MONOTONICITY OF SOME FUNCTIONS IN CALCULUS

G. D. ANDERSON, M. K. VAMANAMURTHY, AND M. VUORINEN

1. Introduction

In the first semester of calculus a student learns that if a function f is continuous on an interval $[a, b]$ and has a positive (negative) derivative on (a, b) then f is increasing (decreasing) on $[a, b]$. This result is obtained easily by means of the Lagrange Mean Value Theorem [Ap, Theorem 5.11, p. 110]. The functions that the student proves monotone in this way are usually polynomials, rational functions, or other elementary functions.

A refinement of the method of proving monotonicity a method that applies to a wide class of quotients of functions, even when the quotient itself is messy to differentiate—can be obtained as an application of the Cauchy Mean Value Theorem, also known as the Generalized Mean Value Theorem (cf. [Ap, p. 110], [Rog, p. 69], [St, p. 178]). Because of the similarity of the hypotheses to those of l'Hôpital's Rule, we refer to this result as the l'Hôpital Monotone Rule, or LMR for short. A proof of this result appears in [AVV2, Theorem 1.25] (see also [AVV1, Lemma 2.2]). The present paper was motivated by the recent interesting work of I. Pinelis (cf. [Pi1], [Pi2]). While some results in this paper are not new, our methods of proof via LMR (Theorem 1.1 below) are much simpler than the proofs appearing in the literature. For completeness we reproduce a proof of LMR here.

1.1. **Theorem (LMR).** *Let $-\infty < a < b < \infty$, and let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b). Let $g'(x) \neq 0$ on (a, b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b), then so are*

$$[f(x) - f(a)]/[g(x) - g(a)] \quad \text{and} \quad [f(x) - f(b)]/[g(x) - g(b)].$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Proof. We only prove the assertion for the first ratio $(f(x) - f(a))/(g(x) - g(a))$, since the other is similar. By the Intermediate Value Theorem for the derivative [Ap, Theorem 5.16, p. 112] we have either $g'(x) < 0$ for all x or $g'(x) > 0$ for all x. We consider only the former case, since the other is similar. Likewise, we consider only the case where $f'(x)/g'(x)$ is increasing, since the other is similar. Next, by the Cauchy Mean Value Theorem, for each $x \in (a,b)$ there exists some $y \in (a,x)$ such that

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(y)}{g'(y)} \le \frac{f'(x)}{g'(x)},$$

or, equivalently,

$$\frac{(g(x)-g(a))f'(x)-(f(x)-f(a))g'(x)}{(g(x)-g(a))g'(x)} \ge 0. \tag{1.2}$$

Now since $g'(x) < 0$ on (a,b), we have $g(x)-g(a) < 0$. Hence $g'(x)/(g(x)-g(a)) > 0$ on (a,b), and multiplying (1.2) by this positive quantity we get

$$\frac{d}{dx}\left[\frac{f(x)-f(a)}{g(x)-g(a)}\right] \ge 0$$

on (a,b). □

The next result extends LMR to the case of indeterminateness at ∞ (cf. [Pi1, Proposition 1.1]).

1.3. **Theorem.** *For $-\infty \le a < \infty$, let f and g be differentiable on (a,∞), and let $f(x)$ and $g(x)$ tend to finite limits L and M, respectively, as x tends to ∞. Let $g'(x)$ never vanish on (a,∞). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,∞), then so is $[f(x)-L]/[g(x)-M]$.*

Proof. By the Intermediate Value Theorem for derivatives [Ap, Theorem 5.16, p. 112] we have $g'(x) < 0$ or $g'(x) > 0$ for all $x \in (a,\infty)$. We assume the former, since the latter case is similar. Hence g is strictly decreasing on (a,∞). Next, we assume that $f'(x)/g'(x)$ is increasing, since the other case is similar. Fix $x \in (a,\infty)$, and let $y \in (x,\infty)$. Then, by the Cauchy Mean Value Theorem, there exists some $z \in (x,y)$ such that

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(z)}{g'(z)} \ge \frac{f'(x)}{g'(x)}.$$

That is,

$$\frac{(f(x)-f(y))g'(x)-(g(x)-g(y))f'(x)}{(g(x)-g(y))g'(x)} \ge 0.$$

Multiplying by the negative factor $g'(x)/(g(x)-g(y))$ and letting $y \to \infty$, we get $(d/dx)[(f(x)-L)/(g(x)-M)] \ge 0$, and the result follows. □

1.4. **Remark.** *Theorem 1.3 obviously has an analogous counterpart if ∞ is replaced by $-\infty$.*

We next consider some applications of Theorem 1.1. First, let f be the complex function defined by

$$f(z) = \begin{cases} z \coth z & \text{for } z \in \mathbb{C}\backslash\{in\pi, n \text{ an integer}, n \neq 0\}, \\ 1 & \text{if } z = 0. \end{cases} \tag{1.5}$$

This function occurs frequently in Real and Complex Analysis, Fourier Analysis, and Number Theory ([L],[Sp]). Beginning with the Bernoulli series [Sp, p. 171, Ex. 163]

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \text{ for } |z| < 2\pi,$$

one can show that

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(2z)^{2n}, \text{ for } |z| < \pi.$$

It is well known that $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, and $B_{2n+1} = 0$ for $n = 1, 2, 3, \ldots$ (cf. [AAR],[T]). In particular, it follows that $(f(z) - 1)/z^2$ tends to $2B_2 = 1/3$ as z tends to 0. By using the Mittag-Leffler Theorem [Sp, p. 192, Ex. 35] we obtain the partial fraction decomposition

$$\frac{f(z) - 1}{z^2} = 2\sum_{n=1}^{\infty} \frac{1}{z^2 + (n\pi)^2}, z \neq in\pi, \ n \text{ a nonzero integer.} \tag{1.6}$$

In particular, it follows that the function $(f(x) - 1)/x^2$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$, with range $(0, 1/3)$. We now use LMR to give elementary proofs of this and related results.

1.7. **Theorem.** *(1) The function* $F(x) \equiv (\coth x - 1/x)/\tanh x$ *for* $x \neq 0$ *and* $F(0) = 1/3$*, is decreasing on* $(-\infty, 0]$ *and increasing on* $[0, \infty)$*, with range* $[1/3, 1)$*. Moreover,* $F'(0) = 0$*.*

(2) The function $G(x) \equiv \log((\sinh x)/x)/\log(\cosh x)$ *for* $x \neq 0$ *and* $G(0) = 1/3$*, is decreasing on* $(-\infty, 0]$ *and increasing on* $[0, \infty)$*, with range* $[1/3, 1)$*. Moreover,* $G'(0) = 0$*.*

(3) The function $H(x) \equiv (f(x) - 1)/x^2 \equiv (\coth x - 1/x)/x$ *for* $x \neq 0$ *and* $H(0) = 1/3$*, is increasing on* $(-\infty, 0]$ *and decreasing on* $[0, \infty)$*, with range* $(0, 1/3]$*, where* f *is as in (1.5). Moreover,* $H'(0) = 0$*.*

Proof. Since all three functions F, G, and H are even, it is enough to establish the assertions on $[0, \infty)$. Since the limiting values are clear by l'Hôpital's Rule, we need only prove the monotonicity. Since each of the functions has the indeterminate form $0/0$ as x tends to 0, we can apply LMR in each case.

For (1), $F(x) = (x - \tanh x)/(x \tanh^2 x)$. The derivative ratio for this function may be simplified to $1/(1 + (4x/\sinh(2x)))$. By LMR, the ratio $4x/\sinh(2x)$ is clearly decreasing on $(0, \infty)$. Hence by LMR the monotonicity for F follows. For

$F'(0)$ we write $(F(x) - F(0))/x = (3x - 3\tanh x - x\tanh^2 x)/(3x^2\tanh^2 x)$. Since $(\tanh x)/x$ tends to 1 as $x \to 0$, by repeated application of l'Hôpital's Rule we get

$$\begin{aligned} F'(0) &= \lim_{x\to 0} \frac{3x - 3\tanh x - x\tanh^2 x}{3x^4} \\ &= \lim_{x\to 0} \frac{3 - 3\,\mathrm{sech}^2 x - \tanh^2 x - 2x\tanh x\,\mathrm{sech}^2 x}{12x^3} \\ &= \lim_{x\to 0} \frac{\tanh^2 x - x\tanh x\,\mathrm{sech}^2 x}{6x^3} = \lim_{x\to 0} \frac{\tanh x - x\,\mathrm{sech}^2 x}{6x^2} \\ &= \lim_{x\to 0} \frac{x\,\mathrm{sech}^2 x\tanh x}{6x} = \lim_{x\to 0} \frac{\mathrm{sech}^2 x\tanh x}{6} = 0. \end{aligned}$$

For (2), the derivative ratio for the function $G(x)$ is precisely $F(x)$. Hence, by LMR, the asserted monotonicity for G follows. Again, $G'(0) = 0$. The proof is similar to that of $F'(0) = 0$, hence omitted.

Finally for (3), $H(x) = (x\cosh x - \sinh x)/(x^2 \sinh x)$, so that the derivative ratio can be simplified to $1/(2 + x/\tanh x)$. Now by LMR, the ratio $x/\tanh x$ is clearly increasing, so that the monotonicity for H follows by LMR. That $H'(0) = 0$ follows easily from the Mittag-Leffler partial fraction decomposition (1.6). This can also be shown directly by repeated application of l'Hôpital's Rule, as in Part (1). We omit the details. □

1.8. **Remark.** *The assertion (2) in Theorem 1.7 improves the result in* [Mit, 3.6.9, p. 270].

2. Results for Trigonometric Functions

It is natural to consider a trigonometric analogue of Theorem 1.7. We have the following.

2.1. **Theorem.** *The functions defined by*

$$f(x) \equiv \frac{x\tan^2 x}{\tan x - x} \quad \textit{and} \quad g(x) \equiv \frac{\log(\sec x)}{\log(x\csc x)}, \text{ for } x \neq 0,$$

and $f(0) = g(0) = 3$, *are decreasing on* $(-\pi/2, 0]$ *and increasing on* $[0, \pi/2)$, *with range* $[3, \infty)$. *In particular,*

$$\cos x < \left(\frac{\sin x}{x}\right)^3$$

for all nonzero x *in* $(-\pi/2, \pi/2)$. *Moreover,* $f'(0) = g'(0) = 0$.

Proof. Since f and g are even functions, it is enough to establish the assertions on $(0, \pi/2)$. By LMR f is increasing if $1+[4x/\sin(2x)]$ is so, which is clear. Next, again by LMR, the result for $g(x)$ follows from that for $f(x)$. The limiting values at the end points and the values of derivatives at 0 follow easily by repeated application of l'Hôpital's Rule. We omit the details. □

In [Wi] J. Wilker posed the problem of proving that

$$(a) \qquad \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \quad 0 < x < \frac{\pi}{2}\,,$$

and of finding

$$(b) \qquad c \equiv \inf_{0<x<\pi/2} \frac{\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2}{x^3 \tan x}\,.$$

In [SJVA] Anglesio showed, by a long elementary proof, that the function in (a) is increasing on $(0, \pi/2)$ and that the value of c in (b) is $16/\pi^4$. A shorter proof, using properties of Bernoulli numbers, was given in [GQQL]. In [Pi2] I. Pinelis obtained these results by using l'Hôpital-type rules for monotonicity. We here give an alternative proof for Part (a) that uses LMR.

2.2. **Theorem.** *The function defined by*

$$f(x) \equiv \begin{cases} \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} & \text{if } x \neq 0, \\ 2 & \text{if } x = 0, \end{cases}$$

is decreasing on $(-\pi/2, 0]$ *and increasing on* $[0, \pi/2)$*, with range* $[2, \infty)$*. Moreover,* $f'(0) = 0$*.*

Proof. Since f is an even function, it is enough to establish the assertion on $[0, \pi/2)$. The limiting values at the end points and the value of the derivative at 0 follow easily by repeated application of l'Hôpital's Rule. We omit the details. Next, $f(x) = g(x)/x^2$, where $g(x) = \sin^2 x + x \tan x$. Now

$$g'(x) = 2 \sin x \cos x + x \sec^2 x + \tan x$$

and

$$\frac{1}{2} g''(x) = \cos^2 x - \sin^2 x + \sec^2 x + x \sec^2 x \tan x = F(x),$$

say. We need to show that F is increasing on $[0, \pi/2)$. Clearly,

$$\begin{aligned}F'(x) &= -4\sin x\cos x + 3\sec^2 x\tan x + 2x\sec^2 x\tan^2 x + x\sec^4 x\\ &= 3\sec^2 x\tan x - 3\sin x\cos x + x\sec^4 x - \sin x\cos x + 2x\sec^2 x\tan^2 x\\ &> 2x\sec^2 x\tan^2 x > 0,\end{aligned}$$

since $\tan x > x > \sin x$, and $\sec x > 1$ on $(0, \pi/2)$. □

2.3. **Remark.** *It can be shown that*

$$\frac{16}{\pi^4} < \frac{\left(\dfrac{\sin x}{x}\right)^2 + \dfrac{\tan x}{x} - 2}{x^3\tan x} < \frac{16}{90} = \frac{8}{45},$$

for all nonzero values of x in $(-\pi/2, \pi/2)$. The lower bound is the limit as x tends to $+\pi/2$ or $-\pi/2$, while the upper bound is the limit as x tends to 0. The lower bound assertion follows from the Wilker-Anglesio result ([SJVA], [Pi2]*). We only need to show that the upper bound is the limit as x tends to 0. Denote this limit by M. Since $(\tan x)/x$ tends to 1 as x tends to 0,*

$$M = \lim_{x\to 0}\frac{\sin^2 x + x\tan x - 2x^2}{x^6} = \lim_{x\to 0}\frac{1-\cos(2x) + 2x\tan x - 4x^2}{2x^6}.$$

Expanding by Taylor series about 0 and simplifying, we get $M = [2^6/6! + 4/15]/2 = 16/90 = 8/45$.

2.4. **Theorem.** *The functions*

$$f(x) \equiv \frac{2\sin x + \tan x}{x} \quad \textit{and} \quad g(x) \equiv \frac{2\sin^2 x + \tan^2 x}{x^2}, \text{for } x \neq 0,$$

$f(0) = g(0) = 3$, are decreasing on $(-\pi/2, 0]$ and increasing on $[0, \pi/2)$, with range $[3, \infty)$.

Proof. Since f and g are even functions, it is enough to establish the assertions on $(0, \pi/2)$. For f, by LMR, we need only prove that $2\cos x + \sec^2 x$ is increasing. The derivative of this expression is

$$2\sec^2 x\tan x - 2\sin x = 2(\sec^3 x - 1)\sin x > 0,$$

since $\sin x > 0$ and $\sec x > 1$.

For g, by LMR applied twice, we need only prove that $\sec^4 x + 2\sec^2 x\tan^2 x + 2\cos^2 x - 2\sin^2 x$ is increasing. The derivative of this function is

$$\begin{aligned}&8\sec^4 x\tan x - 8\sin x\cos x + 4\sec^2 x\tan^3 x\\ &\qquad = 8(\sec^5 x - \cos x)\sin x + 4\sec^2 x\tan^3 x\\ &\qquad > 4\sec^2 x\tan^3 x > 0,\end{aligned}$$

since $\sec x > 1 > \cos x$, and $\tan x > 0$. The limiting values at the end points and derivatives at 0 are obtained easily by repeated application of l'Hôpital's Rule. □

2.5. **Corollary.** *The function* $f(x) \equiv \tan x \sin^2 x - x^3$ *is increasing on* $(-\pi/2, \pi/2)$. *In particular,* $\cos x < \sin^3 x/x^3$, *for all nonzero* x *in* $[-\pi/2, \pi/2]$.

Proof. $f'(x) = \tan^2 x + 2\sin^2 x - 3x^2 > 0$, for all nonzero x in $(-\pi/2, \pi/2)$, by Theorem 2.4 . □

3. Results for Algebraic Functions

3.1. **Theorem.** *For* $a \in (1, \infty)$, *let* $f : [-1, \infty) \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} \dfrac{(1+x)^a - 1}{x} & \text{if } x \neq 0, \\ a & \text{if } x = 0. \end{cases}$$

Then f *is increasing on* $[-1, \infty)$. *In particular, we have the sharp inequalities*

(a) $(1+x)^a > 1 + ax$ *for all nonzero* x *in* $[-1, \infty)$, *and*
(b) $(1+x)^a < 1 + (2^a - 1)x$ *for all* $x \in (0, 1)$.

Proof. Since $f(x) = g(x)/x$, where $g(x) = (1+x)^a - 1$, and $g(0) = 0$, we can apply LMR. Now $g'(x) = a(1+x)^{a-1}$ and $g''(x) = a(a-1)(1+x)^{a-2}$, so that $g''(x) > 0$. Hence $g'(x)$ is increasing, and so $f(x)$ is also increasing by Theorem 1.1. □

Theorem 3.1 can be extended to higher powers as follows (cf. [Mit, Theorem 5, p. 35]).

3.2. **Theorem.** *Let* $a > 1$, *and let* $n = [a] =$ *the greatest integer* $\leq a$. *For each* $m = 0, 1, 2, \ldots, (n-1)$, *let* f *be defined on* $[-1, \infty)$ *by*

$$f(x) = \begin{cases} \dfrac{(1+x)^a - \sum_{k=0}^{m} C(a,k)x^k}{x^{m+1}} & \text{for } x \neq 0, \\ C(a, m+1) & \text{for } x = 0. \end{cases}$$

Here $C(a,k)$ *denotes the binomial coefficient* $a(a-1)(a-2)\ldots(a-k+1)/k!$, *for* $k = 0, 1, 2, \ldots$. *Then* f *is strictly increasing. In particular,* $f(x) > f(0)$ *for all* $x \in (0, \infty)$ *and* $f(x) < f(0)$ *for all* $x \in [-1, 0)$.

Proof. The proof is by LMR and induction on n. First, for $n = 1$, we have $m = 0$, and the result is precisely Theorem 3.1. Next, assume the result for all values $\leq n$ and prove it for $n + 1 = [a]$. Then $m \in \{0, 1, 2, \ldots, n\}$. If $m = 0$ the result again reduces to Theorem 3.1. Hence let $m \geq 1$, so that $m-1 \in \{0,1,2,\ldots,n-1\}$and $n=[a]-1=[a-1]$. Now

$$f(x)=\frac{(1+x)^a-\sum_{k=0}^{m} C(a,k)x^k}{x^{m+1}}$$

has the indeterminate form $0/0$ at $x=0$. The derivative ratio can be simplified to

$$\frac{a}{m+1}\ \frac{(1+x)^{a-1}-\sum_{k=0}^{m-1} C(a-1,k)x^k}{x^m},$$

which is increasing by the induction hypothesis. Further,

$$f(0)=\frac{a}{m+1}\ C(a-1,m)=C(a,m+1).$$

□

3.3. **Theorem.** *For $a \in (1,\infty)$ and $x \in (-1,\infty)$, we have*

$$(1+x)^a < 1+ax(1+x)^{a-1}. \tag{3.4}$$

Proof. First let $x>0$. Then by the Mean Value Theorem there exists $y \in (0,x)$ with

$$((1+x)^a-1)/x = a(1+y)^{a-1} < a(1+x)^{a-1}$$

So (3.4) follows, since $x>0$.

Next, let $x \in (-1,0)$. Then, again by the Mean Value Theorem, there exists $z \in (x,0)$, such that

$$\frac{(1+x)^a-1}{x}=a(1+z)^{a-1} > a(1+x)^{a-1},$$

so that again (3.4) follows, since $x<0$. □

3.5. **Theorem.** *(1) The function $f(x) \equiv (1+x)^{1/x}$ is decreasing and the function $g(x) \equiv (1+1/x)^x$ is increasing on $(0,\infty)$, with range $(1,e)$.*

(2) The function $F(x) \equiv (1+x)^{1+1/x}$ is increasing and $G(x) \equiv (1+1/x)^{1+x}$ is decreasing on $(0,\infty)$, with range (e,∞).

Proof. (1) Since $g(x)=f(1/x)$, we need only prove the assertion for $f(x)$. Now $\log f(x) = (\log(1+x))/x$, which has the indeterminate form $0/0$ at $x=0$. The ratio of derivatives is $1/(1+x)$, which is clearly decreasing on $(0,\infty)$, with limits 1 and 0 as x tends to 0 and ∞, respectively. Hence the assertion follows by LMR.

(2) Since $G(x)=F(1/x)$, we need only prove the assertion for $F(x)$. Now $\log F(x) = [(1+x)\log(1+x)]/x$, which has the indeterminate form $0/0$ at $x=0$. The ratio of derivatives is $1+\log(1+x)$, which is clearly increasing on $(0,\infty)$, with limits 1 and ∞, as x tends to 0 and ∞, respectively. Hence, again the assertion follows from LMR. □

3.6. **Theorem.** *For $p > q > 0$, $x > 0$, $x \neq 1$,*

$$\frac{x^p - 1}{p} > \frac{x^q - 1}{q}. \tag{3.7}$$

Proof. Let $f(x) = (x^p - 1)/(x^q - 1)$ for $x \neq 1$, and let $f(1) = p/q$. Then f has the indeterminate form $0/0$ at $x = 1$, and by L'Hôpital's Rule it has limit p/q as x tends to 1. The derivative ratio is $(p/q)x^{p-q}$, which is clearly an increasing function of x on $(0, \infty)$. Hence, by LMR, f is increasing on $(0, \infty)$. Thus $f(x) < f(1)$ on $(0, 1)$ and $f(x) > f(1)$ on $(1, \infty)$. This yields the required result. □

3.8. **Remark.** *Both sides of (3.7) are equal to* 0 *when* $x = 1$.

4. Counterexamples

In this section we give some counterexamples to show the limitations of Theorem 1.1 and Corollary 1.3. First, the converse of Theorem 1.1 is false. Next, the analogues of Theorem 1.1 and Corollary 1.3 for the infinite indeterminate form ∞/∞ are false.

Example 1. Let f and g be defined on $[0, 1)$ by $f(x) = \operatorname{arctanh}(x)$ and $g(x) = 1/(1 - x)$ and let $F(x) = f(x)/g(x)$. Then F has the infinite indeterminate form ∞/∞ at $x = 1$. However, F is NOT monotone on $(0, 1)$, since $F(0) = 0 = F(1-)$. On the other hand, $f'(x)/g'(x) = (1 - x)/(1 + x)$, which is monotone decreasing on $[0, 1)$.

Example 2. Let $f(x) = x^2 \sin(1/x)$, and let $g(x) = \sin x$. Then the usual l'Hôpital Rule does not apply [St, p. 180]. In fact, as $x \to 0$, $\lim f(x)/g(x) = 0$, while $\lim f'(x)/g'(x)$ does not exist.

Example 3. Let $f(x) = x \int_0^x (1 + \sin(1/t))dt$ and $g(x) = x$. Then $f(x)/g(x)$ is increasing on $(0, \infty)$, while $f'(x)/g'(x)$ is not monotone on $(0, \infty)$.

Example 4. Let $f(x) = 1/x - \log x$, $g(x) = 1/x$. Then $f(x)/g(x) = 1 - x \log x$ and $f'(x)/g'(x) = 1 + x$ are both increasing on $(0, 1/e)$. On the other hand, if $f(x) = 1/x$, $g(x) = \csc x$, then $f(x)/g(x) = (\sin x)/x$ is decreasing on $(0, \pi/2)$, and $f'(x)/g'(x) = (\sin^2 x)/(x^2 \cos x)$ is increasing on $(0, \pi/2)$. Note that in each case $f(x)/g(x)$ has the indeterminate form ∞/∞ at $x = 0$.

Example 5. Let $f(x) = \cosh x$, $g(x) = x$. Then $f(x)/g(x)$ has the indeterminate form ∞/∞ at $x = \infty$. Clearly $f'(x)/g'(x) = \sinh x$ is increasing on $(0, \infty)$, whereas

$f(x)/g(x)$ tends to ∞ as x tends either to 0 or to ∞, so that $f(x)/g(x)$ is not monotone on $(0, \infty)$.

References

[Ah] L. V. Ahlfors: *Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1978.

[AAR] G. E. Andrews, R. Askey, and R. Roy: *Special Functions*, Encyclopedia of Mathematics and its Aplications, 71. Cambridge U. Press, Cambridge, 1999.

[Ap] T. M. Apostol: *Mathematical Analysis: A Modern Approach to Advanced Calculus*, Second edition, Addison-Wesley Publ. Co., Reading, MA, 1975.

[AVV1] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen: *Inequalities for quasiconformal mappings in space*, Pacific J. Math. 160 (1993), 1–18.

[AVV2] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen: *Conformal Invariants, Inequalities, and Quasiconformal Maps*, J. Wiley, New York, 1997.

[CD] M. J. Cloud and B. C. Drachman: *Inequalities. With applications to engineering.* Springer-Verlag, New York, 1998.

[GQQL] B.-N. Guo, B.-M. Qiao, F. Qi, and W. Li: *On new proofs of Wilker's inequalities involving trigonometric functions*, Math. Ineq. Appl. 6 (2003), 19-22.

[HLP] G. H. Hardy, J. E. Littlewood, and G. Pólya: *Inequalities*, Cambridge Univ. Press, London, 1952.

[L] S. Lang: *Complex Analysis*, 4th ed., Springer-Verlag, New York, 1999.

[Mar] A. I. Markushevich: *Theory of Functions of a Complex Variable*, Vol. II, Prentice-Hall, Englewoord Cliffs, NJ, 1965.

[Mit] D. S. Mitrinović: *Analytic Inequalities*, Grundlehren Math. Wiss., Band 165, Springer-Verlag, Berlin, 1970.

[Pi1] I. Pinelis: *L'Hospital type rules for oscillation, with applications*, JIPAM. J. Inequal. Pure Appl. Math. 2 (2001), no. 3, Article 33, 24 pp. (electronic).

[Pi2] I. Pinelis: *L'Hospital Rules for Monotonicity and the Wilker-Anglesio Inequality*, Amer. Math. Monthly 111 (2004), 905–909.

[Rog] K. Rogers: *Advanced Calculus*, Charles E. Merrill, Columbus, OH, 1976.

[Sp] M. R. Spiegel: *Complex Variables*, Schaum, McGraw-Hill, New York, 1974.

[St] K. R. Stromberg: *Introduction to Classical Real Analysis*, Wadsworth, Belmont, CA, 1981.

[SJVA] J. S. Sumner, A. A. Jagers, M. Vowe, and J. Anglesio: *Inequalities involving trigonometric functions*, Amer. Math. Monthly 98 (1991), 264-267.

[T] N. M. Temme: *Special Functons. An Introduction to the Classical Functions of Mathematical Physics*, Wiley, New York, 1996.

[WW] E. T. Whittaker and G. N. Watson: *A Course of Modern Analysis*, 4th ed., Cambridge Univ. Press, London, l958.

[Wi] J. B. Wilker: Problem E3306, Amer. Math. Monthly 96 (1989), 55.

Glen D. Anderson and **M. K. Vamanamurthy** ("Vaman") both received their Ph.D. from the University of Michigan, writing their theses under the supervision of Frederick W. Gehring. Vamanamurthy came to Michigan from India on a Fulbright Scholarship. Anderson finished in 1965 and Vamanamurthy in 1969. Anderson retired as Professor of Mathematics in 2000 from Michigan State University, and

Vamanamurthy retired as Associate Professor of Mathematics from University of Auckland, New Zealand that same year. They have been collaborating in research since 1969. In 1985 they began working with **Matti Vuorinen**, who received his Ph.D. from University of Helsinki under the direction of Jussi Vaisala in 1977. Vuorinen is currently Professor of Mathematics at the University of Turku, Finland. His research and teaching interests include Classical Analysis and the use of computers in mathematics instruction. The three authors have written many papers and a book together. Each has been a guest at the universities of the other two several times.

Anderson: anderson@math.msu.edu
Vamanamurthy: vamanamu@math.auckland.ac.nz
Vuorinen: vuorinen@utu.fi