $[k,l]\in\mathcal{S}$ if and only if $\mathcal{V}^{(l)}_{(k)}(t)$ is non-degenerate (for all t of level k).

Proposition 5.1. *A standard pretopology is topological (resp., regular) if and only if its state is topological (resp., regular).*

Proof. Consider a standard pretopology on T and the corresponding state $\mathcal{S}$. This pretopology is a topology if and only if $\mathcal{V}(t)\subset\mathcal{V}(\mathcal{V}(t))$ for every t, where $\mathcal{V}(t)$ stands for the vicinity filter. This condition holds if and only if $\mathcal{V}^{(m)}_{(l)}(\mathcal{V}^{(l)}_{(k)}(t))$ is non-degenerate, provided that $\mathcal{V}^{(m)}_{(l)}$ and $\mathcal{V}^{(l)}_{(k)}$ are non-degenerate for each $m>l>k$, which, by (4.2), amounts to the following condition on $\mathcal{S}$: if $[k,l],[l,m]\in\mathcal{S}$ then $[k,m]\in\mathcal{S}$.

Let $\mathcal{S}$ be regular. We assume that $t \in \lim \mathcal{U}$, where $\mathcal{U}$ is a free ultrafilter, and want to show that $t \in \lim \mathrm{adh}^{\natural} \mathcal{U}$. If $l_T(t) = k$, then there is $k < m \leq r(T)$ such that $T^{(m)} \in \mathcal{U}$, hence $\mathcal{V}_{(k)}^{(m)}(t) \leq \mathcal{U}$, which means that $\mathcal{V}_{(k)}^{(m)}(t)$ is non-degenerate, that is, $[k, m] \in S$. Because $T^{\uparrow}(t) \in \mathcal{V}(t)$ and $T^{\uparrow}(t)$ is closed, also $T^{\uparrow}(t) \in \mathrm{adh}^{\natural} \mathcal{V}(t)$. If now $k < l < m$ is such that $[l, m] \in \mathcal{S}$, then, by the regularity of $\mathcal{S}$, also $[k, l] \in \mathcal{S}$, hence $\mathcal{V}_{(k)}^{(l)}(t)$ is non-degenerate. Therefore $t \in \lim \mathrm{adh}^{\natural} \mathcal{U}$ because, by virtue of Lemma 4.1, $\left(\mathrm{adh}_{(l)}^{(m)}\right)^{\natural} \mathcal{U} \geq \mathcal{V}_{(k)}^{(l)}(t)$ for every $k < l < m$.

Conversely, if a standard pretopology is regular, $\mathcal{S}$ is the corresponding state, and $[k, m], [l, m] \in S$ for $k < l < m$, then $\mathcal{V}_{(k)}^{(m)}$ and $\mathcal{V}_{(l)}^{(m)}$ are non-degenerate, hence $\left(\mathrm{adh}_{(l)}^{(m)}\right)^{\natural} \mathcal{V}_{(k)}^{(m)}(t) = \mathcal{V}_{(k)}^{(l)}(t)$ is non-degenerate by the regularity of the pretopology, hence $[k, l] \in \mathcal{S}$, which proves the regularity of $\mathcal{S}$. ∎

6. Pretopological combinatorics

If $\mathcal{S}$ is a state, then the *partial regularization* $r\mathcal{S}$ of $\mathcal{S}$ is equal to the union of $\mathcal{S}$ and all the intervals of the form $[k, l]$ where $[k, m], [l, m] \in \mathcal{S}$ and $k < l$. A sequence of non-empty states $\mathcal{S}_j, \mathcal{S}_{j-1}, \ldots, \mathcal{S}_1$ is *regularizing* if $\mathcal{S}_{i-1} = r\mathcal{S}_i$ for each $1 < i \leq j$; a regularizing sequence is *complete* if $\mathcal{S}_1$ is regular. A complete sequence on $[0, n]$ is *maximal* if its length is the maximum of the length of all the regularizing sequences. A state is called *maximally irregular* if it is the initial state for a maximal complete sequence of states on $[0, n]$. If $0 < m < n$ and $\mathcal{S}$ is a state on $[0, n]$, then we define

$$\mathcal{S}^{(m)} = \{I \in \mathcal{S} : I \subset [0, m]\},$$

the restriction of $\mathcal{S}$ to $[0, m]$.

Proposition 6.1. *The maximal length of a complete sequence on* $[0, n]$ *is* n.

Proof. There is one non-empty state on $[0, 1]$, namely that consisting of one interval $[0, 1]$, and thus the length of the regularizing sequence that starts at $\{[0, 1]\}$ is 1. Suppose the claim true for the states on $[0, n]$, and consider a state $\mathcal{S}$ on $[0, n+1]$. If there is in $\mathcal{S}$ no interval ending in $n+1$, then $\mathcal{S}$ is equivalent to a state on $[1, n]$, hence, by the inductive assumption, the maximal length of its sequence is n. If there is in $\mathcal{S}$ only one interval ending with $n+1$, then it will not participate in creating new elements of $r\mathcal{S}$, hence again $\mathcal{S}$ is equivalent to a state on $[0, n]$. If there are at least two such intervals in $\mathcal{S}$, then there will be no interval in $r\mathcal{S}$ ending with $n+1$, hence $r\mathcal{S}$ is equivalent to a state on $[0, n]$, thus by the inductive assumption, the maximal length of a complete sequence starting with $r\mathcal{S}$ is n, hence that starting with $\mathcal{S}$ is $n+1$. ∎

Proposition 6.2. *If* $\mathcal{S}$ *is a maximally irregular state on* $[0, n]$, *then*

(6.1) *for each* $m \in [2, n]$ *there is a unique* $k < m$ *with* $[k, m] \in \mathcal{S} \backslash \{[n-1, n]\}$,

(6.2) *for each* $k \in [0, n-1]$ *there is a unique* $m > k$ *such that* $[k, m] \in \mathcal{S}$.

Proof. Use induction on n. For $n = 1$ the only maximal complete sequence on $[0, 1]$ consists of the regular state $\mathcal{S} = \{[0, 1]\}$. If the claim holds for n and $\mathcal{S}$ is a maximally irregular state on $[0, n+1]$, then $(r\mathcal{S})|_n$ is a maximally irregular state on $[0, n]$, hence by inductive assumption, $[n-1, n] \in r\mathcal{S}$ and there is a unique $k < n-1$ such that $[k, n] \in r\mathcal{S}$. Both $[k, n]$ and $[n-1, n]$ cannot belong to $\mathcal{S}$, because then

$[k, n-1] \in r\mathcal{S}$, contradicting the uniqueness in (6.2). If $[k, n] \notin \mathcal{S}$ and $[n-1, n] \notin \mathcal{S}$, then $[k, n+1], [n-1, n+1], [n, n+1] \in \mathcal{S}$ and thus $[k, n-1] \in r\mathcal{S}$ in contradiction with the uniqueness in (6.2). Therefore either $[n-1, n] \in \mathcal{S}$ and $[k, n] \notin \mathcal{S}$, which implies that $[n, n+1], [k, n+1] \in \mathcal{S}$, or $[k, n] \in \mathcal{S}$ and $[n-1, n] \notin \mathcal{S}$, which implies that $[n, n+1], [n-1, n+1] \in \mathcal{S}$.

The proof will be complete if we show that if $[i, j] \in r\mathcal{S}$ and $j < n-1$ (thus $i \neq k$) then $[i, j] \in \mathcal{S}$. If this were not the case then there would be l such that $[i, l], [j, l] \in \mathcal{S} \subset r\mathcal{S}$, and, by the uniqueness in (6.2) applied to $(r\mathcal{S})|_n$, $l = n+1$, that is, $[i, n+1], [j, n+1] \in \mathcal{S}$. As $[k, n+1] \in \mathcal{S}$ either $[i, k] \in r\mathcal{S}$ or $[k, i] \in r\mathcal{S}$, violating the uniqueness of the inductive assumption (6.2). Thus $[i, j] \in \mathcal{S}$. ∎

Therefore a maximally irregular state $\mathcal{S}$ on $[0, n]$ has exactly n intervals: to every $k \in [0, n-1]$ there corresponds one interval $[k, l_k]$. If $n > 1$ then the only doubleton that must not belong to S is $[0, 1]$, and the one that must belong is $[n-1, n]$.

Remark 6.3. *The doubletons of a maximally irregular state determine the state. The algorithm is as follows: let $\mathcal{A}$ be the set of doubletons of a maximally irregular state $\mathcal{S}$ on $[0, n]$. As just observed, $[n-1, n] \in \mathcal{A}$, and there must be $l < n-1$ such that $[l, n] \in \mathcal{A}$. Notice that l is the greatest among the integers less than n such that $[l-1, l] \notin \mathcal{S}$. The next interval of $\mathcal{S}$ is of the form $[k, l]$, where k is the greatest among the integers less than l such that $[l-1, l] \notin \mathcal{S}$. And so on... In this way, all the integers j from $[0, n-1]$ such that $[j, j+1] \notin \mathcal{S}$, appear as the initial ends of the remaining intervals of $\mathcal{S}$.*

The necessity expressed by Proposition 6.2 is also sufficient. A (partial) map on an ordered space is said to be *contractive (strictly contractive)* if $x \geq f(x)$ (respectively, $x > f(x)$) for every x.

Proposition 6.4. *A state $\mathcal{S}$ on $[0, n]$ is maximally irregular if and only if $[n-1, n] \in \mathcal{S}$, and there exists a strictly contractive bijection $f : [2, n] \to [0, n-2]$ such that $[f(k), k] \in \mathcal{S}$ for every $k \in [2, n]$.*

Proof. The necessity follows from Proposition 6.2.

We show by induction that the conditions of the statement imply maximal irregularity. If $n = 2$, then $\mathcal{S} = \{[0, 2], [1, 2]\}$ is the (only) state fulfilling the conditions. Here $f : [2] \to [0]$. The state $r\mathcal{S} = \{[0, 2], [0, 1], [1, 2]\}$ is regular. Suppose that the statement is true for the states on $[0, k]$ for $k \leq n$ and let $\mathcal{S}$ be a state on $[0, n+1]$ which fulfills the conditions. In particular, there are only two intervals with the same end: $[n, n+1]$ and $[f(n+1), n+1]$. Therefore $r\mathcal{S} = \mathcal{S} \cup \{[f(n+1), n]\}$, hence the collection of the intervals of $r\mathcal{S}$ which do not end in $n+1$ fulfills the conditions of the proposition, hence is maximally irregular state on $[0, n]$ by virtue of the inductive assumption. ∎

We have seen in Remark 6.3 that doubletons determine a maximally irregular state. In terms of the function f of Proposition 6.4, with an auxiliary map g defined by $g(j) = f(j) + 1$ this amounts to the following

Proposition 6.5. *Each contractive bijection $g : [2, n] \to [1, n-1]$ is determined by the fixed points.*

Let $F \subset [1, n-1]$. Then F is a disjoint union of intervals

$$F = \bigcup_{1 \leq i \leq j} [k_i, l_i].$$

By definition, $l_0 = 0$ and $k_{j+1} = n$. We set $g(m) = m-1$ for $l_{i-1}+1 < m < k_i$ and for $1 \le i \le j+1$ and $g(l_i+1) = k_i - 1$ for every i. If $m \in F$ then we set $g(m) = m$. In this way we define a unique contractive bijection g such that $\operatorname{fix} g = F$.

There are exactly 2^{n-1} maximally irregular states on $[0,n]$. This follows from the following

Proposition 6.6. *No state on $[0,1]$ is irregular. The only irregular state on $[0,2]$ is $\{[0,2],[1,2]\}$. If $\mathcal{S}$ is a maximally irregular state on $[0,n]$, then the two maximally irregular states on $[0,n+1]$, the restriction to $[0,n]$ of the regularization of which is $\mathcal{S}$ are*

$$\mathcal{S}\backslash\{[n-1,n]\} \cup \{[n-1,n+1],[n,n+1]\},$$
$$\mathcal{S}\backslash\{[k,n]\} \cup \{[k,n+1],[n,n+1]\},$$

where k is the unique element of $[0,n-1]$ for which $[k,n] \in \mathcal{S}$.

Proposition 6.6 enables one to construct all the maximally irregular states. The scheme below starts with the only irregular state on $[0,2]$.

$$[0,2],[1,2] \to \begin{cases} [0,3],[1,2],[2,3] \to \begin{cases} [0,4],[1,2],[2,3],[3,4] \\ [0,3],[1,2],[2,4],[3,4] \end{cases} \\ [0,2],[1,3],[2,3] \to \begin{cases} [0,2],[1,4],[2,3],[3,4] \\ [0,2],[1,3],[2,4],[3,4] \end{cases} \end{cases}$$

We would need also an algorithm of obtaining maximally irregular states on $[0,n]$ from those on $[1,n]$.

Proposition 6.7. *For a maximally irregular state $\mathcal{S}$ on $[1,n]$, there are two maximally irregular states $\mathcal{S}_0, \mathcal{S}_1$ on $[0,n]$ the restrictions of which to $[1,n]$ are equal to $\mathcal{S}$, namely we set $\mathcal{S}_0 = \mathcal{S} \cup \{[0,2]\}$, and if $[1,k] \in \mathcal{S}$ then $\mathcal{S}_1 = \mathcal{S} \cup \{[0,k],[1,2]\}$.*

Proof. As $\mathcal{S}$ is a maximally irregular state $\mathcal{S}$ on $[1,n]$, by Proposition 6.4 there is a bijection $f : [3,n] \to [1,n-2]$ such that $[f(k),k] \in \mathcal{S}$ for every $k \in [3,n]$. There are two ways of extending f to a bijection from $[2,n]$ to $[0,n-2]$, that is, $f_0(2) = 0$ and $f_0(k) = f(k)$ for $k \in [3,n]$, or when m is such that $f(m) = 1$ then set $f_1(m) = 0, f_1(2) = 1$ and $f_1(k) = f(k)$ for all $k \in [3,n]$ distinct from m, that is, for which $f(k) \neq 1$. ∎

As for pretopologies, we introduce a concept of irregularity spectrum and of irregularity number of a point of $[0,n]$ with respect to a state on $[0,n]$. If $\mathcal{T}$ is a state on $[0,n]$ then p is a spectrum component of k if k is an irregularity point of $r^{p-1}\mathcal{T}$. If now we come back to the standard pretopologies on cascades corresponding to states, we realize the origin of a maximally irregular cascade need not be irregular. We witness a phenomenon of propagation of irregularities in the process of partial regularizations, before they disappear with the full regularization.

Example 6.8. *If $\mathcal{S} = \{[0,2],[1,3],[2,3]\}$, then 1 is irregular, but 0 is regular. As $r\mathcal{S}\backslash\mathcal{S} = \{[1,2]\}$, only 0 is irregular for $r\mathcal{S}$. Because $r^2\mathcal{S}\backslash r\mathcal{S} = \{[0,1]\}$, the state $r^2\mathcal{S}$ is regular. Therefore the irregularity of 0 is 2.*

Example 6.9. *Similarly, only 2 is irregular for $\mathcal{R} = \{[0,2],[1,3],[2,4],[3,4]\}$, only 1 is irregular for $r\mathcal{R}$, because $r\mathcal{R}\backslash\mathcal{T} = \{[2,3]\}$, and only 0 is irregular for $r^2\mathcal{R}$, because $r^2\mathcal{R}\backslash r\mathcal{R} = \{[1,2]\}$. As $r^3\mathcal{R}\backslash r^2\mathcal{R} = \{[0,1]\}$, the state $r^3\mathcal{R}$ is regular. Hence the irregularity of 0 is 3.*

Example 6.10. *If* $\mathcal{T} = \{[0,4],[1,2],[2,3],[3,4]\}$ *then* $r\mathcal{T}\backslash\mathcal{T} = \{[0,3]\}$, $r^2\mathcal{T}\backslash r\mathcal{T} = \{[0,2]\}$ *and* $r^3\mathcal{T} = \{[0,1]\}$. *Therefore* 0 *is irregular for* $\mathcal{T}, r\mathcal{T}$ *and for* $r^2\mathcal{T}$, *hence the irregularity spectrum of* 0 *is* $\{1,2,3\}$ *and its irregularity is* 3.

Theorem 6.11. *If* $\mathcal{S}$ *is a state and* $r^m\mathcal{S}$ *is topological for some* $m \geq 1$, *then* $r\mathcal{S}$ *is regular, hence topological.*

Proof. First prove together the cases $m = 1$ and 2, so let $m = 1$ or 2. Suppose to the contrary that $\mathcal{S}$ is a state on $[0,n]$ such that $r^m\mathcal{S}$ is topological and $[i,j] \in r^2\mathcal{S}\backslash r\mathcal{S}$. Thus there is $k > j$ such that $[i,k],[j,k] \in r\mathcal{S} \subset r^m\mathcal{S}$. The set $\{k \in [1,n] : k > j, [i,k],[j,k] \in r^m\mathcal{S}\}$ is non-empty; we now fix k to be its largest number. It is claimed that $[i,k],[j,k] \in \mathcal{S}$. Suppose not.

If $[i,k] \notin \mathcal{S}$ then there is $l > k$ with $[i,l],[k,l] \in r^{m-1}\mathcal{S} \subset r^m\mathcal{S}$. Then $[j,k],[k,l] \in r^m\mathcal{S}$, so $[j,l] \in r^m\mathcal{S}$ by the topologicity of $r^m\mathcal{S}$, which contradicts the maximality of k.

If $[j,k] \notin \mathcal{S}$ then there is $l > k$ with $[j,l],[k,l] \in r^{m-1}\mathcal{S} \subset r^m\mathcal{S}$. Then $[i,k],[k,l] \in r^m\mathcal{S}$ so $[i,l] \in r^m\mathcal{S}$ as $r^m\mathcal{S}$ is topological, and since also $[j,l] \in r^m\mathcal{S}$, the maximality of k is contradicted.

Thus $[i,k],[j,k] \in \mathcal{S}$ and hence $[i,j] \in r\mathcal{S}$ contrary to the assumption. Therefore $r^2\mathcal{S} = r\mathcal{S}$ if either $r\mathcal{S}$ or $r^2\mathcal{S}$ is topological.

Suppose that for some state $\mathcal{S}$ on $[0,n]$ and some $m > 2$ the state $r^m\mathcal{S}$ is topological, but $r\mathcal{S}$ is not regular. Take the least such m over all states $\mathcal{S}$ on $[0,n]$. As $r^m\mathcal{S} = r^{m-1}(r\mathcal{S})$ is topological, by the minimality of m the state $r(r\mathcal{S}) = r^2\mathcal{S}$ must be regular, hence $r^2\mathcal{S} = r^m\mathcal{S}$ is topological. Thus by the case $m = 2$ already proven, $r\mathcal{S}$ is regular. ∎

In the theorem above, k must not be equal to 0. Indeed,

Example 6.12. *The state* $\mathcal{S} = \{[0,3],[1,4],[2,3],[2,4]\}$ *is topological, but* $r\mathcal{S}\backslash\mathcal{S} = \{[0,2],[1,2]\}$ *and* $r^2\mathcal{S}\backslash r\mathcal{S} = \{[0,1]\}$.

Moreover, we will see that for every ordinal β there exists a Hausdorff topology, the irregularity of which is β.

Proposition 6.13. *For every natural* n *there exists a topological state whose irregularity is* n.

Proof. This is true for $n = 1$, namely $\{[0,2],[1,2]\}$. So suppose that there exists a topological state $\mathcal{R} = \{[a_k,b_k] : 1 \leq k \leq l\}$ on $[0,n]$ whose irregularity is $n-1$. Let $1 < n < c_1 < \ldots < c_l$, and set $I_k = [a_k,c_k]$ and $J_k = [b_k,c_k]$. Then $\mathcal{S} = \{I_k : 1 \leq k \leq l\} \cup \{J_k : 1 \leq k \leq l\}$ is a topological state of irregularity n. Indeed, $\mathcal{S}$ is topological, because there are no consecutive intervals in $\mathcal{S}$. On the other hand, $r\mathcal{S} = \mathcal{R} \sqcup \mathcal{S}$, and if two intervals in $r\mathcal{S}$ have common right ends and have not been simultaneously in $\mathcal{S}$, then they are both in $\mathcal{R}$, because the right ends of the elements of $\mathcal{S}$ are strictly greater than those of the elements of $\mathcal{R}$, so that $r^2\mathcal{S} = r\mathcal{R} \cup \mathcal{S}$. For the same reason $r^{j+1}\mathcal{S} = r^j\mathcal{R} \cup \mathcal{S}$ for each natural j, which proves that the irregularity of $\mathcal{S}$ is n. ∎

The regularization of a topology need not be topological.

Example 6.14. *The state* $\{[0,3],[1,3],[1,2]\}$ *is topological and its partial regularization* $\{[0,3],[1,3],[1,2],[0,1]\}$ *is regular but not topological, because its topologization is* $\{[0,3],[1,3],[1,2],[0,1],[0,2]\}$. *As the topologization of a regular state, the last state must be regular by Proposition 6.15.*

Proposition 6.15. *The topologization of a regular state is regular.*

Proof. Let $\mathcal{S}$ be regular and consider $[j,k] \in rT\mathcal{S}$, that is, there exists $m > k$ such that $[k,m],[j,m] \in T\mathcal{S}$, and thus $[j,m] = \bigcup_{i=1}^{j} J_i$ and $[j,k] = \bigcup_{i=1}^{k} K_i$, where $\{J_i : 1 \le i \le j\}$ and $\{K_i : 1 \le i \le k\}$ are intervals belonging to $\mathcal{S}$ such that the right end of J_i (respectively, K_i) is less than or equal to the left end of J_{i+1} (respectively, K_{i+1}). Use induction on $j+k$. If $j+k=2$, then $J_1 \sim K_1 \in \mathcal{S} \subset T\mathcal{S}$. If $l < j$ and $m < k$ and $J_j = K_k$ then $I = J \sim K = \bigcup_{i=1}^{j} J_i \sim \bigcup_{i=1}^{k} K_i = \bigcup_{i=1}^{l} J_i \sim \bigcup_{i=1}^{m} K_i$, so the inductive assumption ensures $I \in T\mathcal{S}$. Otherwise either $J_j \sim K_k \neq \varnothing$ or $K_k \sim J_j \neq \varnothing$. In either case the non-empty interval belongs to $\mathcal{S}$. In the first case $I = \bigcup_{i=1}^{j-1} J_i \cup (J_j \sim K_k) \sim \bigcup_{i=1}^{k-1} K_i$, in the second $I = \bigcup_{i=1}^{j-1} J_i \left(\bigcup_{i=1}^{k-1} K_i \cup (K_k \sim J_j)\right)$. Appeal to the inductive hypothesis to get $I \in T\mathcal{S}$. ∎

However if we come back to compatible pretopologies on cascades, then we discover that this proposition is not new. Proposition 6.15 is equivalent to the fact that the topologization of a compatible regular pretopology on a cascade of finite rank is regular. But a compatible pretopology is Hausdorff and moreover every sequential cascade (not only of finite rank) is countable. Therefore Proposition 6.15 is a special case of Theorem 3.9.

7. Regularization of pretopologies of countable character

A convergence ξ is of *countable character (first-countable)* if $x \in \lim_\xi \mathcal{F}$ implies the existence of a countably based filter $\mathcal{G} \subset \mathcal{F}$ such that $x \in \lim_\xi \mathcal{G}$. Therefore a pretopology is of countable character, whenever every vicinity filter is countably based. Countable character is a concretely coreflective property of convergences. In particular, every infimum of convergences of countable character is of countable character. This however is no longer the case in the category of pretopologies.

Example 7.1. *Consider a countable fan, that is, the disjoint union $\{\infty\} \cup \{(n,k) : n,k < \omega\}$ and let π_m be a convergence defined by $\{\infty\} = \lim_{\pi_m} \mathcal{F}$ for a free filter $\mathcal{F}$ whenever $\mathcal{F}$ is finer than the cofinite filter of $\{(n,k) : k < \omega, n \le m\}$. The other points are isolated. This defines a descending sequence of Hausdorff pretopologies of countable character (actually sequential), and clearly $\bigwedge_{m<\omega} \pi_m$ is a convergence of countable character. But the infimum in the lattice of pretopologies $\bigwedge_{m<\omega}^{P} \pi_m = P\left(\bigwedge_{m<\omega} \pi_m\right)$ is the well-known fan topology, which is Fréchet but not of countable character.*

Countable character is preserved by the partial regularization. In fact, if ξ is of countable character and if $x \in \lim_{r\xi} \mathcal{F}$, then there is a countably based filter $\mathcal{G}$ such that $x \in \lim_\xi \mathcal{G}$ and $\mathrm{adh}_\xi^\natural \mathcal{G} \le \mathcal{F}$. Of course, $x \in \lim_{r\xi}(\mathrm{adh}_\xi^\natural \mathcal{G})$ and $\mathrm{adh}_\xi^\natural \mathcal{G}$ is countably based. And since the countable character is stable for infima, every iterated partial regularization, hence also the regularization, of a convergence of countable character is of countable character [9, Proposition 7.1].[4] However an infinitely iterated partial pretopological regularization of a pretopology of countable character need not be of countable character, which is due to the fact that the pretopological infimum in general does not preserve the character. Indeed, consider

[4]Actually this statement holds for convergences of any character.

Example 7.2. *Let $A = \bigsqcup_{n,m<\omega} A_{n,m}$ be a sink endowed with its natural convergence. We extend the convergence of A to $X = \{\infty\} \cup A$ so that $\mathcal{F}_0$ converges to ∞. This defines a topology τ of countable character. Notice that $\operatorname{adh}^{\natural}_{r^k \tau} \mathcal{F}_0 = \bigwedge_{m \leq k} \mathcal{F}_k$, thus $\bigwedge_{k<\omega} \mathcal{F}_k$ is the coarsest free filter that converges to ∞ in $r^{\omega}_{P}\tau$, which shows that $r^{\omega}_{P}\tau$ is not of countable character.*

Proposition 7.3. *If a pretopology π of countable character is Hausdorff, then $r\pi$ is Hausdorff.*

Proof. If x is isolated, then the singleton $\{x\}$ constitutes a base for π at x, hence $\operatorname{adh}_{\pi}\{x\} = \{x\}$ is a base of x for $r\pi$. If x is not isolated then there is a base $(V_n)_n$ of the vicinity $\mathcal{V}_{\pi}(x)$ such that $V_n \backslash V_{n+1} \neq \varnothing$ for every $n < \omega$, and $\bigcap_{n<\pi} V_n = \{x\}$. As π is Hausdorff,

$$\{x\} = \lim_{\pi} \mathcal{V}_{\pi}(x) = \operatorname{adh}_{\pi} \mathcal{V}_{\pi}(x) = \bigcap_{n<\omega} \operatorname{adh}_{\pi} V_n,$$

hence $r\pi$ is Hausdorff, because the intersection of the base $\{\operatorname{adh}_{\pi} V_n : n < \omega\}$ of the vicinity filter of x in $r\pi$ is $\{x\}$. ∎

Proposition 7.4. *Let π be a pretopology of countable character. An element x is irregular with respect to π if and only if there exists a sequence $(x_n)_n$ such that*

$$x \in \lim_{r\pi}(x_n)_n \backslash \operatorname{adh}_{\pi}(x_n)_n. \tag{7.1}$$

Proof. An element x is irregular for π if and only if $\operatorname{adh}^{\natural}_{\pi} \mathcal{V}_{\pi}(x_{\varnothing})$ does not converge to x, that is, whenever there is $V \in \mathcal{V}_{\pi}(x)$ and a decreasing filter base (V_n) of $\mathcal{V}_{\pi}(x)$ such that for every $n < \omega$ there is $x_n \in \operatorname{adh}_{\pi} V_n \backslash V$. Hence $(x_n)_n$ converges to x in $r\pi$ but $x \notin \operatorname{adh}_{\pi}(x_n)$, which implies that x is irregular for π. ∎

We observe that no separation axiom has been used in Proposition 7.4. The characterization above cannot be extended to arbitrary convergences (not even pseudotopologies) of countable character. It holds however for paratopologies of countable character. A convergence is a *paratopology* [1] whenever $x \notin \lim \mathcal{F}$ implies the existence of a countably based filter $\mathcal{H}$ that meshes with $\mathcal{F}$ such that $x \notin \operatorname{adh} \mathcal{H}$.

Proposition 7.5. *Let π be a pretopology of countable character. An element x is irregular with respect to π if and only if there exists a sequence (x_n) and a bisequence $(x_{n,k})$ such that $(x_{n,k})_k$ is free for each $n < \omega$, $x \notin \operatorname{adh}_{\pi}(x_n)_n$, but $x_n \in \lim_{\pi}(x_{n,k})_k$ for every $n < \omega$, and*

$$x \in \lim_{\pi} \int_{(n)} (x_{n,k})_k.$$

Proof. Indeed, by Proposition 7.4 there is a sequence (x_n) such that (7.1) holds. In particular, if (V_m) is a decreasing base of $\mathcal{V}_{\pi}(x)$ then for every $m < \omega$ there is $n_m > n_{m-1}$ such that $x_n \in \operatorname{adh}_{\pi} V_m$ for $n \geq n_m$. Consequently, for each such an n there exists a sequence $(x_{n,k})_k$ on V_m for which $x_n \in \lim_{\pi}(x_{n,k})_k$. Since $\int_{(n)}(x_{n,k})_k$ is finer than $\mathcal{V}_{\pi}(x)$, it converges to $x_{\varnothing}$ in π. If $(x_{n,k})_k$ were not free for infinitely many n, then $\int_{(n)}(x_{n,k})_k$ would be coarser than a subsequence of $(x_n)_n$, which must not converge to x in π in view of (7.1). Therefore, $(x_{n,k})_k$ is free for almost all n, hence for all n after having dropped a finite number of them. ∎

Proposition 7.5 will be now extended to

Lemma 7.6. *If π is a pretopology of countable character, and $x_\varnothing \in \lim_{r\pi} \mathcal{F}$, then there is $F \in \mathcal{F}$ and for each $x \in F$ there is a sequential filter $\mathcal{E}(x)$ such that $x \in \lim_\pi \mathcal{E}(x)$ and $x_\varnothing \in \lim_\pi \mathcal{E}(\mathcal{F})$. If moreover $x_\varnothing \notin \mathrm{adh}_\pi \mathcal{F}$, then we can choose $\mathcal{E}(x)$ to be free.*

Proof. If $(V_m)_{m<\omega}$ is a decreasing base of the vicinity filter $\mathcal{V}_\pi(x_\varnothing)$, then $x_\varnothing \in \lim_{r\pi} \mathcal{F}$ amounts to $\mathrm{adh}^\natural_\pi \mathcal{V}_\pi(x_\varnothing) \leq \mathcal{F}$, that is, $\mathrm{adh}_\pi V_m \in \mathcal{F}$ for each $m < \omega$. Let $V_\infty = \bigcap_{m<\omega} \mathrm{adh}_\pi V_m$, and decompose $\mathcal{F}$ into $\mathcal{F}_0 = \mathcal{F} \vee V^c_\infty$ and $\mathcal{F}_1 = \mathcal{F} \vee V_\infty$, where either $\mathcal{F}_0$ or $\mathcal{F}_1$ can be degenerate.

If $x \in \mathrm{adh}_\pi V_m \setminus \mathrm{adh}_\pi V_{m+1}$ (we do not exclude the case where the difference is empty), then there is a sequential filter $\mathcal{E}(x)$ such that $V_m \in \mathcal{E}(x)$ and $x \in \lim_\pi \mathcal{E}(x)$. As $\mathrm{adh}_\pi V_m \in \mathcal{F}$ for each $m < \omega$, then $\mathcal{E}(\mathcal{F}_0) \geq \mathcal{V}_\pi(x_\varnothing)$ provided that $\mathcal{F}_0$ is non-degenerate.

On the other hand, if $x \in V_\infty = \bigcap_{m<\omega} \mathrm{adh}_\pi V_m = \mathrm{adh}_\pi \mathcal{V}_\pi(x_\varnothing)$ (the latter holds because π is a pretopology), then there is a sequential filter $\mathcal{E}(x) \geq \mathcal{V}_\pi(x_\varnothing)$ such that $x \in \lim_\pi \mathcal{E}(x)$, hence $\mathcal{E}(V_\infty) \geq \mathcal{V}_\pi(x_\varnothing)$. Hence if $\mathcal{F}_1$ is non-degenerate, then $\mathcal{E}(\mathcal{F}_1) \geq \mathcal{E}(V_\infty) \geq \mathcal{V}_\pi(x_\varnothing)$. Therefore $\mathcal{E}(\mathcal{F}) = \mathcal{E}(\mathcal{F}_0) \wedge \mathcal{E}(\mathcal{F}_1) \geq \mathcal{V}_\pi(x_\varnothing)$.

If $x_\varnothing \notin \mathrm{adh}_\pi \mathcal{F}$ and there is $H \in \mathcal{F}^\#$ such that $\mathcal{E}(x)$ is not free for every $x \in H$, then the principal filter $\mathcal{N}_\iota(x)$ of x is finer than $\mathcal{E}(x)$, hence $x_\varnothing \in \lim_\pi \mathcal{N}_\iota(\mathcal{F} \vee H)$ by the first part of the proof, which yields a contradiction, because $\mathcal{N}_\iota(\mathcal{F} \vee H) = \mathcal{F} \vee H$. ∎

Classical simplest examples of non-regular topologies are of countable character.

Example 7.7. [5, Example 1.5.6] *Consider the unit interval* $[0, 1]$ *in which a basic family of closed sets consists of the closed sets for the natural topology and of* $\{\frac{1}{n} : n < \omega\}$. *In this topology* $x = 0$ *is irregular. Then* $x_n = \frac{1}{n}$ *and* $x_{n,k} = \frac{1}{n} + \frac{1}{k}$ *verify Proposition 7.5.*

Example 7.8. *Consider the unit disc in* $\mathbb{R}^2$, *the interior of which carries the natural topology, while a neighborhood base of an element* x_∞ *of the border is of the form*

$$\{x : \|x\| < 1, \|x - x_\infty\| < \frac{1}{n}\} \cup \{x_\infty\}.$$

To illustrate Proposition 7.5 take any sequence (x_n) *of distinct terms on the border converging to* x_∞ *in the natural topology, and let* $(x_{n,k})_k$ *be a sequence converging to* x_n *from inside. We can also ask that the family* $\{x_{n,k} : k < \omega\}$ *where* $n < \omega$ *be discrete.*

The contour $\int_{(n)}(x_{n,k})_k$ in the proposition above is not countably based. As π is of countable character, the trace of $\mathcal{V}_\pi(x)$ on $\{x_{n,k} : n, k < \omega\}$ is countably based, coarser than $\int_{(n)}(x_{n,k})_k$, and converges to x in π. This fact suggests that regularity can be studied with the aid of sequential cascades endowed with some basic irregular pretopologies of countable character. Standard pretopologies on cascades are not of countable character. We can however introduce on sequential cascades some other pretopologies, which are of countable character, and which preserve most of other properties useful in our study of regularity. In the case of an extended bisequence $\{\varnothing\} \cup \omega \cup \{(n, k) : n, k < \omega\}$, we define a fundamental irregular pretopology so that the elements (n, k) are isolated, $V_m = \{n\} \cup \{(n, k) : k \geq m\}$ is a neighborhood base of n, and $V_m = \{\varnothing\} \cup \{(n, k) : n, k \geq m\}$ is a neighborhood base of $\varnothing$. We shall call this space a *fundamental irregular bisequence.* The partial regularization of the fundamental irregular bisequence is the natural topology of the extended bisequence.

Proposition 7.9. *An element x of a Hausdorff pretopology of countable character is irregular if and only if there is at x a homeormorphically embedded fundamental irregular bisequence.*

Proof. Let π be a Hausdorff pretopology of countable character, and x an irregular point. By Proposition 7.5 there is a sequence (x_n) and a bisequence $(x_{n,k})$ such that $x_n = \lim_\pi (x_{n,k})_k$ for each $n < \omega$, $x = \lim_\pi \int_{(n)} (x_{n,k})_k$ but $x \notin \mathrm{adh}_\pi (x_n)_n$. Therefore, by taking a subsequence if necessary, we can assume that all the terms of $(x_n)_n$ are distinct, because $r\pi$ is Hausdorff and *a fortiori* T_1. As $\{x_n\} \cup \{x_{n,k} : k < \omega\}$ is compact in π for every $n < \omega$, and $x = \lim_\pi \int_{(n)} (x_{n,k})_k$, we can, by taking subsequences of $(x_n)_n$ and of $(x_{n,k})_k$ for $n < \omega$ if necessary, find a neighborhood base $(V_n)_n$ of x such that $x_{n,k} \in V_n \backslash V_{n+1}$ for every $n < \omega$. It is clear that the pretopology induced $\{x\} \cup \{x : n < \omega\} \cup \{x_{n,k} : n, k < \omega\}$ coincides with the fundamental irregular topology. ∎

If T is a sequential cascade fully embedded in $\omega^{<\omega}$ then for each $t \in T$ there is $k < \omega$, and a sequence $(n_1, \ldots, n_k)$ of natural numbers so that t is identified with $(n_1, \ldots, n_k)$. A *fundamental* topology of a sequential cascade T fully embedded in $\omega^{<\omega}$ is defined by the following neighborhood bases $\mathcal{B}(t) = \{V_{t,m} : m < \omega\}$ of $t \in T$, where

$$V_{t,m} = \{t\} \cup \{(t, n, s) \in T : n \geq m\}.$$

Of course, $V_{t,m} \subset T^\uparrow(t)$. We observe that $V_{t,m}$ is open. Indeed if $t \neq r \in V_{t,m}$ then there is a finite (possibly empty) sequence p such that $r = (t, n, p)$ with $n \geq m$, hence $V_{r,0} = T^\uparrow(r) \subset V_{t,m}$. On the other hand, $V_{t,m}$ is closed, for if $s \notin V_{t,m}$, then let $r = \min\{t, s\}$. If $r < t$ then there is m such that $(r, m) \leq t$ and then $V_{r,m+1}$ is a neighborhood of s and $t \notin V_{r,m+1}$. If $r = t$ then there exist $n < m$ and a finite (possibly empty) sequence p such that $s = (t, n, p)$ and thus $T^\uparrow(s) \cap V_{t,m} = \varnothing$. We infer that the fundamental topology of a sequential cascade is Hausdorff, regular and of countable character. It is coarser than the natural topology.

We have analogous situation as for standard pretopologies. If $\mathcal{N}(t)$ stands for the neighborhood filter of t for the fundamental topology of a (monotone) sequential cascade, then denote by $\mathcal{N}_{(k)}^{(l)}(t)$ the restriction of the neighborhood filter of t, of level k, to the level $T^{(l)}$ of T. The closure (from the level l to the level k) is defined, as for the natural topology, by

$$t \in \mathrm{cl}_{(k)}^{(l)} A \Leftrightarrow A \in \left(\mathcal{N}_{(k)}^{(l)}(t)\right)^{\#}.$$

If T is of finite rank, then we can decompose the closure

$$\mathrm{cl}\, A = \bigcup_{k \leq l \leq r(T)} \mathrm{cl}_{(k)}^{(l)} A.$$

It is straightforward that for the fundamental topology of a monotone cascade of finite rank,

$$\left(\mathrm{cl}_{(l)}^{(m)}\right)^{\natural} \mathcal{N}_{(k)}^{(m)}(t) = \mathcal{N}_{(k)}^{(l)}(t) \tag{7.2}$$

for $k < l < m$.

Let $\mathcal{N}(t)$ be the neighborhood filter of t for the fundamental topology, and let $\mathcal{V}(t)$ be a vicinity filter at $t \in T$ of a pretopology on a sequential cascade T. If t is of level k and $l > k$, then $\mathcal{V}_{(k)}^{(l)}(t)$ denotes the restriction of $\mathcal{V}(t)$ to the level l of T. A pretopology on T is said to be *fundamental* if for every $0 \leq k < l \leq r(T)$, either $\mathcal{V}_{(k)}^{(l)}(t) = \mathcal{N}_{(k)}^{(l)}(t)$ or $\mathcal{V}_{(k)}^{(l)}(t)$ is degenerate for every t of level k. As for a standard pretopology, we associate a state with a fundamental pretopology so that $[k, l]$ belongs to the state if and only if $\mathcal{V}_{(k)}^{(l)}(t)$ is non-degenerate for each t of level k.

It follows from (7.2) that

Proposition 7.10. *A fundamental pretopology on a cascade of finite rank is regular if and only if the corresponding state is regular.*

Let us stress that (4.2) does not hold for the fundamental topology of a sequential cascade. One has only the inequality $\leq$. Roughly speaking, the equality is characteristic to topologies, which are the topologizations of some minimal pretopologies.

In view of Proposition 7.9 one could expect that if the irregularity of an element x of a Hausdorff pretopology of countable character is n, then there exists at x a homeomorphically embedded maximally irregular fundamental multisequence of rank $n+1$ (hence of irregularity n). However none of the two fundamental maximally irregular pretopologies of rank 3 (and irregularity 2) can be homeomorphically embedded in a topology, because they are non-topological. In fact, the restriction of a topology is topological, because of the reflective character of topologicity. Here we have an example of a Hausdorff topology of countable character, and of irregularity 2.

Example 7.11. *Let T be a (monotone) sequential cascade of rank 3. For every $t \in T$ of rank 2 split $T^+(t)$ into two disjoint infinite sets $T_0^+(t), T_1^+(t)$. We shall say that the elements belonging to $T_0^+(t)$ with t of level 2, and those of $T_1^+(t)$ with t of level 2 are of level 3_1. Let $\mathcal{M}(t)$ stand for the neighborhood filter of t with respect to the fundamental topology. We define a topology on T by:*

$$\mathcal{N}_{(0)}^{(3_0)}(\varnothing) = \mathcal{M}_{(0)}^{(3_0)}(\varnothing),$$
$$\mathcal{N}_{(2)}^{(3_0)}(t) = \mathcal{M}_{(2)}^{(3_0)}(t),$$
$$\mathcal{N}_{(1)}^{(3_1)}(t) = \mathcal{M}_{(1)}^{(3_1)}(t),$$
$$\mathcal{N}_{(2)}^{(3_1)}(t) = \mathcal{M}_{(2)}^{(3_1)}(t).$$

The other free restrictions of $\mathcal{M}$ are degenerate. This is a topology of countable character, and of irregularity 2.

8. Ramified fundamental cascades

Proposition 7.9 characterizes irregular elements of Hausdorff pretopologies of countable character in terms of a homeomorphically embedded fundamental irregular bisequence. In an attempt at characterizing elements of irregularity $n > 1$ of such spaces, one encounters a new phenomenon already for irregularity 2.

Indeed, let x be an element of irregularity 2 of a Hausdorff pretopology π of countable character on X. This means that x is irregular for $r\pi$, which is of countable character and Hausdorff by Proposition 7.3, and thus by Proposition 7.9, there is a homeomorphically embedded fundamental irregular bisequence $f : T \to X$ such that $f(\varnothing) = x$; in particular, $f(n) = \lim_{r\pi} f(n,k)_k$ and $x = \lim_{r\pi} f(\mathcal{V}^{(2)}_{(0)}(\varnothing))$, but $x \notin \operatorname{adh}_{r\pi} f(n)_n$.

Case 1. Now, if $x \in \operatorname{adh}_\pi f(\mathcal{V}^{(2)}_{(0)}(\varnothing))$, then by taking a subcascade if necessary, we can assume that $x = \lim_\pi f(\mathcal{V}^{(2)}_{(0)}(\varnothing))$.

Case 2. Otherwise by Lemma 7.6, T can be extended to a fundamental cascade S of rank 3, and f to a map $F : S \to X$ so that $F(t,k)_k$ is free and $F(t) = \lim_\pi F(t,k)_k$ for every $t \in \max T$, and $x = \lim_\pi F(\mathcal{V}^{(3)}_{(0)}(\varnothing))$.

Consider now another alternative regarding $f : T \to X$.

Case A. If $f(n) = \lim_\pi f(n,k)_k$ for infinitely many n, then by taking a subcascade corresponding to those n, we may suppose that this holds for each $n < \omega$.

Case B. If on the contrary, there is n_0 such that $f(n) \neq \lim_\pi (f(n,k))_k$ for $n \geq n_0$, then by taking a subcascade corresponding to those n, we can assume that the property holds for each $n < \omega$. This means that $f(n)$ is irregular (with respect to π) for each n, and thus by Proposition 7.9, there is an extension V of rank 3 of T, and an extension G of f to V such that $G|_{V\uparrow(n)}$ is a homeomorphically embedded fundamental irregular bisequence for each $n < \omega$.

If Cases 1. and A. occurred simultaneously, then we would get a characterization of the irregularity 1 of x, that is, $[0,2],[1,2]$. If Cases 1. and B. hold then the multisequence G is of the type $[0,2],[1,3],[2,3]$. If Cases 2. and A. hold then the multisequence F is of the type $[0,3],[1,2],[2,3]$. As for the simultaneity of Cases 2. and B., there is no type which corresponds. This is the situation of Example 7.11. In this case, the map $F \cup G : S \cup V \to X$ presents a new type of embedding, which will be referred to as $[0,3_0],[2,3_0],[1,3_1],[2,3_1]$. It is what we will call a ramified trisequence.

Let us notice that the existence of a homeomorphically embedded fundamental maximally irregular multisequence f of rank 3 does not imply that $f(\varnothing)$ has the irregularity 2.

Example 8.1. *Consider a cascade S of rank 3 endowed with a fundamental pretopology of the type $[0,2],[1,3],[2,3]$, and a cascade T of rank 2 endowed with a fundamental pretopology of the type $[0,2],[1,2]$. Let us identify the elements of level 0 and 1 of the two cascades, but distinguish the other levels. A way of representing the resulting pretopology is to form a set of subintervals of a tree*

$$0 < 1 \begin{array}{l} < 2_0 < 3_0 \\ < 2_1 \end{array} .$$

This set is the following: $\mathcal{T} = \{[0,2_0],[1,3_0],[2_0,3_0],[0,2_1],[1,2_1]\}$. Notice that $\mathcal{T}$ includes $\{[0,2_0],[1,3_0],[2_0,3_0]\}$, that is, the corresponding pretopology admits at 0 a homeomorphically embedded maximally irregular multisequence of rank 3. But $r\mathcal{T} \backslash \mathcal{T} = \{[1,2_0],[0,1]\}$, so that $r\mathcal{T}$ is regular.

The trisequence in Example 8.1 is not openly embedded, because $[0,2_1],[0,2_1]$ correspond to filters convergent to some elements of that trisequence from outside.

A *ramified level tree* L is the binary tree of height ω, that is, such that for each $l \in L$, the set $L^+(l)$ of immediate successors of l contains two elements. A ramified level tree can be represented as the tree of finite sequences, the terms of which are

0 or 1. As every tree the ramified level tree admits the level (ordinal) function: the root is of level 0, and if the level $h_L(l)$ has been defined till $m < \omega$, then the minimal elements of $\{l \in L : h_L(l) > m\}$ are of level $m+1\}$. A *ramified type* is a downwards closed subtree of L with finite branches. Therefore each non-maximal element of a ramified type has either one or two immediate successors.

Let L be a ramified type. A (sequential) *ramified cascade* T of type L is a monotone sequential cascade for which a map $\lambda : T \to L$ is defined so that

$$\lambda(\varnothing) = \varnothing,$$
$$\lambda(T^+(t)) = L^+(\lambda(t)),$$
$$l \neq \varnothing \Rightarrow \operatorname{card} \lambda^{-1}(l) = \infty.$$

If $t \in T$ then $\lambda(t)$ is called the *ramified level* of t.

A *standard pretopology* of a sequential ramified cascade T of type L is defined analogously as that for a sequential cascade, that is, for every $r, s \in L$ with $r < s$ either $\mathcal{V}_r^s(t) = \mathcal{N}_r^s(t)$ or $\mathcal{V}_r^s(t)$ is degenerate for every t with $\lambda(t) = r$. Here $\mathcal{V}$ stands for the vicinity system of a pretopology and $\mathcal{N}$ for the neighborhood system of the natural topology of T, while $\mathcal{V}_r^s(t)$ and $\mathcal{N}_r^s(t)$ stand respectively for the restrictions of the vicinity and the neighborhood filters of t with $\lambda(t) = r$ to the elements of ramified level s. Also the definition of *fundamental* pretopologies obviously extends that for usual sequential cascades.

A map $f : T \to W$ (from one ramified cascade to another) is *level-preserving* if there is a map $\varphi : \lambda_T(T) \to \lambda_W(W)$ (called the *level map* of f) such that

$$\lambda_W \circ f = \varphi \circ \lambda_T.$$

Proposition 8.2. *A level-preserving map $f : T \to W$ is continuous if and only if $f(\mathcal{V}_r^s(t)) \geq \mathcal{V}_{\varphi(r)}^{\varphi(s)}(f(t))$ for every couple $r < s$ of ramified levels of T and for each t of level r, where φ is the level map of f.*

A *state* $\mathcal{S}$ on a ramified level tree L is a finite set of intervals of cardinality at least 2 of L. The (ramified) *type* of $\mathcal{S}$ is the downwards closure of the elements of $\mathcal{S}$. Of course, it is a subtree of L. The *rank* of a state is that of its ramified type.

A state $\mathcal{T}$ associated with a standard (or a fundamental) pretopology is defined by $[r, s] \in \mathcal{T}$ if and only if $\mathcal{V}_r^s$ is non-degenerate.[5] Regularity, partial regularization and topologicity of a state on a ramified type are defined in the same way as for a state on an interval of natural numbers. It is a straightforward generalization of Proposition 5.1 that

Proposition 8.3. *A standard (fundamental) pretopology on a ramified cascade is topological (resp., regular) if and only if its state is topological (resp., regular).*

Let T, W be ramified cascades (considered either with standard or fundamental pretopologies) and $\mathcal{T}, \mathcal{W}$ be the corresponding ramified states.

If a map $f : T \to W$ between sequential cascades is continuous, then

$$f(T^+(t)) \backslash W^+(f(t) \cup \{f(t)\}$$

is finite for every $t \in T \backslash \max T$. By removing, for every t, the finite number of successors that derogate from that inclusion, we get a restriction of f, which is order-preserving. Consider a level-preserving map $f : T \to W$, and its level map $\varphi : \lambda_T(T) \to \lambda_W(W)$. Then

[5] that is, $\mathcal{V}_r^s(t)$ is degenerate for each t of ramified level r.

Proposition 8.4. *If a level-preserving map $f : T \to W$ is continuous then its level map fulfills $\varphi(\mathcal{T}) \subset \mathcal{W}$.*

Proof. Let f be continuous and let $[r, s] \in \mathcal{T}$. This means that $\mathcal{V}_r^s(t)$ is non-degenerate for every $t \in \lambda_T^-(r)$. As f is level-preserving, $\lambda_W(f(t)) = \varphi(r)$ and $\lambda_W(f(v)) = \varphi(s)$ for every $v \in \lambda_T^-(s)$, and $f(\mathcal{V}_r^s(t)) \geq \mathcal{V}_{\varphi(r)}^{\varphi(s)}(f(t))$, because f is continuous. This implies that $\mathcal{V}_{\varphi(r)}^{\varphi(s)}(w)$ is non-degenerate (for each w of level $\varphi(r)$), hence $[\varphi(r), \varphi(s)] \in \mathcal{W}$. ∎

A state on a ramified type is *maximally irregular* if its irregularity is maximal among all states of a given rank, and if it is minimal (with respect to inclusion) among the states of this irregularity. There are ramified states of maximal irregularity that are not minimal. This is in contrast with the situation of states on an interval.

Example 8.5. *Let $\mathcal{T}$ consist of $[0, 3_0], [2, 3_0], [1, 2], [1, 3_1], [2, 3_1]$. Then $r\mathcal{T}\backslash\mathcal{T} = \{[0, 2]\}$ and $r^2\mathcal{T}\backslash r\mathcal{T} = \{[0, 1]\}$ and $r^2\mathcal{T}$ is regular, hence the irregularity of $\mathcal{T}$ is 2 and is maximal for the rank 3. However $\{[0, 3_0], [2, 3_0], [1, 2]\}$ is a maximally irregular state on $\{0, 1, 2, 3_0\}$, so that $\mathcal{T}$ is not minimal.*

Let $\mathcal{T}$ be a state on P. The set $\operatorname{act}\mathcal{T}$ consists of *active* elements of $\mathcal{T}$, that is, of intervals that were used in obtaining $r\mathcal{T}\backslash\mathcal{T}$. If $I \in \operatorname{act}\mathcal{T}$ then there is $J \in \operatorname{act}\mathcal{T}$ such that $J \neq I$ and $\max I = \max J$. Let

$$\mathcal{T}_{\max} = \{I \in \mathcal{T} : \max I \cap \max P \neq \varnothing\}.$$

The set $\mathcal{T}_{\max}$ does not produce anything new during the second regularization, that is,

$$r^2\mathcal{T}\backslash r\mathcal{T} = r(r\mathcal{T}\backslash\mathcal{T}_{\max})\backslash r\mathcal{T}. \tag{8.1}$$

If $\mathcal{T}$ is maximally irregular then $\mathcal{T}_{\max} \subset \operatorname{act}\mathcal{T}$. Indeed, if $I \in \mathcal{T}_{\max}\backslash \operatorname{act}\mathcal{T}$ then $r^k(\mathcal{T}\backslash\{I\}) = r^k\mathcal{T}\backslash\{I\}$ for each $k < \omega$. This means that $\mathcal{T}$ would not be a minimal state corresponding to its irregularity.

It may happen that $\operatorname{act}\mathcal{T} \cap \operatorname{act}(r\mathcal{T}) \neq \varnothing$, even for a non-ramified state $\mathcal{T}$. Of course, such a state $\mathcal{T}$ must not be maximally irregular, because $\operatorname{act}\mathcal{T} = \mathcal{T}_{\max}$ for a non-ramified maximally irregular state $\mathcal{T}$.

Example 8.6. *Let $\mathcal{T} = \{[0, 3], [2, 3], [1, 4], [3, 4]\}$. Then $\operatorname{act}\mathcal{T} = \mathcal{T}, r\mathcal{T}\backslash\mathcal{T} = \{[0, 2], [1, 3]\}$, and $\operatorname{act}(r\mathcal{T}) = \{[0, 3], [2, 3], [1, 3]\}$. As mentioned $\mathcal{T}$ is not maximally irregular, because $(r\mathcal{T})^{(3)} = \{[0, 3], [2, 3], [0, 2], [1, 3]\}$ has 4 elements on $[0, 3]$.*

We notice that

Proposition 8.7. *If a state $\mathcal{T}$ is maximally irregular, then the elements of $\bigcup\mathcal{T}$ of level ≤ 2 form a chain.*

Proof. In fact, if $\mathcal{T}$ is a state and $\mathcal{T}_0 = \mathcal{T} \cap L^{\uparrow}(1_0)$ and $\mathcal{T}_1 = \mathcal{T} \cap L^{\uparrow}(1_1)$, then $(r\mathcal{T})_0 = r\mathcal{T}_0$ and $(r\mathcal{T})_1 = r\mathcal{T}_1$, so that if $\mathcal{T}_0$ and $\mathcal{T}_1$ were both non-empty, it would be enough to keep the one of the irregularity equal to that of the whole $\mathcal{T}$, contrary to the (set-theoretic) minimality of $\mathcal{T}$. On the other hand, if $L^+(\varnothing) = \{1\}$ and $L^+(1) = \{2_0, 2_1\}$, then let $\mathcal{T}_0 = \{I \in \mathcal{T} : 2_0 \leq \max I\}$ and $\mathcal{T}_1 = \{I \in \mathcal{T} : 2_1 \leq \max I\}$. In this case, because $[0, 1]$ can be obtained merely as an element of either $r^k\mathcal{T}_0$ or of $r^k\mathcal{T}_1$ for some k, the irregularity of $\mathcal{T}$ will be equal to the irregularity either of $\mathcal{T}_0$ or of $\mathcal{T}_1$, hence will not be maximal. ∎

Contrary to non-ramified maximally irregular states there can be ramified states that have active non-maximal intervals.

Example 8.8. *Consider* $\mathcal{S} = \{[0,5_0],[1,4_1],[2,4_1],[2,5_1],[3,5_1],[3,4_0],[4_0,5_0]\}$. *Then* $r\mathcal{S}\backslash\mathcal{S} = \{[0,4_0],[1,2],[2,3]\}$*, which shows that* $[1,4_1],[2,4_1] \in \operatorname{act}\mathcal{S}\backslash\mathcal{S}_{\max}$. *On the other hand* $r^2\mathcal{S}\backslash r\mathcal{S} = \{[0,3]\}, r^3\mathcal{S}\backslash r^2\mathcal{S} = \{[0,2]\}, r^4\mathcal{S}\backslash r^3\mathcal{S} = \{[0,1]\}$ *and* $r^4\mathcal{S}$ *is regular. It is easily seen that* $\mathcal{S}$ *is maximally irregular.*

As we will see, only a special subclass of maximally irregular states is sufficient to characterize finite irregularity of pretopologies of countable character. We define elementary states by induction on the rank. If $\mathcal{T}$ a state starting at 1 with the property that there is a unique ramified level t such that $[1,t] \in \mathcal{T}$, then

$$\mathcal{T}^* = \mathcal{T}\backslash[1,t] \cup [0,t].$$

The *elementary* state of rank 1 is the unique state of rank 1 that is $\{[0,1]\}$. The *elementary* state of rank 2 is the unique maximally irregular state of rank 2, that is $\{[0,2],[1,2]\}$. Suppose that we have defined elementary states $\mathcal{S}$ of rank less than or equal to m with the property that

$$\bigcup \mathcal{S} \cap \{l \in L : h_L(l) \leq 2\} \text{ is a chain,} \tag{8.2}$$

$$\exists!_{s_0,s_1 \in L}\ [0,s_0] \in \mathcal{S}, [1,s_1] \in \mathcal{S}, \tag{8.3}$$

an *elementary* state $\mathcal{S}$ of rank $m+1$ is of the form

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_0^*, \tag{8.4}$$

where $\mathcal{S}_0, \mathcal{S}_1$ are elementary states starting from 1, of ranks $1 \leq r(\mathcal{S}_0), r(\mathcal{S}_1) \leq m$ with $m = \max(r(\mathcal{S}_0), r(\mathcal{S}_1))$. It is clear that $\mathcal{S}$ given by (8.4) fulfills (8.2) and (8.3).

Proposition 8.9. *Each elementary state of rank* ≥ 2 *is maximally irregular.*

Proof. This is true for $m = 2$, where the irregularity of $\{[0,2],[1,2]\}$ is 1. Now if $\mathcal{S}_0, \mathcal{S}_1$ are elementary states starting at 1 such that $1 \leq r(\mathcal{S}_0), r(\mathcal{S}_1) \leq m$ and $m = \max(r(\mathcal{S}_0), r(\mathcal{S}_1))$, then the state $\mathcal{S}$ from (8.4) is of rank $m+1$ and fulfills

$$r^k\mathcal{S} = r^k\mathcal{S}_1 \cup r^k\mathcal{S}_0^*,$$

for each $k \leq m-1$ and the first k such that both $[0,1] \in r^k\mathcal{S}$ and $[0,2] \in r^k\mathcal{S}$ is equal to $\max(\rho(\mathcal{S}), \rho(\mathcal{T}))$, hence to $m-1$. ∎

We call *elementary* a fundamental pretopology on a sequential cascade if the corresponding state is elementary.

Theorem 8.10. *If* π *is a Hausdorff pretopology of countable character,* $m \geq 1$ *and*

$$x \in \lim\nolimits_{r^m\pi}(x_n)_n \backslash \operatorname{adh}_{r^{m-1}\pi}(x_n)_n, \tag{8.5}$$

then there exists homeomorphically embedded in π*, an elementary fundamental ramified multisequence* f *of rank* $m+1$ *such that* $f(\varnothing) = x$ *and* $f(n) = x_n$ *for each* $n < \omega$.

Proof. This is true for $m = 1$ because of Propositions 7.4 and 7.9. So suppose that the claim holds for $m \geq 1$, and let π be a Hausdorff pretopology of countable character on X such that

$$x \in \lim\nolimits_{r^{m+1}\pi}(x_n)_n \backslash \operatorname{adh}_{r^m\pi}(x_n)_n.$$

Because $r^m\pi$ is of countable character and Hausdorff (by Proposition 7.3), we can apply Proposition 7.9 to $r^m\pi$ to infer the existence of an elementary fundamental bisequence f on X, which is homeomorphically embedded in $r^m\pi$ with $f(\varnothing) = x$ and $f(n) = x_n$. Moreover, by Hausdorffness, we can require that there be a collection $(W_n)_n$ of disjoint subsets of X such that $W_n \in \mathcal{V}_{r^m\pi}(x_n)$ for every $n < \omega$.

Let p be the least natural number such that $x_n \in \lim_{r^p\pi}(x_{n,k})_k$ for almost each $n < \omega$, where $x_{n,k} = f(n,k)$. Because $p \leq m$, by inductive assumption, for every such n there is homeomorphically embedded an elementary fundamental ramified multisequence f_n from a ramified cascade T_n of rank $p+1$ to W_n, such that $f_n(\varnothing) = x_n$ and $f_n(k) = x_{n,k}$ for each $n, k < \omega$. Of course, in case $p = 0$, the multisequences f_n are of rank 1, so that the underlying sequential cascade is endowed with a regular topology. Because there are finitely many types of elementary fundamental ramified cascades of finite rank, by taking a subsequence of $(n)_n$ if necessary, we can assure that T_n are all of the same type.

Let q be the least natural number such that the filter $\mathcal{F} \approx \{\{f(n,k) : k < \omega\} : n < \omega\}$ converges to x in $r^q\pi$. Then $m = \max(p,q)$ for otherwise $x \in \operatorname{adh}_{r^m\pi}(x_n)_n$, contrary to the assumption.

By Hausdorffness, we can assume that there is a collection $\{W_{n,k} : n, k < \omega\}$ of disjoint sets such that $W_{n,k} \in \mathcal{V}_{r^q\pi}(x_{n,k})$.

If $q = 0$ then $x \in \lim_\pi \mathcal{F}$. Otherwise, by Lemma 7.6, for every (n,k) there exists a free sequential filter $\mathcal{E}(n,k) \approx (x_{n,k,l})_l$, which converges to $x_{n,k}$ in $r^{q-1}\pi$, and such that the filter $\mathcal{G} \approx \{\{x_{n,k,l} : k, l < \omega\} : n < \omega\}$ converges to x in $r^{q-1}\pi$.

Let v be the least natural number such that $\mathcal{E}(n,k)$ converges to $x_{n,k}$ in $r^v\pi$ for almost n, k. Of course, $v \leq q-1$. Hence, by inductive assumption, there is homeomorphically embedded an elementary fundamental ramified multisequence $f_{n,k}$ (from a ramified cascade $S_{n,k}$ to $W_{n,k}$) of rank $v+1$ such that $f_{n,k}(\varnothing) = x_{n,k}$ and $f_{n,k}(l) = x_{n,k,l}$ for each $n, k, l < \omega$. Of course, in case $v = 0$, each $f_{n,k}$ of rank 1, hence the underlying sequential cascade is endowed with a regular topology. Because there are finitely many types of elementary fundamental ramified cascades of finite rank, by taking a subcascade R of $\{\varnothing\} \cup \{(n) : n < \omega\} \cup \{(n,k) : n, k < \omega\}$ if necessary, we can assure that $S_{n,k}$ are all of the same type.

Let w be the least natural number such that $x \in \lim_{r^w\pi} \mathcal{G}$. Of course, $q - 1 = \max(v, w)$ for otherwise $x \in \lim_{r^q\pi} \mathcal{F}$. If $w = 0$ then we stop the construction. Otherwise we continue on applying Lemma 7.6 to $\mathcal{G}$, and so on.

We construct now a ramified cascade by taking the disjoint union of $R, T_n, S_{n,k}$ and possibly of other cascades resulting from the described construction. Then we quotient so that $\varnothing_{T_n}$ coincide with $n \in R$, $T_n^+(\varnothing_{T_n})$ coincide with $R^+(n)$, $\varnothing_{S_{n,k}}$ coincide $(n,k) \in R$, and so on. The constructed component multisequences coincide at the points that we have identified. Moreover we took care that the individual component cascades have ranges in disjoint vicinities of distinct points. Therefore the constructed multisequence is an injection. The pretopology of the constructed cascade is induced with the component cascades with the exception of the vicinity of the least element $\varnothing$. There is only one non-zero ramified level s for which $\mathcal{V}_0^{(s)}(\varnothing)$ is non-degenerate. If in our construction $q = 0$, then it will correspond to the filter $\mathcal{F}$, if $w = 0$ then it will correspond to the filter $\mathcal{G}$, and so on. The constructed cascade is elementary of rank $m+1$.

Because a map between pretopologies of countable character, is continuous if and only if it is sequentially continuous, the constructed injective multisequence is a homeomorphic embedding. ∎

Corollary 8.11. *If m is the irregularity of an element x of a Hausdorff pretopology of countable character, then there exists homeomorphically embedded, a maximally irregular fundamental ramified multisequence f of rank $m+1$ such that $f(\varnothing) = x$.*

Example 8.1 shows that the converse is true only for irregularity 1. However,

Proposition 8.12. *If there exists homeomorphically openly embedded a maximally irregular fundamental ramified multisequence f of rank $m+1$ such that $f(\varnothing) = x$, then m is irregularity of x.*

Example 8.13. *Consider the state*

$$\mathcal{S} = \{[0, 5_0], [1, 4_1], [2, 4_1], [2, 5_1], [3, 5_1], [3, 4_0], [4_0, 5_0]\}$$

from Example 8.8. By Corollary 8.11, there is a homeomorphically embedded elementary state. We claim that

$$\mathcal{R} = \{[0, 5_{0,0}], [3_0, 4_{0,0}], [4_{0,0}, 5_{0,0}], [1, 3_1], [2, 3_1], [2, 4_{0,1}], [3_0, 4_{0,1}]\}$$

is one. It is an elementary state obtained by the extension procedure as in Proposition 8.9, that is, $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1^*$, where $\mathcal{R}_0 = \{[1, 3_1], [2, 3_1]\}$ and

$$\mathcal{R}_1 = \{[1, 5_{0,0}], [3_0, 4_{0,0}], [4_{0,0}, 5_{0,0}], [2, 4_{0,1}], [3_0, 4_{0,1}]\}.$$

In turn, $\mathcal{R}_0$ and $\mathcal{R}_1$ are elementary states, and $\mathcal{R}_1 = \mathcal{R}_{1,0} \cup \mathcal{R}_{1,1}^*$, where $\mathcal{R}_{1,0} = \{[2, 4_{0,1}], [3_0, 4_{0,1}]\}$ and $\mathcal{R}_{1,1} = \{[2, 5_{0,0}], [3_0, 4_{0,0}], [4_{0,0}, 5_{0,0}]\}$. The homeomorphic embedding can be defined as follows:

$$\varphi([0, 5_{0,0}]) = [0, 5_0], \varphi([3_0, 4_{0,0}]) = [3, 4_0], \varphi([4_{0,0}, 5_{0,0}]) = [4_0, 5_0], \varphi([1, 3_1]) = [1, 4_1],$$
$$\varphi([2, 3_1]) = [2, 4_1], \varphi([2, 4_{0,1}]) = [2, 5_1], \varphi([3_0, 4_{0,1}]) = [3, 5_1].$$

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