Dirichlet-to Neumann techniques for
the plazma-waves in a slot-diod

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Abstract

We suggest to calculate the amplitudes of the plasma waves in a slot-diod in a presence of few governing electrodes, via reduction of the linearized hydrodynamic equation to the second order differential equation with an operator weight, defined by the Dirichlet-to-Neumann map. In case of the straight slot this equation admitts further reduction to an integral equation with a trace-class integral operator. The eigenvalues of it are calculated via finite-dimensional approximation of the corresponding determinant.

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1 Introduction: geometry and the basic equations

The hydrodynamical analogy was suggested for plazma waves in [1, 2] and was intensely used for description of the plazma-current in a two-dimensional slot of comparative simple configuration, see for instance recent papers [3, 4]. In [4] the hydrodynamic electron transport model is used for description of plazma oscillations in gated 2D channel in high electron mobility transistor (HMET). Analysis of the plasma oscillations spectrum based on hydro-dynamic analogy is applicable also to other HMET-based teraherz devices, see [5, 6].

Mathematically the problem is reduced to calculation of the self-consistent electric potential $\varphi(x,z,t) = \varphi_0(x,y,z) + \int \varphi_\omega(x,y,z)e^{i\omega t}\, d\omega$ from the system of three basic equations (1,2,3). The three-dimensional Poisson equation

$$\triangle_3 \varphi = \frac{4\pi e}{\kappa}\, \Sigma\, \delta(z)\, \mathcal{X}_\Gamma \tag{1}$$

connects the potential with the non-zero concentration $\Sigma(x,y,z,t) = \Sigma(x,y,0,t) = \Sigma_0(x,y) + \int \Sigma_\omega(x,y)e^{-i\omega t}\, d\omega$ localized on a two-dimensional slot Γ situated on a smooth surface S between the electrodes $\Gamma_\pm \in S$. The function $\mathcal{X}_\Gamma$ is the indicator of the slot: $\mathcal{X}_\Gamma(x,y) = 1$, if $(x,y) \in \Gamma$, otherwise $\mathcal{X}_\Gamma(x,y) = 0$. The variables Σ, φ fulfil the continuity equation

$$\frac{\partial \Sigma}{\partial t} + \mathrm{div}_2 \Sigma u = 0,\ (x,y) \in \Gamma, \tag{2}$$

and the Euler equation on the slot:

$$\frac{\partial u}{\partial t} + \langle u, \nabla_2 \rangle u = \frac{e}{m}\nabla_2 \varphi - \nu u,\ (x,y) \in \Gamma. \tag{3}$$

These equations describe plazma waves on the slot. They connect the electron's concentration with the electron's velocity u in tangent direction $u(x,y) \in T_\Gamma(x,y)$ on the slot. We assume that several governing electrodes γ_s, $s = 1,2,\dots$ are present. The total number of them may be any, even countable infinite, but we assume that distances between the electrodes $\Gamma_\pm$, γ_s and the distances from the surface S are strictly positive. The complement $R^3 \setminus \left\{\Gamma_- \cup \Gamma \cup \Gamma_+ \cup \gamma_1 \cup \gamma_2 \dots\right\} := \Omega$ plays the role of the basic domain where the electric potential is defined. We assume that the geometry and the physical parameters of the "device" constituted by the details $\Gamma_\pm$, Γ, γ_1, γ_2, $\dots$ are chosen in such a way that the plazma current is observed only on the two-dimensional slot $\Gamma \subset S$ between the basic electrodes $\Gamma_{\pm}$. In simplest case considered previously in [3, 4] the role of the surface S is played by the horizontal plane $z = 0$, the slot is a straight channel $-L < x < L$, $-\infty < y < \infty$ and the running waves are spreading in the lateral direction y with amplitudes - standing waves - defined by the eigenfunctions $f_l(x)$ on the cross-section of the slot. In general case the structure of waves may be more sophisticated, but we assume that (x, y) are the coordinates on the slot $\Gamma \subset S$ and z is the normal coordinate. In that case each small open neighborhood of the slot $\Omega_\varepsilon \supset \Gamma$ is cut by the surface S into two parts: the upper part $\Omega_\varepsilon^+ = \{\Omega_\varepsilon \cap (z > 0)\}$ and the lower part $\Omega_\varepsilon^- = \{\Omega_\varepsilon \cap (z < 0)\}$. For the functions defined on Ω_ε^+ we can consider the upper and lower limits as $\lim_{z\to 0^{\pm}} f(x, y, z) = f_{\pm}(x, y, 0)$.

Assuming that the speed u only slightely deviates from the stationary speed $u_0(x, y)$ of the equilibrium process,

$$u(x, y, t) = u_0(x, y) + \int u_\omega(x, y)e^{-i\omega t} d\omega,$$

$\int |u_\omega(x, y)|d\omega << |u_0(x, y)|$, and similar conditions are fulfilled for the potential and the concentration on the slot, one can derive from the above basic equations (1,2,3) stationary equations for equilibrium values of the parameters Σ_0, φ_0, u_0:

$$\text{div}_2\Sigma_0 u_0 = 0,\ \langle u_0, \nabla_2\rangle u_0 = \frac{e}{m}\nabla_2\varphi_0 - \nu u_0,\ \triangle_3\varphi_0 = \frac{4\pi e}{\kappa}\Sigma_0\delta(z)\mathcal{X}_\Gamma. \tag{4}$$

We assume that this non-linear system of partial differential equations, with appropriate boundary conditions on the electrodes $\Gamma_{\pm}, \gamma_s$

$$\varphi_0(x, y, 0)\Big|_{(x,y)\in\Gamma_{\pm}} = V_{\pm},\ \varphi_0(x, y, 0)\Big|_{(x,y)\in\gamma_s} = V_s,\ s = 1, 2, \dots$$

is already solved, and consider the linear system for the amplitudes Σ_ω, u_ω, φ_ω of the first order correcting terms. Neglecting terms of higher order we may connect directly the amplitude $\Sigma_\omega(x, y)$ of the electron's concentration, with the amplitude u_ω, φ_ω of the velocity and one of the potential:

$$-i\omega\Sigma_\omega(x) + \text{div}_2\left[\Sigma_0(x, y)u_\omega + u_0(x, y)\Sigma_\omega\right] = 0,$$

$$\triangle_3\varphi_\omega = \frac{4\pi e}{\kappa}\ \Sigma_\omega\ \delta(z)\ \mathcal{X}_\Gamma$$

$$(\nu - i\omega)u_\omega = \frac{e}{m}\nabla_2\varphi_\omega. \tag{5}$$

The first of these equations can be interpreted based on physical meaning of the concentration: one should take into account that the concentation of electrons on the slot is originated by the supply of electrons from Γ_- and is spread on the slot due to the drift defined by the stationary speed u_0. Hence the corresponding first order differential equation should be supplied with boundary data for Σ_ω on the boundary of the electrode Γ_-, where the stationary speed u_0 looks into the outgoing direction, toward Γ_+. In that case the concentration is obtained inside the slot via integration on characteristics of the first equation. Note that this algorithm of calculation of the concentration is in agreement with the algorithm of the construction of solution of partial differential equations of second order with a small coefficient in front of the higher derivatives. This algorithm is suggested also in the mathematical paper [7]. In our case it can be obtained, if the complete Navier-Stokes equation with the small viscosity $\varepsilon \to 0$ is considered, instead of the Euler equation. In our case makes sense to set the boundary conditions for the amplitude as $\Sigma_\omega\big|_{\Gamma_-} = 0$. Then the system (5) has a unique solution for given ν, ω if the corresponding homogeneous system with zero boundary conditions has only trivial solution $u_\omega = \Sigma_\omega = \varphi_0 = 0$. Thus the question on solvability of the system is reduced to the corresponding spectral problem for the system (5) with zero boundary conditions on the electrodes. The first differential equation equation can be presented as

$$\text{div}\left(u_0\Sigma_\omega e^{-i\omega\int_{\Gamma_-}^{(x,y)}\frac{u_0}{|u_0|^2}ds} + \Sigma_0\frac{e}{m(\nu - i\omega)}\nabla_2\varphi_\omega\right) = 0,$$

where the integration $\int_{\Gamma_-}^{(x,y)} \frac{u_0}{|u_0|^2} ds$ in the exponent goes along the streamline of the stationary speed. The solution of the first equation is defined up to an auxilliary solenoidal field $(-f_y,\, f_x) := F$ on the slot, which should be defined from physical requirements. Then we obtain:

$$\Sigma_\omega = \left[\frac{\langle u_0,\, F\rangle}{|u_0|^2} - \Sigma_0 \frac{e\langle u_0,\, \nabla_2 \varphi_\omega\rangle}{m(\nu - i\omega)|u_0|^2} \right] e^{i\omega \int_{\Gamma_-}^{(x,y)} \frac{u_0}{|u_0|^2} ds}.$$

On the other hand, in the left side of the middle equation stays the Laplacian. When integrating on the short "vertical" interval $(-\delta,\, \delta)$ and taking the limit $\delta \to 0$ we obtain in the left side the jump of the normal defivative of the potential on the slot. This way the system of differential equatioons (5) is reduced to the equation containing the solenoidal field F:

$$\left[\frac{\partial \varphi_\omega}{\partial n}\right] = \frac{4\pi e}{\kappa} \Sigma_\omega = \left[\frac{\langle u_0,\, F\rangle}{|u_0|^2} - \Sigma_0 \frac{e\langle u_0,\, \nabla_2 \varphi_\omega\rangle}{m(\nu - i\omega)|u_0|^2} \right] e^{i\omega \int_{\Gamma_-}^{(x,y)} \frac{u_0}{|u_0|^2} ds} := \mathcal{L}\varphi_\omega,\ (x,y) \in \Gamma. \tag{6}$$

Note that in the left side of the equation stays the construction defined by the values $\Lambda_\pm$ of the Dirichlet-to-Neumann map (DN-map) of the Laplacian , see [15, 16], on the upper and lower shores of the slot, for instance $\Lambda_- : \varphi\big|_\Gamma \to \lim_{z=\to 0^-} \frac{\partial \varphi}{\partial z}\big|_\Gamma$. Then, assuming that tha potential is continuous on the slot, we obtain an equation for the potential φ_ω:

$$\left[\Lambda_+ - \Lambda_-\right] \varphi_\omega + \frac{4\pi e}{\kappa}\, \mathcal{L}\, \varphi_\omega = 0. \tag{7}$$

. In principle, if the Dirichlet-to-Neumann map is known, this equation may help to find critical values of the parameters for which the original boundary problem does not have a solution, or has a non-unique solution. Unfortunately the equation (7) is not a standard equation of mathematical Physics, since it contains the solenoidal field F, still to be defined. Nevertheless, one can choose the basic parameters and the geometry of the electrodes such that the above equation takes more convenient form. Now we discuss some reduced forms of the basic equation (7).

1.Assume first that the concentration Σ_ω is slowly varying along the streamlines of the stationary velocity u_0, $|\langle u_0 \nabla \Sigma_\omega\rangle| << \Sigma_\omega$. In that case the first differential equation for the concentration is reduced to the algebraic equation

$$-i\omega \Sigma_\omega(x) + \mathrm{div}_2\, \Sigma_0(x,y) u_\omega + \Sigma_\omega \mathrm{div}_2 u_0(x,y) =\approx 0,$$

which implies

$$\Sigma_\omega(x) \approx \frac{1}{i\omega - \mathrm{div}_2 u_0(x,y)}\ \mathrm{div}_2 \Sigma_0(x,y) u_\omega =$$
$$\frac{e}{m(i\omega - \mathrm{div}_2 u_0(x,y))(\nu - i\omega)}\, \Sigma_0(x,y) \nabla_2 \varphi_\omega. \tag{8}$$

This already implies a differential equation for the poteltial φ_ω.

2. We can also add similar assumption of the slow variation of the stationary speed: u_0 : $|\mathrm{div} u_0| << |\omega|$. Then

$$\Sigma_\omega(x) = \frac{e}{i\omega\, m\, (\nu - i\omega)} \Sigma_0(x,y) \nabla_2 \varphi_\omega.$$

This implies a differential equation for the potential with the spectral parameter $\frac{4\pi\, e^2}{i\omega\, m\, \kappa\, (\nu - i\omega)} := \frac{2}{q}$

$$\left[\Lambda_+ - \Lambda_-\right] \varphi_\omega = \frac{2}{q} \mathrm{div}_2 \Sigma_0(x,y) \nabla_2 \varphi_\omega. \tag{9}$$

Subject to the above assumptions, the derived equation is equivalent to the initial problem on the plazma current for slowly changing electron's velocity not only in case of the flat slot Γ and electrodes $\Gamma_\pm$, but also in general case when the slot and electrodes have arbitrary geometry. Nevertheless hereafter we explore the most important case of the flat geometry when S is a horizontal plane $z = 0$, but consider the case when the governing electrons are present.

In paper [21] the problem on plazma current on the flat two-dimensionalslot is considered in form of the equation for the potential:

$$\triangle_3 \varphi_\omega = \left(\frac{4\pi e^2 \Sigma_0}{m\kappa\omega(\omega - i\nu)}\right)\left(\frac{\partial \varphi_\omega^2}{\partial x^2} + \frac{\partial \varphi_\omega^2}{\partial y^2}\right) \delta z = \frac{2}{q} \Sigma_0 \triangle_2 \varphi_\omega \delta(z), \tag{10}$$

with $\Sigma_0 = const$, but the approach to solving the differs from our suggestion above: instead of using the DN- map in the left side of the equation, the authors of [21] use the inverse operator in the right side:

$$\varphi_\omega(x, y) = \frac{2}{q} \int_\Gamma G(x, y; \xi, \eta) \left(\frac{\partial \varphi_\omega^2}{\partial \xi^2} + \frac{\partial \varphi_\omega^2}{\partial \eta^2}\right) d\xi\, d\eta, \ (x, y) \in \Gamma,$$

assuming that $G(x, y; \xi, \eta)$ is the free Green function. In fact this suggestion gives a right answer in case of flat geometry with no governing electrodes because the restriction of the free Green function onto the slot coincides with the restriction onto the slot of the Green function of the Neumann problem. On the other hand, the map defined by the Neumann Green function on the boundary of the domain is the inverse of the Dirichlet-to-Neumann map for the domain, see for instance [16]. In [21], due to the symmetry we have on the slot, $\Lambda_+ - \Lambda_- = 2\Lambda_+$. Then

$$2\left[\Lambda_+ - \Lambda_-\right]^{-1} * = \Lambda_+^{-1} * = \int_\Gamma G^N(x, y|\xi, \eta) * d\xi\, d\eta = \int_\Gamma G(x, y|\xi, \eta) * d\xi\, d\eta$$

where G is the free Green-function. Though in that case the substitution of the Neumann Green function by the free Green function is possible, in general case, when the governing electrodes are present and/or Γ, Γ_+ are non-flat, either DN-map or the corresponding inverse operator must be used.

Note that the Green-function $G^D(x, y, z; \xi, \eta, \zeta)$ of the homogeneous Dirichlet problem for Laplacian in the 3-d space with electrodes/ additional electrodes present, is the main tool for solution of the problem on plazma current, because all important maps used in course of derivation/solution of the equations can be obtained from it. In particular, the kernel of the Poisson map $\mathcal{P}$ is obtained via differentiation of the Green function of Dirichlet problem in outward direction on the boundary, in our case:

$$\mathcal{P}(x, y, z; \xi, \eta, 0) = -\frac{\partial G^D}{\partial n_{\xi,\eta,0}}(x, y, z; \xi, \eta, 0)$$

where $(x, y, z) \in \Omega$, $(\xi, \eta, 0) \in \Gamma$. The generalized kernel of the DN-map can be presented as a formal integral operator on Γ with the generalized kernel:

$$\Lambda(x, y, 0; \xi, \eta, 0) = -\frac{\partial^2 G_0^D}{\partial n_{x,y,0}\, \partial n_{\xi,\eta,0}}(x, y, 0; \xi, \eta, 0),$$

where the outward normals on Γ with respect to the upper or lower neighborhoods Ω_ε^+ of the slot Γ are used respectively for $\Lambda_\pm$. On the other hand, construction of the Green function in a domain with few stanard exclusions like $\Gamma_\pm$, γ_1, γ_2, ... may be obtained via simple iteration process, see [10]. In particular this way the Dirichlet Green function may be constructed for the "device" assembled of a straight horizontal slot between basic horizontal electrodes and few governing electrodes in form of straight cylindrical rods suspended parallel to the horizontal plane as governing electrodes, see next section.

In second section of this paper we review the spectral properties of the simplest problem with an infinite straight slot and no governing electrodes. In the third section we consider the modified problem with several governing electrodes. In the forth section, assuming that the equilibrium concentration has bumps at some cross-sections of the slot caused by the governing electrodes γ_s, we reveal resonance phenomena in scattering of lateral waves depending on geometry and potentials on γ_s. This observation permits, in principle, to manipulate the transmission coefficients the lateral waves in the slot. In Appendix A basic features of the Dirichlet-to-Neumann map are reviewd. In Appendix B few cross-section eigenfunctions in the slot are calculated numerically. In Appendix C a convenient formula for Poisson map for the device with the flat geometry is derived.

In this paper we collected few mathematical facts and tools which may facilate mathematical modelling of the manipulated plazma channel. If these tools are prepared, we are able to calculate the admittance Y_ω based on the corresponding formula for the current, see, for instance [3, 4] concerning the case of simplest geometry with no additional electrodes and constant electron concentation Σ_0:

$$J_\omega = \frac{e^2\Sigma_0}{m(\nu - i\omega)} \int_{-l}^{l} g(x) \frac{\partial \varphi_\omega}{\partial x}\Big|_{z=0} dx - i\omega c V_\omega, \tag{11}$$

with proper form-factor, for instance $g(x) = \frac{1}{\pi\sqrt{l^2-x^2}}$, [17] and the dimensionless geometric capacitance κ

$$Y_\omega = i\omega \left[\frac{\kappa J_0(q_\omega l)}{4\pi \sin q_\omega l}\right].$$

we will dicuss the corresponding formulae in case of non-trivial geometry in forthcoming publications.

2 Basic spectral problem for simplest geometry

Consider the device constructed of flat basic electrodes $\Gamma_\pm$ and a slot $\Gamma : -l < y < l,\ -\infty < x < \infty$ situated on the horizontal plane $z = 0$. The corresponding spectral problem is reduced to the differential equation (10). To re-write it in form (9), we need the DN-map Λ_+ of the upper half-space $R_+ : z > 0$. The DN-map of the upper half-space $z > 0$ is a generalized integral operator with the distribution kernel:

$$\Lambda_+ (x, y; \xi, \eta) = -\frac{\partial}{\partial z}\mathcal{P}_+ = \frac{1}{4\pi^2}\int_\infty^\infty \int_\infty^\infty e^{ip(x-\xi)}\, e^{iq(y-\eta)}\, \sqrt{p^2+q^2}\, dp\, dq. \tag{12}$$

The Laplacian on the slot with zero boundary conditions at the electrodes $\Gamma_\pm$ has continuous spectrum with step-wise growing multiplicity $2m$ on the spectral bands $\left[\frac{\pi^2 m^2}{4l^2}\right]$, $m = 1, 2, 3, \ldots$ and eigenfunctions $\psi_{m,p}(y, x) = \frac{1}{\sqrt{2\pi\, l}}\, e^{i\,p\,x} \sin \frac{\pi\, m\,(y+l)}{2l}$, $m = 1, 2, \ldots$ which correspond to the values of the spectral parameter $\lambda = \frac{\pi^2 m^2}{4l^2} + p^2$,

$$-\triangle_\Gamma = \sum_{m=1}^\infty \int_{-\infty}^\infty dp \left[\frac{\pi^2 m^2}{4l^2} + p^2\right] \psi_{m\,p}(y, x)\rangle \langle \psi_{m\,p}(y, x).$$

We rewrite the equation (10) in form (9) as an infinite linear system $\mathcal{K}\phi = q^{-1}\phi$ with the generalized matrix kernel:

$$\int_{-\infty}^\infty dp \int_{-l}^l dy \int_{-\infty}^\infty dq \int_{-l}^l d\eta \frac{\sin\frac{m\pi(y+l)}{2l}}{\sqrt{\frac{\pi^2 m^2}{4l^2} + p^2}}\, e^{i\,(px+qy)}\, \frac{\sqrt{p^2+q^2}}{4\pi^2 l} e^{-i(p\xi+q\eta)}\, \frac{\sin\frac{n\pi(\eta+l)}{2l}}{\sqrt{\frac{\pi^2 n^2}{4l^2} + p^2}} := \mathcal{K}_{m,n}(x, \xi),$$

or, separating the Fourier transform $\mathcal{F}u(x) \to \tilde{u}(p)$

$$\mathcal{K}_{m,n}(x, \xi) =$$

$$\mathcal{F}^+ \int_{-\infty}^\infty dq \int_{-l}^l dy \int_{-l}^l d\eta \frac{\sin\frac{m\pi(y+l)}{2l}}{\sqrt{\frac{\pi^2 m^2}{4l^2} + p^2}}\, e^{i\,qy}\, \frac{\sqrt{p^2+q^2}}{2\pi l}\, e^{-iq\eta}\, \frac{\sin\frac{n\pi(\eta+l)}{2l}}{\sqrt{\frac{\pi^2 n^2}{4l^2} + p^2}} \mathcal{F} := \left\{\mathcal{F}^+\, \tilde{\mathcal{K}}(p)\, \mathcal{F}\right\}_{m,n}, \tag{13}$$

where $\tilde{\mathcal{K}}(p)$ is the multiplication operator by the infinite matrix $\tilde{\mathcal{K}}_{m,n}(p)$. We will find the eigenvalues and eigenvectors of the matrix $\mathcal{K}(p)$. Then the spectral modes φ of the equation (10) are found by inverse Fourier transform $\varphi_\omega = \mathcal{F}\phi$ from the eigen-functions of the equation

$$\tilde{\mathcal{K}}(p)\phi = \frac{2}{q}\, \Sigma_0 \phi. \tag{14}$$

We will show that the matrix-function $\mathcal{K}$ is compact for each p. Then denoting by $\kappa_1(p)$, $\kappa_2(p)$, $\kappa_3(p), \dots$ the eigenvalues of $\tilde{\mathcal{K}}(p)$ and by $\phi_1, \phi_2, \phi_3, \dots$ the corresponding normalized eigenvectors, we form the eigenmode corresponding to $\kappa_m(s)$ as $\mathcal{F}\delta(p-s)\ \phi_m = (2\pi)^{-1/2} e^{isx} \phi_m(s)$. Hence the spectrum of the multiplication operator $\tilde{\mathcal{K}}$ has a band-structure with thresholds $\max_r \kappa_m(r) = \kappa_m(r)$. It is more convenient to substitute now the exponential Fourier transform by the trigonometrical Fourier transform:

$$\delta(y-\eta) = \frac{1}{2\pi} \int_{\infty}^{\infty} e^{q(y-\eta)} dq = \frac{1}{2\pi} \int_{\infty}^{\infty} [\cos qy\ cosq\eta + \sin qy\ \sin q\eta]\, dq.$$

Then calculation of the matrix $\mathcal{K}$ is reduced to calculation of the elementary integrals obtained via the change of variable : $y+1 \to y$:

$$J_r^s(q) = \int_0^{2l} \sin q(y-l)\ \sin\ \frac{\pi r y}{2l}\, dy = \cos q\, l \int_0^{2l} \sin q\, y\ \sin\ \frac{\pi r y}{2l}\, dy - \sin q\, l \int_0^{2l} \cos q\, y\ \sin\ \frac{\pi r y}{2l}\, dy$$

and

$$J_r^c(q) = \int_0^{2l} \cos q(y-l)\ \sin\ \frac{\pi r y}{2}\, dy = \cos p\, l \int_0^{2l} \cos q\, y\ \sin\ \frac{\pi r y}{2l}\, dy + \sin q\, l \int_0^{2l} \sin q\, y\ \sin\ \frac{\pi r y}{2l}\, dy.$$

We have, with $y/l := \hat{y}$:

$$\int_0^{2l} \cos qy \sin\ \frac{\pi r y}{2l} dy = \frac{l}{2} \int_0^2 [\sin(ql + r\,\pi/2)\hat{y} - \sin(ql - r\,\pi/2)\hat{y}]\, d\hat{y} =$$
$$\frac{l}{2ql + r\,\pi} [-\cos(2q + r\,\pi) + 1] + \frac{l}{2ql - r\,\pi} [\cos(2q - r\,\pi) - 1] =$$
$$(-1)^r\ \cos 2ql\ \left(\frac{l}{2ql - r\,\pi} - \frac{1}{2ql + r\,\pi}\right) + \left(\frac{l}{2ql + r\,\pi} - \frac{l}{2ql - r\,\pi}\right) =$$
$$\left[(-1)^r\ \cos 2ql - 1\right] \frac{2\pi r l}{4q^2 l^2 - \pi^2\, r^2}. \tag{15}$$

Similarly we obtain

$$\int_0^{2l} \sin qy \sin\ \frac{\pi r y}{2l} dx = \frac{l}{2} \int_0^2 [\cos(ql - r\,\pi/2)\hat{y} - \cos(ql + r\,\pi/2)\hat{y}]\, d\hat{y} =$$
$$\left[\sin 2ql\, (-1)^r\right]\ \frac{2\pi r l}{4q^2 l^2 - \pi^2\, r^2}. \tag{16}$$

Substituting (15,16) into $J_r^c(p)$, $J_r^s(p)$ we see, that all terms J_r^s with odd r and all terms J_r^c with even r are equal to zero, and all remaining terms are equal to

$$J_{2m}^s = \frac{2\pi m}{q^2 l^2 - \pi^2 m^2} \sin q\, l, \quad J_{2m+1}^c = -\frac{\pi(2m+1)}{q^2 l^2 - \pi^2 (m+1/2)^2} \cos q\, l. \tag{17}$$

Then for the operator $K = K_\infty$ framed by projections onto $L_2(\gamma)$ we obtain the matrix elements:

$$\tilde{K}_{rt}(p) = \frac{1}{2\pi\, l} \int_{-\infty}^{\infty} J_r^s(q)\, \frac{\sqrt{p^2+q^2}}{\sqrt{\frac{\pi^2 r^2}{4l^2} + p^2}\sqrt{\frac{\pi^2 t^2}{4l^2} + p^2}}\, J_t^s(q) dq + \frac{1}{2\pi\, l} \int_{-\infty}^{\infty} \frac{\sqrt{p^2+q^2}}{\sqrt{\frac{\pi^2 r^2}{4l^2} + p^2}\sqrt{\frac{\pi^2 t^2}{4l^2} + p^2}}\, J_r^c(q)\, p\, J_t^c(q) dq =$$

$$\begin{cases} \dfrac{2\pi\, m\, n}{\sqrt{\frac{\pi^2 r^2}{4l^2}+p^2}\sqrt{\frac{\pi^2 t^2}{4l^2}+p^2}} \displaystyle\int_{-\infty}^{\infty} \dfrac{\sqrt{p^2+q^2}\sin^2 q\, l}{l\,(q^2\, l^2 - \pi^2 m^2)(q^2\, l^2 - \pi^2 n^2)} dq, & \text{if } r = 2m,\ t = 2n \\[2ex] \dfrac{4\pi(m+1/2)(n+1/2)}{\sqrt{\frac{\pi^2 r^2}{4l^2}+p^2}\sqrt{\frac{\pi^2 t^2}{4l^2}+p^2}} \displaystyle\int_{-\infty}^{\infty} \dfrac{\sqrt{p^2+q^2}\ \cos^2 q\, l}{l\,(q^2\, l^2 - \pi^2 (m+1/2)^2)(p^2 - \pi^2 (n+1/2)^2)} dq, & \text{if } r = 2m+1,\ t = 2n+1, \end{cases} \tag{18}$$

and 0 for complementary sets of indices. For the matrix $\tilde{\mathcal{K}}(0)$ we obtain, due to the definition (13)

$$\tilde{\mathcal{K}}_{r,t}(0) = \begin{cases} \frac{4}{\pi l} \int_0^\infty s \frac{\sin^2 s}{(s^2-\pi^2 m^2)(s^2-\pi^2 n^2)} d\, s, & \text{if } r = 2m,\ t = 2n \\ \frac{4}{\pi l} \int_0^\infty s \frac{\cos^2 s}{(s^2-\pi^2 (m+1/2)^2)(s^2-\pi^2 (n+1/2)^2)} d\, s, & \text{if } r = 2m+1,\ t = 2n+1, \end{cases} \tag{19}$$

One can see from (19) that the matrix $\mathcal{K}$ is a sum of two matrixes acting in invariant subspaces spanned by vectors with non-zero components with even and odd components only. Thus the problem of spectral analysis splits into two parts in corresponding subspaces on the slot $-l < \eta < l$:

$$\bigvee_m \sin\frac{m\pi\eta}{l} = E_{odd},\ \bigvee_m \cos\frac{(2m+1)\ \eta\pi}{2\, l} = E_{even}.$$

The subspace sanned E_{odd}, for odd $r = 2m+1$, is spanned by even functions on the slot, and the subspace E_{even}, for even $r = 2m$, is spanned by odd functions on the slot. The spectral analysis of $\mathcal{K}$ can be accomplished in these spaces separately.

Based on matrix representation (19) we can prove that the operator $\mathcal{K}$ belongs to Hilbert-Schmitd class, hence is has discrete spectrum, and its square has a finite trace, hence the infinite determinant can be approximated by determinants of finite cut-off matrices. We derive these facts from asymptotic behavior of elements of $\mathcal{K}_{rt}$ for large (r,t).

Theorem 2.1 *Elements of the matrix $\tilde{\mathcal{K}}(0)$ have the following asymptotic for large r,t:*

$$\frac{\pi}{4\, l}\ \tilde{\mathcal{K}}_{rt} = Const\ \frac{\ln r\, t^{-1}}{(r-t)(r+t)},\ r,t > 0. \tag{20}$$

Proof will be given for the part of the operator $\tilde{\mathcal{K}}(0)$ in the subspace of anti-symmetric modes, $r = 2m,\ t = 2n$. The asymptotic of elements of the part of $\mathcal{K}$ in the symmetric subspace $r = 2m+1,\ t = 2n+1$ is derived in a similar way.

We present the integrand in the first integral (19) the following way :

$$s\,\frac{\sin^2 s}{(s^2-\pi^2 m^2)(s^2-\pi^2 n^2)} = \frac{1}{\pi^2 (m^2-n^2)}\left[\frac{s}{s^2(s^2-\pi^2\, m^2)} - \frac{s}{s^2(s^2-\pi^2\, n^2)}\right]$$

Then the corresponding integral is presented as

$$\frac{2}{\pi^3 (m^2-n^2)}\left[\int_0^\infty \frac{s\, \sin^2\, s\ ds}{s^2(s^2-\pi^2\, m^2)} - \int_0^\infty \frac{s\, \sin^2\, s\ ds}{s^2(s^2-\pi^2\, n^2)}\right] := \frac{2}{\pi^3 (m^2-n^2)}\left[\mathcal{J}_m - \mathcal{J}_n\right]. \tag{21}$$

Each of integrals in the right side can be presented, due to Jordan lemma as an integral on the imaginary axis p, e.g.:

$$\mathcal{J}_m = \frac{1}{2}\int_0^{i\infty} \frac{1-e^{2is}}{s\,(s^2-\pi^2\, m^2)}\, ds = -\frac{1}{2}\int_0^\infty \frac{1-e^{-2t}}{t(t^2-\pi^2\, m^2)}\ dt$$

The last integral can be presented as a sum of two integrals $\int_0^A + \int_A^\infty := \mathcal{J}_m^A + \mathcal{J}_m^\infty$. The first of them is estimated for large m by $\mathrm{Const}\, m^{-2}$, and the second may be calculated explicity after neglecting the exponential for large A:

$$\mathcal{J}_m^\infty \approx \frac{1}{4}\ \ln\frac{A^2+\pi^2\, m^2}{A^2} \approx \frac{1}{2}\ln m. \tag{22}$$

Taking into account only the dominating term for large m we obtain, due to (21) the following asymptotic for the integral (19) for $m,n\to\infty$

$$\frac{\pi\, l}{4}\ \tilde{\mathcal{K}}_{2m,2n}(0) = \int_0^\infty s\,\frac{\sin^2 s}{(s^2-\pi^2 m^2)(s^2-\pi^2 n^2)} ds \approx \frac{\ln m/n}{\pi^3 (m^2-n^2)}. \tag{23}$$

End of the proof.
Corollary. The operator $\tilde{\mathcal{K}}(0)$ is from Hilbert-Schmidt class because the series $\sum_{rt} |\tilde{\mathcal{K}}_{rt}(0)|^2 = \mathrm{Trace}\tilde{\mathcal{K}}^+ \tilde{\mathcal{K}}$ is convergent. Convergence of this series, due to smoothness of the asymptotic (23), is equivalent to the convergence of the corresponding integral on the first quadrant outside the unit disc:

$$\frac{1}{\pi^6}\int_{m^2+n^2\geq 1} \frac{|\ln m/n|^2}{(m^2-n^2)^2}\, dm\, dn =$$

$$\frac{1}{\pi^6}\int_{\rho\geq 1} \frac{d\rho}{\rho^3}\int_{\theta=0}^{\pi/2} \frac{|\ln \tan\theta|^2}{|\cos 2\theta|^2}\, d\theta.$$

It is convergent because the integrand is a bounded continuous function of θ. Similar statement is true for $\tilde{\mathcal{K}}(p)$, $-\infty < p, \infty$ as well. This statement allows us to calculate the eigenvalues of the operator $\tilde{\mathcal{K}}(p)$ approximating $\tilde{\mathcal{K}}(p)$ by finite cut-off matrices, see the corresponding calculation for $\tilde{\mathcal{K}}(0)$ in Appendix B.

Summarising above results we conclude that in case of simplest geometry of the device with only two basic electrodes the spectrum of the problem (10) has band-structure with tresholds defined by maxima of the eivenvalues $\kappa(p)$ of the operator $\tilde{\mathcal{K}}(p)$. One can guess that these maximal values are acieved at $p = 0$, then the upper thresholds can be calculated from the data given in Appendix B.

3 The slot-device with governing electrodes

In this section we will show that the presence of the electrodes defines the resonance properties of the device. These properties can be used for selecting excitations spreading in the slot. The corresponding device with periodic array of governing electrodes can possess even more interesting spectral properties defined by the resonance band-gaps, see for instance a series of mathematical papers concerning spectral properties of periodic and a-periodic lattices caused by resonance "decoration" at the nodes, [22, 23, 24, 25, 26, 27].

In this section we consider the simplest device with two governing electrodes γ_1, γ_2, two basic electrdes Γ_1, Γ_2 and one plazma-channel Γ squeezed between Γ_1, Γ_2. It is convenient to begin with slightly more general Dirichlet problem with boundary data on the surface $\partial\gamma_0 := S \supset \left\{\Gamma \cup \Gamma_+ \cup \Gamma_- \cup \Gamma\right\}$ and on the governing electrodes γ_s, $s = 1, 2, \ldots$. To apply the general construction of the Dirichlet problem via series of iterations, suggested in [10], we need to have the Poisson maps in the complements Ω_0, Ω_1, Ω_2 of $S := \gamma_0$ and governing electrodes. In special case when γ_s, $s = 1, 2, \ldots$ are circular cylinders and S is horizontal the kernels of the corresponding Poisson maps are known, see [11]. For instance, the Poisson-kernel for the half-space $z > 0$ is:

$$\mathcal{P}_0(x, y, z; \xi, \eta, 0) = \frac{1}{4\pi^2}\int\int dp\, dq e^{-\sqrt{p^2+q^2}\,z} e^{ip(x-\xi)+iq(y-\eta)}, \tag{24}$$

and the Poisson-kernel of the complement $R_3\backslash\gamma_s$ of the cilinder γ_s radius ρ_s is

$$\mathcal{P}_s(\varphi, \rho, y; \theta, \eta) = \int_{-\infty}^{\infty} dq \sum_{k=-\infty}^{\infty} e^{ik(\varphi-\theta)} e^{iq(y-\eta)} \frac{\mathcal{H}^1_k(iq\rho)}{\mathcal{H}^1_k(iq\rho_s)}, \tag{25}$$

where $\mathcal{H}^1$ is the conventional Bessel function of the first kind. We assume that the slot with non-trivial plazma current on it is a domain $\Gamma \subset S$ between the basic electrodes $\Gamma_\pm$, with the voltages $V_\pm$ on them. Then the plazma - current will develop on Γ if the electric field on the slot is strong enough:

$$E_- < \frac{d_-}{d_\pm}\left[V_+ - V_-\right]. \tag{26}$$

Here E_- is the ioniozation thresholds (the electron's exit work) on Γ_-, $d_\pm$ is the distance between Γ_+, Γ_- and d_- the thickness of the layer of dimensional quantization near the edge of Γ_-. Physically the plazma-current can develop also between the governing electrodes and the basic electrode Γ_+. We assume that now it is not the case, because the potentials V_s lie between $V_\pm$ and the ioniozation thresholds E_s, E_- (the electron's exit work) on the govering elecrtodes and the negative electrode is large enough, compared with the voltage between the basic and govering elecrodes,

$$E_s > \frac{d_s}{d_{s,+}}\left[V_+ - V_s\right],\ E_- > \frac{d_-}{d_{-,s}}\left[V_s - V_-\right]. \tag{27}$$

Here $d_{s,+}$, $d_{-,s}$ are distances from γ_s to Γ_+ and from Γ_- to γ_s, and d_s is the thickness of the surface layer on γ_s.

We postpone discussion of the above physical limitations (26,27) for typical materials to the forthcoming publication, but will concentrate now on calculation of the amplitudes of oscillations of the characteristics values Σ_ω, u_ω, φ_ω of the plazma-current about the equilibrium values Σ_0, φ_0, u_0 of these variables, which are supposed already known.

The central problem met in mathematical design of the device for manipulating running waves is the plazma channel Γ, is the consytuction of the corresponding Green function for Laplacian on the basic domain $\Omega = R_3\setminus\left\{\Gamma_+ \cup \Gamma_- \cup \gamma_1 \cup \gamma_2 \cup \ldots\right\}$. This problem is equivalent to the construction of the corresponding Poisson map for boundary data supported by the shores of the slot and electrodes. We will use the above notations $\mathcal{P}_0$, $\mathcal{P}_s$, $s = 1, 2, \ldots$ for Poisson maps of the half-space $Z > 0$ and ones of the cylinders in the whole space. Then according to [10] the solution of the Dirichlet problem for Laplace equation

$$-\triangle u = 0$$

in $R_3\setminus \cup_{s\geq 0}\gamma_s := R_3\setminus\gamma$ with the boundary data $\mathbf{u}_\gamma(\zeta) = \{u_s\}$ on $\partial\gamma = \cup_{s\geq 0}\partial\gamma_s$, can be obtained via appropriate iteration process suggested in [10]. The normalization procedure we suggest below is based on the iteration process, but gives a compact formula for the Poisson map of the system.

Denote by $\mathcal{P}_s$ the Poisson maps in $R_3\setminus\omega_s$ and construct the solution of the Laplace equation in $R_3\setminus(\gamma_1 \cup \gamma_2)$ with data u_s on $\partial\gamma_s$ in form

$$u = \mathcal{P}_1\hat{u}_1 + \mathcal{P}_2\hat{u}_2. \tag{28}$$

Then we obtain the following linear system for "re-normalized" boundary values u_s

$$\hat{u}_1 + \mathcal{P}_{12}\hat{u}_2 = u_1 \tag{29}$$

$$\mathcal{P}_{21}\hat{u}_1 + \hat{u}_2 = u_2, \tag{30}$$

where $\mathcal{P}_{st} = \big|_{\partial\gamma_s}\mathcal{P}_t$ defines the restriction of $\mathcal{P}_t\hat{u}_t$ onto $\partial\gamma_s$. The operators $\mathcal{P}_{st}$ for $s \neq t$ are contracting in $C_{\partial\gamma_s} \times C_{\partial\gamma_t}$, due to maximum principle, hence the system (29) has unique solution

$$\hat{u}_1 = \frac{I}{I - \mathcal{P}_{12}\mathcal{P}_{21}}\left[u_1 - \mathcal{P}_{12}\,u_2\right] \tag{31}$$

$$\hat{u}_2 = \frac{I}{I - \mathcal{P}_{21}\mathcal{P}_{12}}\left[u_2 - \mathcal{P}_{21}\,u_1\right] \tag{32}$$

defined by the renorm-matrix corresponding to the joining $\gamma_{12} = \gamma_1 \cup \gamma_{-2}$ of the electrodes:

$$\begin{pmatrix} \frac{I}{I-\mathcal{P}_{12}\mathcal{P}_{21}} & -\frac{I}{I-\mathcal{P}_{12}\mathcal{P}_{21}}\mathcal{P}_{12} \\ -\frac{I}{I-\mathcal{P}_{12}\mathcal{P}_{21}}\mathcal{P}_{21} & \frac{I}{I-\mathcal{P}_{12}\mathcal{P}_{21}} \end{pmatrix} := \mathcal{R}_{\gamma_{12}}. \tag{33}$$

This matrix transforms the boundary data u_1, u_2 into re-normalized data $\hat{u}_1$, $\hat{u}_2$ which can be used for construction of the solution of the original boundary problem by the formula (28) based on partial Poisson maps $\mathcal{P}_1$, $\mathcal{P}_2$. Then the Poisson map $\mathcal{P}_{\gamma_1\cup\gamma_2}$ in the complement $R_3\setminus(\gamma_{12})$ of the electrodes is obtained as the matrix product row by column:

$$\{\mathcal{P}_1,\,\mathcal{P}_2\}\,\mathcal{R}_{\gamma_1\gamma_2} := \mathcal{P}_{(12)}. \tag{34}$$

The corresponding DN-map is obtained via differentiation with respect to the the outward normal:

$$\Lambda_{(12)} = \{\Lambda_1,\,\Lambda_2\}\,\mathcal{R}_{\gamma_1\gamma_2}. \tag{35}$$

Now including the electrode γ_0 into the scheme can be done by induction : first we construct corresponding renorm-matrix of restrictions $\mathcal{P}_{(12)0}$ and $\mathcal{P}_{0(12)}$ of the Poisson maps $\big|_{\gamma_{12}} \mathcal{P}_0$ and $\big|_{\gamma_0} \mathcal{P}_{12}$, respectively:

$$\mathcal{R}_{\gamma_0\, \gamma_{(12)}} = \begin{pmatrix} \frac{I}{I-\mathcal{P}_{0(12)}\mathcal{P}_{(21)0}} & -\frac{I}{I-\mathcal{P}_{0(12)}\mathcal{P}_{(21)0}}\mathcal{P}_{0(12)} \\ -\frac{I}{I-\mathcal{P}_{0(12)}\mathcal{P}_{(21)0}}\mathcal{P}_{0(21)} & \frac{I}{I-\mathcal{P}_{(12)0}\mathcal{P}_{0(21)}} \end{pmatrix},$$

then the corresponding Poisson map is obtained as the matrix product

$$\mathcal{P}_{(012)} = \{\mathcal{P}_0\,,\, \mathcal{P}_{12}\}\, \mathcal{R}_{\gamma_0\, \gamma_{(12)}}\,.$$

The corresponding Dirichlet-to-Neumann map is obtainend as:

$$\Lambda_{(012)} = \{\Lambda_0\,,\, \Lambda_{12}\}\, \mathcal{R}_{\gamma_0\, \gamma_{(12)}}\,. \tag{36}$$

Convenient approximate formulae are obtained via perlacement inverse operators by a finite sum of the corresponding Neumann series, for instance: $[I - \mathcal{P}_{12}\,\mathcal{P}_{21}]^{-1} = I + \mathcal{P}_{12}\,\mathcal{P}_{21} + \mathcal{P}_{12}\,\mathcal{P}_{21}\mathcal{P}_{12}\,\mathcal{P}_{21}\ldots$.

Summarize now the derivation of the equation (9) based on formulae obtained for the DN-map $\Lambda_{(0,1,2)}$.Assume that $\gamma_1 \cup \gamma_2 \in \Omega^+$, , and $R_3\backslash\Omega^+ := \Omega^-$, $\gamma_s \in \Omega^+$, $\gamma_s \cap \Omega^- = \emptyset$. Due to the translation symmetry of $\Omega^+ = \Omega_0 \backslash [\gamma_1 \cup \gamma_2]$, the kernel $\Lambda_{(0,1,2)}(x,y,z)$ is connected to the DN-map $\Lambda^{\perp}_{(0,1,2)}(x,0,z;q)$ of the Helmholtz equation $\triangle_2 u = q^2\, u$ on the orthogonal cross-section of $R_2 \cap \Omega^+$ by the formula

$$\Lambda^{+}_{(0,1,2)}(x,y,z;\xi,\zeta,\eta) = \int_{-\infty}^{\infty} \Lambda^{+}_{(0,1,2)}(x,0,z);(\xi,0,\zeta);q)e^{iq(y-\eta)}dq. \tag{37}$$

Here (x,y,z), $(\xi,\zeta,\eta) \in \partial\Omega^+$, $(x,0,z)$, $(\xi,0,\zeta) \in R_2 \cap \partial\Omega$. Similarly the DN-map $\Lambda^{-}_{(0,1,2)}$ is defined by fromula in Ω_{-} similar to (12).

Corsider the perturbed Laplacian on the slot

$$L_\Gamma = -\mathrm{div}_2 \Sigma_0(x,y)\nabla_2,$$

with zero boundary conditions on the border $y = \pm l$. The electron's concentration $\Sigma_0(x,y)$ on the slot is a function of two variables x, y which has, in case of two governing electrodes, the asymptotic Σ_0 at infinity, $x \to \pm\infty$. On the compact part of the slot $\Sigma(x,y)$ is defined by the configuration of the governing electrodes, and can be defined in course of solution of the auxilliary stationary problem (4). We will not solve this problem now, but we may expect that, under the above conditions on stationary potentials, the stationary concentration is suppressed on the slot near to the governing electrodes. We assume that it depends only on the variable x along the channel. Then the spectral problem for L_Γ admitts separation of variables

$$\mathcal{L}\Psi = -\frac{\partial}{\partial x}\Sigma(x)\,\frac{\partial\Psi}{\partial x} - \Sigma(x)\frac{\partial^2}{\partial y^2}\Psi = \lambda\Psi. \tag{38}$$

For positive rapidly stabilizing concentration $\Sigma_0(x) \to \Sigma_0$, , , $x \to \pm\infty$, the spectrum of the problem is pure continuous. It has band-structure with step-wise growing multiplicity:

$$\sigma(\mathcal{L}) = \cup_{r=1}^{\infty}\sigma_r,$$

with branches $\sigma_r = \left[\Sigma_0\, \frac{\pi^2 r^2}{4\,l^2},\ \infty\right)$. The corresponding scattered waves $\Psi(x,y) = \Psi_r(x,y,\lambda) = \frac{1}{\sqrt{l}}\sin\frac{\pi\, r\,(y+l)}{2l}\,\psi^+_r(x)$ fulfil (38), and the amplitude $\psi^+_r(x)$ of the scattered wave in the open channel, $\lambda > \Sigma_0\frac{\pi^2 r^2}{4\,l^2}$ is a bounded solution of the spectral problem in the channel

$$-\frac{d}{dx}\Sigma(x)\,\frac{d\psi_r(x)}{dx} + \Sigma(x)\frac{\pi^2 r^2}{4\,l^2}\,\psi_r(x) = \lambda\psi_r(x)$$

with appropriate asymptotics at infinity. For the plazma waves incoming from $+\infty$ of $x - axis$, in open channels $\lambda\,\Sigma_0^{-1} - \frac{\pi^2 r^2}{4l^2} > 0$

$$\overleftarrow{\psi}_r \approx e^{iQx} + \overrightarrow{S} e^{-iQx} \text{ when} x \to +\infty,$$

and

$$\overleftarrow{\psi}_r \approx \overleftarrow{S} e^{iQx} \text{ when} x \to -\infty.$$

Here $Q = \sqrt{\lambda\,\Sigma_0^{-1} - \frac{\pi^2 r^2}{4l^2}}$. For plazma-waves incoming from $-\infty$ the asymptotiks are

$$\overrightarrow{\psi}_r \approx e^{-iQx} + \overleftarrow{S} e^{iQx} \text{ when } x \to -\infty,$$

$$\overrightarrow{\psi}_r \approx \overrightarrow{S} e^{-iQx} \text{ when } x \to +\infty,$$

The system of all scattered waves $\overleftarrow{\psi}_r$, $\overrightarrow{\psi}_r$, $\sigma_0 \pi^2 r^2 (2\,l)^{-2} < \lambda < \infty$ is complete and orthogonal in each channel (for each r). The whole system of eigenfunctions

$$\overleftarrow{\Psi}_r = \frac{1}{\sqrt{l}} \sin \frac{\pi\, r\,(y+l)}{2l} \overleftarrow{\psi}_r, \; r = 1, 2, \ldots$$

$$\overrightarrow{\Psi}_r(x,\,\lambda) = \frac{1}{\sqrt{l}} \sin \frac{\pi\, r\,(y+l)}{2l} \overrightarrow{\psi}_r, \; r = 1, 2, \ldots \tag{39}$$

in all open channels, $r = 1, 2, \ldots$ is complete and orthogonal in the space $L_2(\Gamma)$ of all square-integrable functions on the slot. Then the Green function of $\mathcal{L}$ is presented in spectral form as

$$[\mathcal{L} - \mu I]^{-1} = \sum_r \int_0^\infty \frac{d\lambda}{\lambda - \mu} \left[\overrightarrow{\Psi}_r(x,\,\lambda)\rangle\langle\overrightarrow{\Psi}_r(\xi,\,\lambda) + \overleftarrow{\Psi}_r(x,\,\lambda)\rangle\langle\overleftarrow{\Psi}_r(\xi,\,\lambda) \right] \frac{1}{2\pi\, l\,\sqrt{\lambda}}\;.$$

We will use this formula for the regular point $\mu = 0$. It is convenient, following the previous section, to re-write the spectral problem (7) in form of equation similar to (14):

$$\mathcal{L}^{-1/2} \left[\Lambda_- - \Lambda_+\right] \mathcal{L}^{-1/2} u = \frac{2}{q} u. \tag{40}$$

If the operator $\Lambda_- - \Lambda_+$, reduced onto the slot, commutes with $\mathcal{L}$, then the operator in the left side of the equation (40) can be reduced to the multiplication by the 2×2 matrix. But verification of that condition requires deeper analysis of the Dirichlet-to-Neumann maps. We will do it in forthcoming paper.

4 Solvable model

.

Consider the special case when two cylindrical governing electrodes $\Gamma_\pm$ ares present in upper and lower half-spaces $\Omega_\pm$ respectrively. We assume that the electrodes are parallel to each other and to the horizontal plane $S: z = 0$, and skew-orthogonal to the slot Γ situated between the electrodes $\Gamma_\pm \subset S$ on the horizontal plane S. We will not calculate the electron's concentration Σ_0, but assume that it depends only on the variable x along the slot and is suppressed near the governing electrode:

$$\Sigma_0(x) = \begin{cases} \sigma_0, \text{ if } & -l < x < l, \\ \Sigma_0, \text{ if } & |x| > l, \end{cases} \tag{41}$$

We assume that Σ_0 coincides with euilibrium electron concentration on the slot without goiverning electrodes, and the concentration is suppressed near the slot: $0 < \sigma_0 << \Sigma_0$. The scattered waves of the spectral problem

$$-\frac{d}{dx}\Sigma_0(x)\frac{du}{dx} + \Sigma_0(x)\frac{d^2 u}{dy^2} = \lambda u \tag{42}$$

are found via separation of variables

$$-\frac{d}{dx}\Sigma_0(x)\frac{du}{dx}+\Sigma_0(x)\frac{\pi^2 r^2}{4l^2}=\lambda u \tag{43}$$

and matching of exponentials with trigonometric functions based on the boundary conditions at $x=\pm l$, for instance:

$$\frac{1}{\sigma_0}\frac{\partial u}{\partial x}\Big|_{l-0}=\frac{1}{\Sigma_0}\frac{\partial u}{\partial x}\Big|_{l+0}.$$

It is convenient to calculate the scattered waves $\overleftarrow{\psi}$ of the above partial problem () in the r-channel spanned by $\sin\frac{\pi r y}{2l}$ as a linear combination of the incoming scattered waves $\psi^r_D(x)=e^{iQ_0x}+S^r_D e^{-iQ_0x}$, $x>l$, $\psi^r_N(x)=e^{iQ_0x}+S^r_N e^{-iQ_0x}$, $x>l$ of the corresponding spectral problems on $(-\infty,\infty)$ with homogeneous Dirichlet or Neumann boundary condition at the origin, respectively:

$$\overleftarrow{\psi}_r(x)=\frac{1}{2}[\psi^r_D(x)+\psi^r_N(x)],$$

$$S^r_D=e^{2iQ_0 l}\,\frac{e^{iQ_0 l}-\mathcal{M}^r_D(\lambda)}{e^{iQ_0 l}+\mathcal{M}^r_D(\lambda)},\ S^r_N=e^{2iQ_0 l}\,\frac{e^{iQ^r_0 l}-\mathcal{M}^r_N(\lambda)}{e^{iQ^r_0 l}+\mathcal{M}^r_N(\lambda)}$$

with $Q^r_0=\sqrt{\lambda\,\Sigma_0^{-1}-\frac{\pi^2 r^2}{4l^2}}$ and

$$\mathcal{M}^r_D(\lambda)=Q^r_0\,\Sigma_0\,\sigma_0^{-1}\ \cot Q^r_0 l,\ \mathcal{M}_N(\lambda)=-Q^r_0\,\Sigma_0\,\sigma_0^{-1}\ \tan Q_0 l.$$

The resonance properties of the transmission and reflection coefficients of the scattered waves $\overleftarrow{\psi}_r(x)$

$$\overleftarrow{S}_r=\frac{1}{2}[S^r_N-S^r_D],\ \overrightarrow{S}_r=\frac{1}{2}[S^r_D+S^r_N]$$

are defined by simgularities of $S_{N,D}$, or by eigenvalues of some auxiliary spectral problem on the interval $[-l,l]$.

To derive a formula for the corresponding operator $\mathcal{K}$ we represent the generalized kernel of the formal integral operator in the left part of the equation (9) using the translation invariance of the system of electrodes:

$$\left[\Lambda_- -\Lambda_+\right](x,y,0;\xi,\eta,0)=\int_{-\infty}^{+\infty}e^{iq(y-\eta)}\left[\lambda_-(q)(x,\xi)-\lambda_+(q)(x,\xi)\right]\,dq.$$

The kernels $\lambda_\pm(p,q)$ as Fourier transforms of the kernels of the DN-maps $\Lambda_\pm$, which are calculated via iteration process as suggested in previous section. Multiplying the left side of (9) by $\mathcal{L}^{-1/2}$ from both sides, and using the notations introduced in section 3

$$\int_{-l}^{l}e^{iqy}\,\sin\frac{\pi\,r\,(y+l)}{2l}dy=\mathcal{J}^c_r(q)+i\mathcal{J}^s_r(q):=\mathbf{J}_r(q),$$

we obtain the operator $\mathcal{L}^{-1/2}\left[\Lambda_- -\Lambda_+\right]\mathcal{L}^{-1/2}:=\mathcal{K}$ in form of an infinite matrix integral operator with the generalized kernel:

$$\int_{-\infty}^{\infty}dq\int_0^\infty\int_0^\infty\frac{d\lambda\,d\mu}{\lambda\,\mu}\mathbf{J}_r(q)\begin{pmatrix}\langle\overleftarrow{\psi}_r\left[\lambda_-(q)-\lambda_+(q)\right]\overleftarrow{\psi}_t(\lambda), & \overleftarrow{\psi}_r\left[\lambda_-(q)-\lambda_+(q)\right]\overrightarrow{\psi}_t(\mu)\\ \overrightarrow{\psi}_r\left[\lambda_-(q)-\lambda_+(q)\right]\overleftarrow{\psi}_t(\lambda), & \overrightarrow{\psi}_r\left[\lambda_-(q)-\lambda_+(q)\right]\overrightarrow{\psi}_t(\mu)\end{pmatrix}\bar{\mathbf{J}}_r\,\mathcal{K}. \tag{44}$$

One can see from (44) that the operator $\mathcal{K}$ contains the scattering matrix of the spectral problem (38), which defines the resonance and transport properties of the plazma channel. These transport properties may be manipulated via varying the potential(s) on the governing electrodes.

Note that the solvable model suggested in this section is based on strong assumprion concerning the distribution of the concentration Σ_0 of electrons. The problem of calculation of the distribution requires solving an advanced non-linear problem (4). We hope to return to this problem in forthcoming publications.

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6 Appendix A: Dirichlet-to-Neumann map - basic facts

We describe here general features of the DN-map, see also [13, 15, 16], for Laplace Operator, or, more generally, for Schrödinger Operator (with real bounded measurable potential q) defined in the space $L_2(\Omega)$ of square - integrable functions by the differential expression

$$L_D v = - \triangle v,$$

or

$$L_D v = - \triangle v + q(x) v,$$

on the class of twice differentiable functions $- \triangle v \in L_2(\Omega)$ vanishing on the piecewise smooth boundary $\Gamma = \partial\Omega$ of the domain Ω. In this section $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are two-dimensional variables. (Recall that in the previous section we used another notations $(x_1, x_2) = (x, z)$). If the boundary of the domain has inner angles, in particular, if the domain is the complement of the interval $[-L - l, L + l]$, then we assume that functions from the domain are submitted to the additional Meixner boundary condition in form $\int_\Omega | \bigtriangledown v|^2 dx < \infty$. This condition guarantees uniqueness of solution of the non-homogeneous equation $L_D v - \lambda v = f \in L_2(\Omega)$ for complex values of the spectral parameter λ. Together with the operator $L := L_D$ we may consider the operator L_N defined by the same differential expression L with homogeneous Neumann conditions on the boundary

$$\left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0,$$

Both $L := L_D$ and L_N are *self-adjoint operators* in $L_2(\Omega)$. Corresponding resolvent kernels $G_{N,D}(x, y, \lambda)$ and the Poisson kernel

$$\mathcal{P}_\lambda(x, y) = -\frac{\partial G_D(x, y, \lambda)}{\partial n_y}, \; y \in \Gamma,$$

for regular values of the spectral parameter λ are locally smooth if $x \neq y$ and square integrable in Ω with boundary values $G_{N,D}(x, y, \lambda)$, $\mathcal{P}(x, y, \lambda)$ from proper Sobolev classes. Behavior of $G_N(x, y, \lambda)$ when both x, y at a smooth point of the boundary $\Gamma = \partial\Omega$ is described by the following asymptotic which may be derived from the integral equations of potential theory:

$$G_N(x, x_\Gamma, \lambda) = \frac{1}{\pi} \log \frac{1}{|x - x_\Gamma|} + \mathbf{Q}_\lambda + o(1). \tag{45}$$

Here the term $\mathbf{Q}_\lambda$ contains a spectral information, [18]. If the domain is compact, then the spectra $\sigma_{N,D}$ of operators $L_{N,D}$ are discrete and real. Solutions of classical boundary problems for operators $L_{N,D}$ may be represented for regular λ (from the complement of the spectrum) by the "re-normalized" simple layer potentials - for the Neumann problem

$$Lu = \lambda u, \; \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \rho,$$

$$u(x) = \int_{\partial\Omega} G_N(x, y, \lambda) \rho(y) d\Gamma, \tag{46}$$

and by the re-normalized double-layer potentials - for Dirichlet problem:

$$Lu = \lambda u, \; u\Big|_{\partial\Omega} = \hat{u}, \; u(x) = \int_{\partial\Omega} \mathcal{P}_D(x, y, \lambda) \hat{u}(y) d\Gamma. \tag{47}$$

Generally the DN-map is represented for regular points λ of the operator L_D as the derivative of the solution of the Dirichlet problem in the direction of the outer normal on the boundary of the domain Ω:

$$(\Lambda(\lambda)\,\hat{u}\,)(x_\Gamma) = \frac{\partial}{\partial n}\bigg|_{x=x_\Gamma} \int_{\partial\Omega} \mathcal{P}_D(x,\, y,\, \lambda)\hat{u}(y)d\Gamma. \tag{48}$$

The inverse map may be presented at the regular points of the operators $L_N^{in,out}$:

$$\begin{gathered}\left(Q^{in,out}(\lambda)\rho^{in,\,out}\right)(x_\Gamma) = \\ \pm \int_\Gamma G_N^{in,\,out}(x,\, y,\, \lambda)\rho^{in,\,out}(y)d\Gamma.\end{gathered} \tag{49}$$

The following simple statement, see [16], shows, that the singularities of the DN-map $\Lambda_{in}(\lambda)$ as an unbounded operator in $L_2(\Gamma)$ and the poles at the eigenvalues of the inner Dirichlet problems may be separated from each other:

Theorem 6.1 *Consider the Schrödinger operator $L = -\triangle + q(x)$ in $L_2(\Omega)$ with real measurable essentially bounded potential q and homogeneous Dirichlet boundary condition at the C_2-smooth boundary Γ of Ω. Then the DN-map Λ_{in}^C of L has the following representation on the complement of the corresponding spectrum Σ_L in complex plane λ, $M > 0$:*

$$\Lambda_{in}(\lambda) = \Lambda_{in}(-M) - (\lambda + M)\mathcal{P}_{-M}^{+}\mathcal{P}_{-M} - (\lambda + M)^2\mathcal{P}_{-M}^{+}R_\lambda\mathcal{P}_{-M}, \tag{50}$$

where R_λ is the resolvent of L, and $\mathcal{P}_{-M}$ is the Poisson kernel of it. The operator $\mathcal{P}_{-M}^{+}\mathcal{P}_{-M}(x_\Gamma, y_\Gamma)$ is bounded in Sobolev class of boundary values of twice differentiable functions $\{u : \triangle u \in L_2(\Omega)\}$ and the operator

$$\begin{gathered}\left(\mathcal{P}_{-M}^{+}R_\lambda\mathcal{P}_{-M}\right)(x_\Gamma, y_\Gamma) = \\ \sum_{\lambda_s\in\Sigma_L} \frac{\frac{\partial\varphi_s}{\partial n}(x_\Gamma)\frac{\partial\varphi_s}{\partial n}(y_\Gamma)}{(\lambda_s + M)^2(\lambda_s - \lambda)}\end{gathered}$$

is compact in $W_2^{3/2}(\Gamma)$.

Similar statement is true for DN-map in exterior domain,

$$\begin{gathered}\Lambda_{out}(\lambda) = \Lambda_{out}(-M) + (\lambda + M)\mathcal{P}_{-M}^{+}\mathcal{P}_{-M} + \\ (\lambda + M)^2\mathcal{P}_{-M}^{+}R_\lambda\mathcal{P}_{-M},\end{gathered} \tag{51}$$

with only difference that first terms of the decomposition contain the DN-map and Poisson kernel for the exterior domain and the last term may contain both the sum over discrete spectrum and the the integral over the absolutely continuous spectrum $\sigma_L^a = [0, \infty)$ of L, with the integrand combined of the normal derivatives of the corresponding scattered waves $\psi(x, |k|, \nu)$, $k = |k|\nu$, $|\nu| = 1$:

$$\mathcal{P}_{-M}^{+}R_\lambda\mathcal{P}_{-M}(x_\Gamma,\, y_\Gamma) = \frac{1}{(2\pi)^3}\int_{|k|^2\in\Sigma_L^a} \frac{\frac{\partial\psi}{\partial n}(x_\Gamma,\, |k|, \nu)\frac{\partial\bar{\psi}_s}{\partial n}(y_\Gamma\, |k|, \nu)}{(|k|^2 + M)^2(|k|^2 - \lambda)} d^3k.$$

Example 1 Consider the non-compact domain Ω_a on the complex plane $z = x_1 + ix_2$ obtained via removing of the interval $\Gamma = [-a < x_1 < a]$ from the real axis $x_2 = 0$. The Green function of the Laplace Operator with zero boundary condition on both sides of Γ can be calculated explicitly via conformal map

$$\frac{z}{a} = \frac{1}{2}\left[u + \frac{1}{u}\right] \tag{52}$$

of the exterior $\hat{D}_1$ of the unit disk $D_1 = \{|u| < 1\}$ onto Ω_1. The inverse function is defined as

$$u = \frac{z}{a} + \sqrt{\left(\frac{z}{a}\right)^2 - 1}, \tag{53}$$

with the branch of the square root defined by the asymptotic condition $\sqrt{\left(\frac{z}{a}\right)^2 - 1} \approx \frac{z}{a}$ if $z \to \infty$.

The Green function $G(u,v)$ of the Laplacian with zero boundary condition in $\hat{D}_1$ is obtained as the real part of the function mapping the domain Ω_1 onto the exterior of the unit disk with the point v transferred to infinity :

$$G(z,s) = \frac{1}{2\pi} \ln \left| \frac{1 - u\bar{v}}{u - v} \right|,$$

with $u = \frac{z}{a} + \sqrt{\left(\frac{z}{a}\right)^2 - 1}$, $v = \frac{s}{a} + \sqrt{\left(\frac{s}{a}\right)^2 - 1}$. Then, according to the previous formula (48) we obtain the generalized kernel of the DN-map in Ω_1 as an operator in proper Sobolev class on the boundary $\Gamma = \Gamma_+ \cup \Gamma_-$:

$$\Lambda_L(z,s)\Big|_{z,s\in\Gamma} = -\frac{\partial^2 G(z,s)}{\partial n_z \, \partial n_s},$$

where n_z, n_s - the outer normals on Γ, that is: the normal "up" on the upper shores and the normal "down" on the lower shores of Γ.

Consider now the jump of the derivative $\frac{\partial}{\partial x_2}$ over the slot $(-l,l)$, assuming that $a = l + L > l$. Denoting the points on the opposite shore of the slot by $s_1 \pm i0 := s_1^{\pm}$, we can calculate the jump of the derivative of the electric field over the slot as

$$\frac{\partial \varphi}{\partial x_2}\Big|_{s_2^+} - \frac{\partial \varphi}{\partial x_2}\Big|_{s_2^-} = -\left[\Lambda_+ - \Lambda_-\right] \varphi =$$

$$\int_{-l}^{l} \left[\frac{\partial^2 G(z,s)}{\partial n_{x_2} \, \partial n_{s_2}}\Big|_{s_2^+} - \frac{\partial^2 G(z,s)}{\partial n_{x_2} \, \partial n_{s_2}}\Big|_{s_2^-} \right] \varphi(s_2) ds_2 := K_L \varphi. \tag{54}$$

More simple example is the DN-map for the upper or lower half-plane.

Example 2 Consider the case when $a = \infty$. In this the Poisson maps $\mathcal{P}_{\pm}$ for the upper and lower half-planes, $z = x_1 + ix_2$, $x_2 > 0$, $x_2 < 0$ are equal to :

$$\mathcal{P}_{\pm} \varphi(x_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ip(x_1 - s_1)} \, e^{\mp |p| x_2} \varphi(s_1, 0) ds_1 dp$$

and hence the DN-maps in upper and lower half-planes are given by the formulae :

$$\Lambda_{\pm} \varphi(x_1) = \mp \frac{\partial \varphi}{\partial x_2} = \frac{\pm 1}{2\pi} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} ds_1 \, e^{ip(x_1 - s_1)} \, |p| \varphi(x, 0),$$

hence they are pseudo-differential operators with the symbols $\pm|p|$ respectively. The corresponding jump of the derivatives is calculated as a positive pseudo-differential operator degree 1. On functions φ vanishing on real axis outside the slot $(-l,l)$ the integration on spacial variable is reduced to the slot, hence

$$-\Lambda_+ \varphi(x_1) + \Lambda_- \varphi(x_1) = \left(\frac{\partial \varphi}{\partial x_2}\Big|_{0^+} - \frac{\partial \varphi}{\partial x_2}\Big|_{0^-} \right)\Big|_{(-l,l)} = K_\infty \varphi(x_1) =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dp \int_{-l}^{l} ds_1 \, e^{ip(x_1 - s_1)} \, 2|p| \, \varphi(s_1, 0). \tag{55}$$

Both operators K_L, K_∞ are positive.

7 Appendix B: cross-section eigenfunctions in the straignt horizontal slot

In the paper [3] the cros-section eigenfunctions in the slot are found just from the ordinary differential equation obtained via replacement the non-trivial left side Λ in the equation (9) by the constant. Then the eigenfunctions are found in explicit form of trigonometric functions. In this paper we developed DN-mashinery to construct realistic equation for the non-equilibrium part of the quantum current and were able to prove, see section 2, that the spectral problem for cross-section component in the slot is reduced to spectral analysis of a Hilbert-Schmidt operator. Nevertheless, it appeared that the eigenfunctions of that operator look very much the same as the eigenfunctions of the corresponding differential equation in [3].

First 10 eigenvalues of the even series of the operator K are

$$0.02452621365, 0.02729557291, 0.03073000783, 0.03506244362, 0.04081584109,$$

$$0.04880536434, 0.06069115269, 0.08024242096, 0.1184600786, 0.2274134275.$$

Here are first 10 eigenfunctions of the even series of the operator K, approximated by 10×10 matrix:

$$\begin{aligned}\mathrm{fi1}(x) := &-.9933100453 \sin((x+1)\pi) + .1013682884 \sin(2(x+1)\pi)\\ &+ .04258288175 \sin(3(x+1)\pi) + .02489271434 \sin(4(x+1)\pi)\\ &+ .01670171207 \sin(5(x+1)\pi) + .01212120481 \sin(6(x+1)\pi)\\ &+ .009267077948 \sin(7(x+1)\pi) + .007347750075 \sin(8(x+1)\pi)\\ &+ .005988729822 \sin(9(x+1)\pi) + .005006906427 \sin(10(x+1)\pi)\end{aligned}$$

$$\begin{aligned}\mathrm{fi2}(x) := &\ .09322284072 \sin((x+1)\pi) + .9863898828 \sin(2(x+1)\pi)\\ &- .1167594867 \sin(3(x+1)\pi) - .05143930063 \sin(4(x+1)\pi)\\ &- .03139491641 \sin(5(x+1)\pi) - .02181172171 \sin(6(x+1)\pi)\\ &- .01629985716 \sin(7(x+1)\pi) - .01275337857 \sin(8(x+1)\pi)\\ &- .01017908494 \sin(9(x+1)\pi) - .008507649126 \sin(10(x+1)\pi)\end{aligned}$$

$$\begin{aligned}\mathrm{fi3}(x) := &\ .04756215312 \sin((x+1)\pi) + .1019542428 \sin(2(x+1)\pi)\\ &+ .9830180170 \sin(3(x+1)\pi) - .1241699449 \sin(4(x+1)\pi)\\ &- .05572263389 \sin(5(x+1)\pi) - .03468574022 \sin(6(x+1)\pi)\\ &- .02457732792 \sin(7(x+1)\pi) - .01865238936 \sin(8(x+1)\pi)\\ &- .01472675112 \sin(9(x+1)\pi) - .01111892307 \sin(10(x+1)\pi)\end{aligned}$$

$$\begin{aligned}\mathrm{fi4}(x) := &\ .03099781436 \sin((x+1)\pi) + .05402325905 \sin(2(x+1)\pi)\\ &+ .1055198724 \sin(3(x+1)\pi) + .9809057605 \sin(4(x+1)\pi)\\ &- .1292962600 \sin(5(x+1)\pi) - .05853282214 \sin(6(x+1)\pi)\\ &- .03692961614 \sin(7(x+1)\pi) - .02642910664 \sin(8(x+1)\pi)\\ &- .02013315988 \sin(9(x+1)\pi) - .01410025753 \sin(10(x+1)\pi)\end{aligned}$$

$$\begin{aligned}\mathrm{fi5}(x) := &-.02257248428 \sin((x+1)\pi) - .03646170107 \sin(2(x+1)\pi)\\ &- .05667955653 \sin(3(x+1)\pi) - .1081023576 \sin(4(x+1)\pi)\\ &- .9793085229 \sin(5(x+1)\pi) + .1335081775 \sin(6(x+1)\pi)\\ &+ .06094414938 \sin(7(x+1)\pi) + .03867020766 \sin(8(x+1)\pi)\\ &+ .02761120408 \sin(9(x+1)\pi) + .02051259443 \sin(10(x+1)\pi)\end{aligned}$$

$$\begin{aligned}\mathrm{fi6}(x) := & -.01754266704\sin((x+1)\pi) - .02732531298\sin(2(x+1)\pi)\\ & - .03883467766\sin(3(x+1)\pi) - .05845804051\sin(4(x+1)\pi)\\ & - .1102840715\sin(5(x+1)\pi) - .9778220785\sin(6(x+1)\pi)\\ & + .1380082005\sin(7(x+1)\pi) + .06304962682\sin(8(x+1)\pi)\\ & + .03953768277\sin(9(x+1)\pi) + .03371598339\sin(10(x+1)\pi)\end{aligned}$$

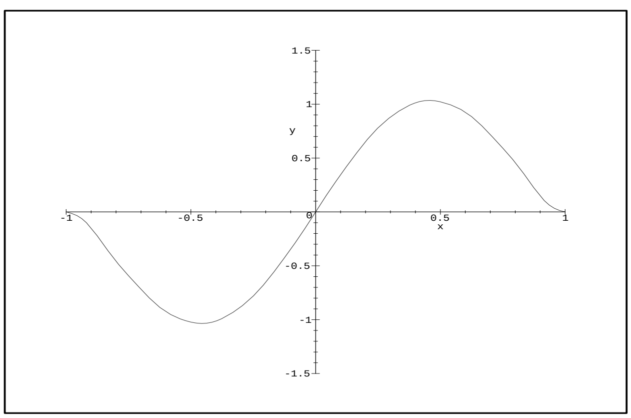
$$\begin{aligned}\mathrm{fi7}(x) := & -.01439693694\sin((x+1)\pi) - .02198938350\sin(2(x+1)\pi)\\ & - .02985722341\sin(3(x+1)\pi) - .04082197296\sin(4(x+1)\pi)\\ & - .06046139201\sin(5(x+1)\pi) - .1132102138\sin(6(x+1)\pi)\\ & - .9765507050\sin(7(x+1)\pi) + .1444466843\sin(8(x+1)\pi)\\ & + .06463188344\sin(9(x+1)\pi) + .03982173960\sin(10(x+1)\pi)\end{aligned}$$

$$\begin{aligned}\mathrm{fi8}(x) := & .01238501069\sin((x+1)\pi) + .01870817889\sin(2(x+1)\pi)\\ & + .02474802277\sin(3(x+1)\pi) + .03218252719\sin(4(x+1)\pi)\\ & + .04306956519\sin(5(x+1)\pi) + .06304196537\sin(6(x+1)\pi)\\ & + .1192713277\sin(7(x+1)\pi) + .9757371008\sin(8(x+1)\pi)\\ & - .1459126166\sin(9(x+1)\pi) - .06663427723\sin(10(x+1)\pi)\end{aligned}$$

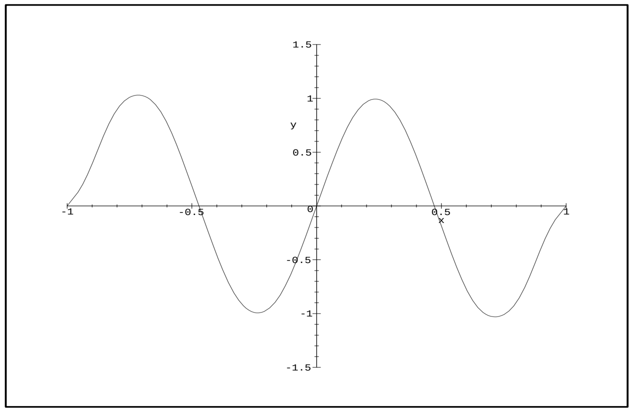
$$\begin{aligned}\mathrm{fi9}(x) := & -.01037295393\sin((x+1)\pi) - .01544717972\sin(2(x+1)\pi)\\ & - .02036106172\sin(3(x+1)\pi) - .02573784222\sin(4(x+1)\pi)\\ & - .03227953659\sin(5(x+1)\pi) - .04174318132\sin(6(x+1)\pi)\\ & - .06362375599\sin(7(x+1)\pi) - .1165340202\sin(8(x+1)\pi)\\ & - .9665573785\sin(9(x+1)\pi) + .2095972077\sin(10(x+1)\pi)\end{aligned}$$

$$\begin{aligned}\mathrm{fi10}(x) := & .01166663095\sin((x+1)\pi) + .01726310214\sin(2(x+1)\pi)\\ & + .02135086391\sin(3(x+1)\pi) + .02592332573\sin(4(x+1)\pi)\\ & + .03394970562\sin(5(x+1)\pi) + .04747997701\sin(6(x+1)\pi)\\ & + .05472869579\sin(7(x+1)\pi) + .08220948194\sin(8(x+1)\pi)\\ & + .1929034873\sin(9(x+1)\pi) + .9736866450\sin(10(x+1)\pi)\end{aligned}$$

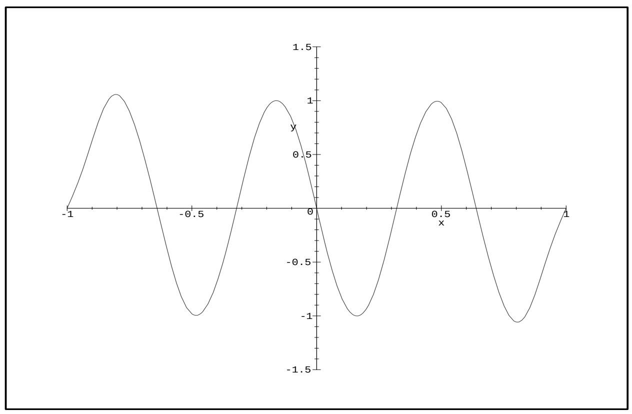
```
> plot(fi1(x),x=-1..1,y=-1.5..1.5);
```



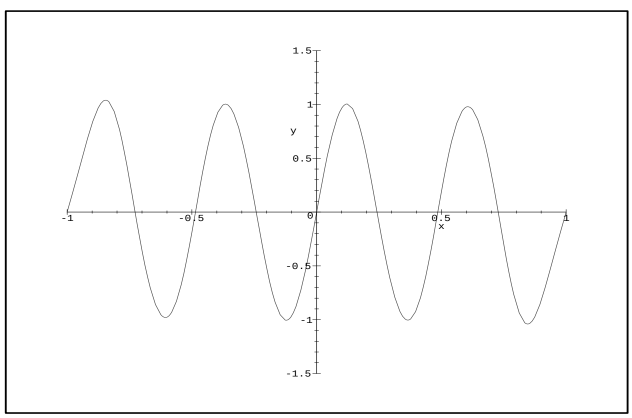

```
> plot(fi2(x),x=-1..1,y=-1.5..1.5);
```



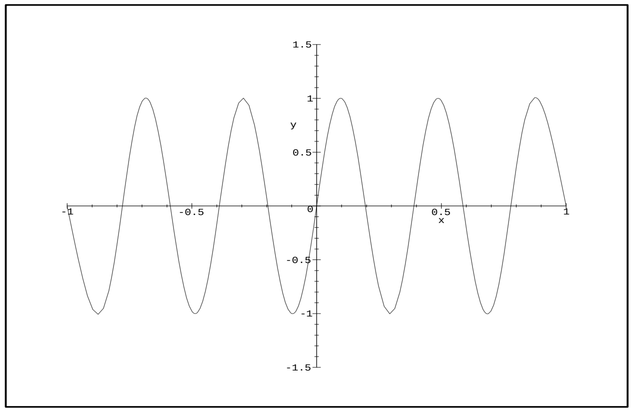

```
> plot(fi3(x),x=-1..1,y=-1.5..1.5);
```



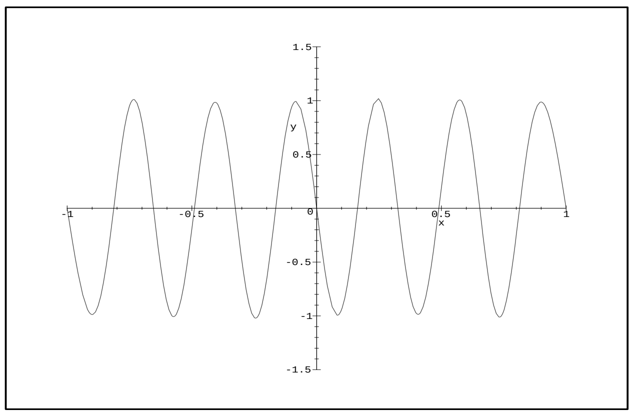

```
> plot(fi4(x),x=-1..1,y=-1.5..1.5);
```



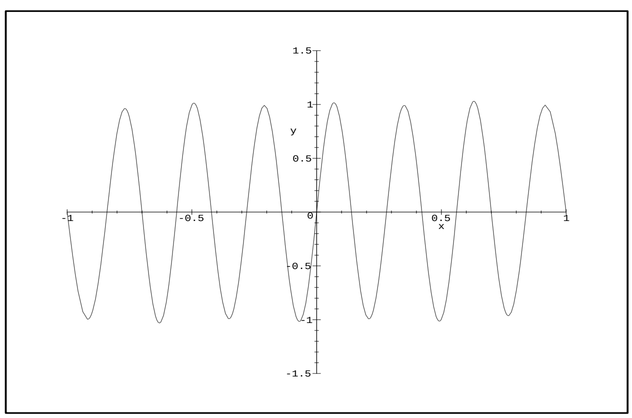

```
> plot(fi5(x),x=-1..1,y=-1.5..1.5);
```



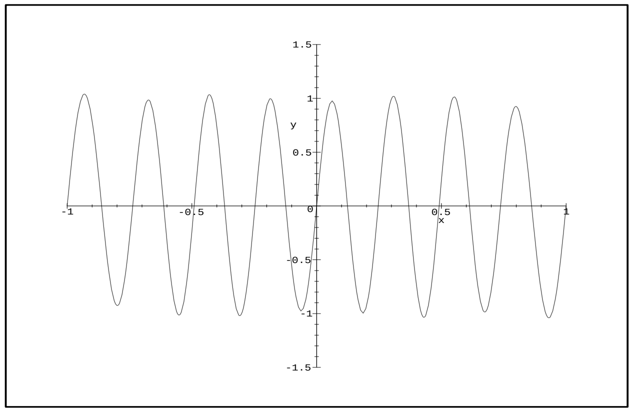

```
>  plot(fi6(x),x=-1..1,y=-1.5..1.5);
```



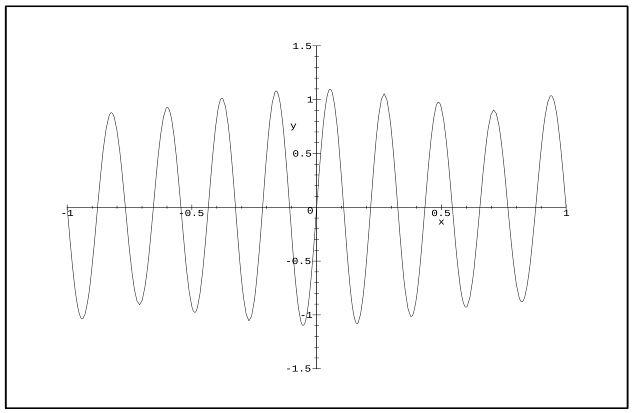

```
>  plot(fi7(x),x=-1..1,y=-1.5..1.5);
```



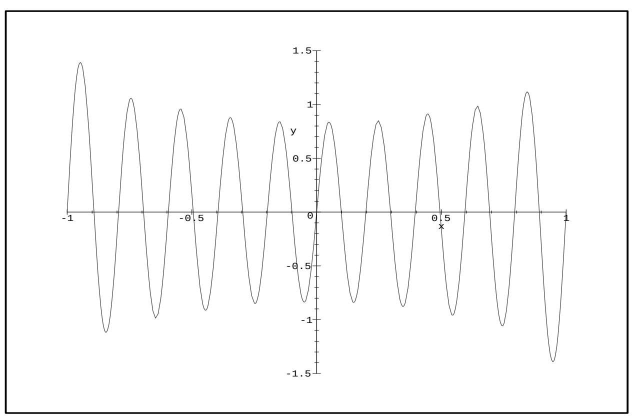

```
>  plot(fi8(x),x=-1..1,y=-1.5..1.5);
```



> plot(fi9(x),x=-1..1,y=-1.5..1.5);



> plot(fi10(x),x=-1..1,y=-1.5..1.5);



Here are first 10 eigenvalues of the odd series of the operator K:

$$0.02580642925, 0.02889694675, 0.03272986513, 0.03766434614, 0.04437798102,$$

$$0.05397919120, 0.06886402733, 0.09504925144, 0.1532098038, 0.3914815726,$$

and first 10 eigenfunctions of the odd series :

$$
\begin{aligned}
\text{fiod1}(x) := {} & -.9910874586 \sin(.5\,(x+1)\,\pi) + .1212875218 \sin(1.5\,(x+1)\,\pi) \\
& + .04436147780 \sin(2.5\,(x+1)\,\pi) + .02399915669 \sin(3.5\,(x+1)\,\pi) \\
& + .01530090916 \sin(4.5\,(x+1)\,\pi) + .01070626954 \sin(5.5\,(x+1)\,\pi) \\
& + .007958603246 \sin(6.5\,(x+1)\,\pi) + .006172535946 \sin(7.5\,(x+1)\,\pi) \\
& + .004945885162 \sin(8.5\,(x+1)\,\pi) + .004054924722 \sin(9.5\,(x+1)\,\pi)
\end{aligned}
$$

$$
\begin{aligned}
\text{fiod2}(x) := {} & .1117423454 \sin(.5\,(x+1)\,\pi) + .9818943229 \sin(1.5\,(x+1)\,\pi) \\
& - .1343729349 \sin(2.5\,(x+1)\,\pi) - .05608113124 \sin(3.5\,(x+1)\,\pi) \\
& - .03307212574 \sin(4.5\,(x+1)\,\pi) - .02241035233 \sin(5.5\,(x+1)\,\pi) \\
& - .01641399421 \sin(6.5\,(x+1)\,\pi) - .01263518518 \sin(7.5\,(x+1)\,\pi) \\
& - .01012904963 \sin(8.5\,(x+1)\,\pi) - .008267623797 \sin(9.5\,(x+1)\,\pi)
\end{aligned}
$$

$$
\begin{aligned}
\text{fiod3}(x) := {} & .05292785974 \sin(.5\,(x+1)\,\pi) + .1162634038 \sin(1.5\,(x+1)\,\pi) \\
& + .9789142759 \sin(2.5\,(x+1)\,\pi) - .1380115902 \sin(3.5\,(x+1)\,\pi) \\
& - .06011078796 \sin(4.5\,(x+1)\,\pi) - .03670653101 \sin(5.5\,(x+1)\,\pi) \\
& - .02559615342 \sin(6.5\,(x+1)\,\pi) - .01916493668 \sin(7.5\,(x+1)\,\pi) \\
& - .01510111946 \sin(8.5\,(x+1)\,\pi) - .01224449592 \sin(9.5\,(x+1)\,\pi)
\end{aligned}
$$

$$
\begin{aligned}
\text{fiod4}(x) := {} & .03274896712 \sin(.5\,(x+1)\,\pi) + .06043162781 \sin(1.5\,(x+1)\,\pi) \\
& + .1157994972 \sin(2.5\,(x+1)\,\pi) + .9773738676 \sin(3.5\,(x+1)\,\pi) \\
& - .1405447629 \sin(4.5\,(x+1)\,\pi) - .06244408352 \sin(5.5\,(x+1)\,\pi) \\
& - .03885164416 \sin(6.5\,(x+1)\,\pi) - .02744174349 \sin(7.5\,(x+1)\,\pi) \\
& - .02092526245 \sin(8.5\,(x+1)\,\pi) - .01593307543 \sin(9.5\,(x+1)\,\pi)
\end{aligned}
$$

$$
\begin{aligned}
\text{fiod5}(x) := {} & .02298961699 \sin(.5\,(x+1)\,\pi) + .03999406558 \sin(1.5\,(x+1)\,\pi) \\
& + .06174483906 \sin(2.5\,(x+1)\,\pi) + .1160041782 \sin(3.5\,(x+1)\,\pi) \\
& + .9762227109 \sin(4.5\,(x+1)\,\pi) - .1431966138 \sin(5.5\,(x+1)\,\pi) \\
& - .06436782790 \sin(6.5\,(x+1)\,\pi) - .04031008795 \sin(7.5\,(x+1)\,\pi) \\
& - .02893430576 \sin(8.5\,(x+1)\,\pi) - .02193605556 \sin(9.5\,(x+1)\,\pi)
\end{aligned}
$$

$$
\begin{aligned}
\text{fiod6}(x) := {} & -.01742840040 \sin(.5\,(x+1)\,\pi) - .02954457432 \sin(1.5\,(x+1)\,\pi) \\
& - .04197416634 \sin(2.5\,(x+1)\,\pi) - .06265915699 \sin(3.5\,(x+1)\,\pi) \\
& - .1169875459 \sin(4.5\,(x+1)\,\pi) - .9751305478 \sin(5.5\,(x+1)\,\pi) \\
& + .1466970244 \sin(6.5\,(x+1)\,\pi) + .06605302331 \sin(7.5\,(x+1)\,\pi) \\
& + .04195037765 \sin(8.5\,(x+1)\,\pi) + .03044411621 \sin(9.5\,(x+1)\,\pi)
\end{aligned}
$$

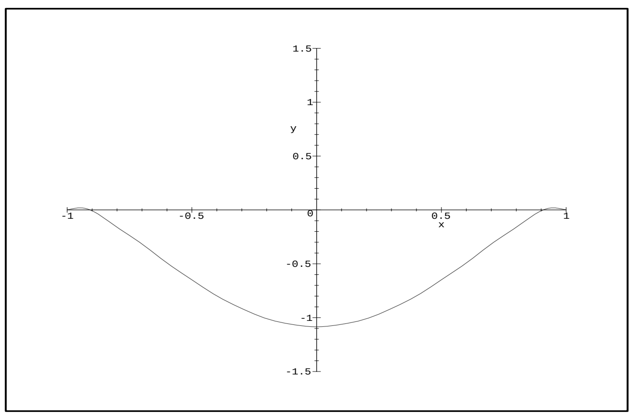
$$
\begin{aligned}
\text{fiod7}(x) := {} & -.01400129365 \sin(.5\,(x+1)\,\pi) - .02343290315 \sin(1.5\,(x+1)\,\pi) \\
& - .03194447996 \sin(2.5\,(x+1)\,\pi) - .04348097673 \sin(3.5\,(x+1)\,\pi) \\
& - .06406576018 \sin(4.5\,(x+1)\,\pi) - .1192784284 \sin(5.5\,(x+1)\,\pi) \\
& - .9740929360 \sin(6.5\,(x+1)\,\pi) + .1508365385 \sin(7.5\,(x+1)\,\pi) \\
& + .06934795728 \sin(8.5\,(x+1)\,\pi) + .03992749231 \sin(9.5\,(x+1)\,\pi)
\end{aligned}
$$

$$\text{fiod8}(x) := -.01159376012 \sin(.5\,(x+1)\,\pi) - .01926693060 \sin(1.5\,(x+1)\,\pi) - .02565271099 \sin(2.5\,(x+1)\,\pi) - .03328625898 \sin(3.5\,(x+1)\,\pi) - .04447434165 \sin(4.5\,(x+1)\,\pi) - .06528856960 \sin(5.5\,(x+1)\,\pi) - .1217965760 \sin(6.5\,(x+1)\,\pi) - .9716971126 \sin(7.5\,(x+1)\,\pi) + .1590061693 \sin(8.5\,(x+1)\,\pi) + .08470609206 \sin(9.5\,(x+1)\,\pi)$$

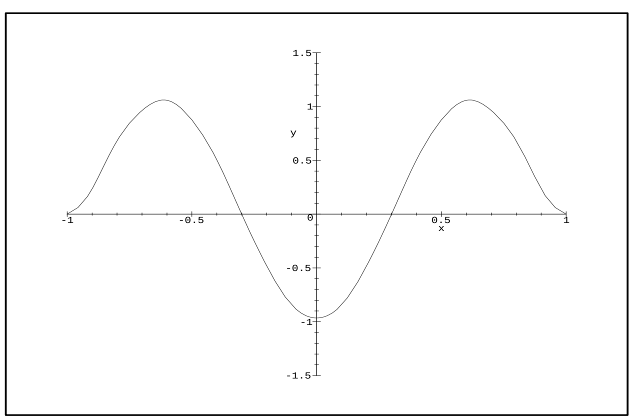
$$\text{fiod9}(x) := -.01037249141 \sin(.5\,(x+1)\,\pi) - .01720988796 \sin(1.5\,(x+1)\,\pi) - .02256025153 \sin(2.5\,(x+1)\,\pi) - .02849055602 \sin(3.5\,(x+1)\,\pi) - .03604306273 \sin(4.5\,(x+1)\,\pi) - .04779420377 \sin(5.5\,(x+1)\,\pi) - .07093271523 \sin(6.5\,(x+1)\,\pi) - .1278131209 \sin(7.5\,(x+1)\,\pi) - .9708913305 \sin(8.5\,(x+1)\,\pi) + .1751985944 \sin(9.5\,(x+1)\,\pi)$$

$$\text{fiod10}(x) := .01072997564 \sin(.5\,(x+1)\,\pi) + .01774293501 \sin(1.5\,(x+1)\,\pi) + .02305558991 \sin(2.5\,(x+1)\,\pi) + .02790429532 \sin(3.5\,(x+1)\,\pi) + .03503740995 \sin(4.5\,(x+1)\,\pi) + .04446837940 \sin(5.5\,(x+1)\,\pi) + .05582553719 \sin(6.5\,(x+1)\,\pi) + .09700962865 \sin(7.5\,(x+1)\,\pi) + .1545571928 \sin(8.5\,(x+1)\,\pi) + .9791013768 \sin(9.5\,(x+1)\,\pi)$$

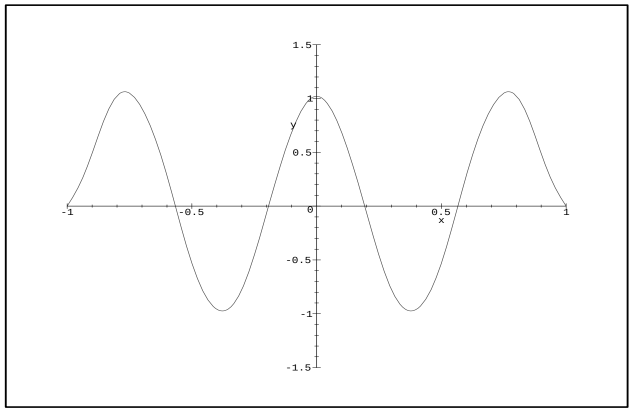
```
> #Drawing plots of the eigenfunctions
> plot(fiod1(x),x=-1..1,y=-1.5..1.5);
```



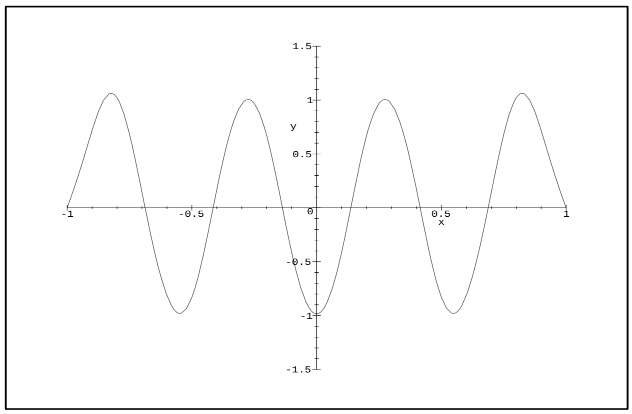

```
> plot(fiod2(x),x=-1..1,y=-1.5..1.5);
```



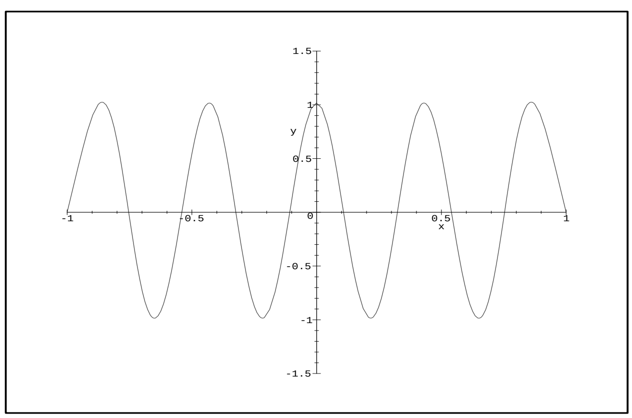
```
> plot(fiod3(x),x=-1..1,y=-1.5..1.5);
```



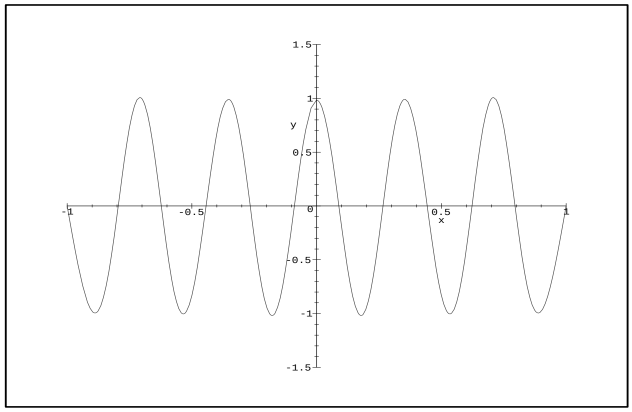

```
> plot(fiod4(x),x=-1..1,y=-1.5..1.5);
```



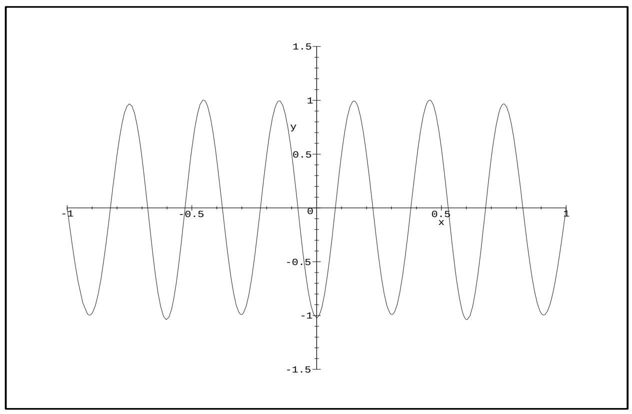

```
> plot(fiod5(x),x=-1..1,y=-1.5..1.5);
```



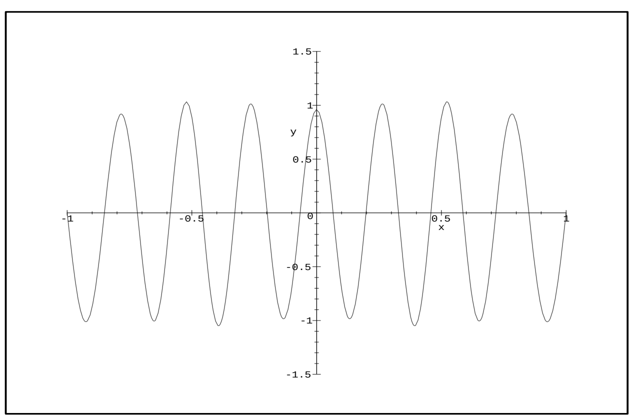

```
> plot(fiod6(x),x=-1..1,y=-1.5..1.5);
```



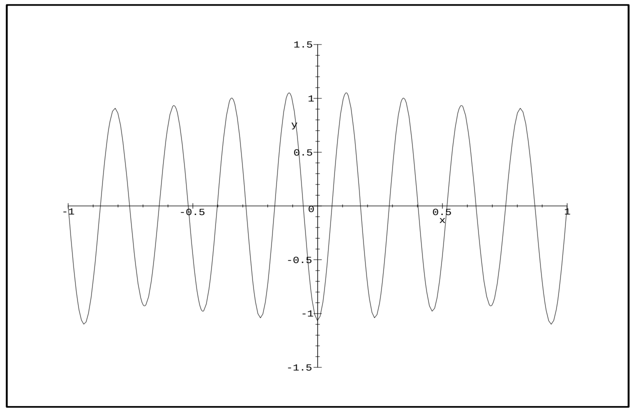
> plot(fiod7(x),x=-1..1,y=-1.5..1.5);



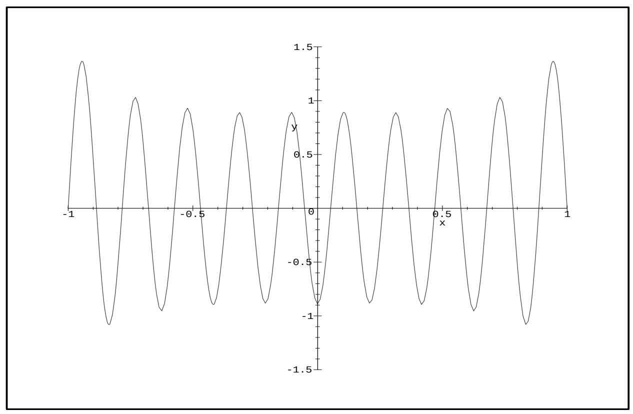
> plot(fiod8(x),x=-1..1,y=-1.5..1.5);



> plot(fiod9(x),x=-1..1,y=-1.5..1.5);




```
>  plot(fiod10(x),x=-1..1,y=-1.5..1.5);
```



In both cases the operator K is substituted by the matrix 10×10. Note that the appearance of the first group of the eigenfunctions of the operator K is almost the same as one of eigenfunctions of the differential operator $\mathcal{L}$. Mathematically this means that it is possible to neglect the surrounding space when calculating the eigenfunctions of , replacing $\Lambda_- - \Lambda_+$ by a constant. This observation was successfuly used in previous papers about slot-devices, see for instance [2] and helped us to obtain realistic results.

8 Appendix C. Basic Dirichlet problem with basic electrodes

The device with two basic electrodes on the horizontal plane is the most simple construction which is solved explicitly. In this section we calculate the relevant Poisson map. The corresponding DN-map is obtained via normal differentiation.

Consider the complex plane $\{\mathbf{z}\} = \mathbf{C}$ with two cuts $\Gamma_\pm = [l, \infty), (-\infty, -l]$ removed , $\Gamma = \Gamma_+ \cup \Gamma_-$:

$$\Omega_l = \mathbf{C} \backslash \left(\Gamma_+ \cup \Gamma_+\right) := \mathbf{C} \backslash \Gamma.$$

Our aim is: to construct a real harmonic function $\Phi(x_1, x_2)$ which takes the (real) boundary values on $\Gamma_\pm$:

$$\Phi\Big|_{\Gamma_\pm} = \pm V,\, V > 0. \tag{56}$$

This problem can be solved in elementary functions based on conformal map similar to (52,53):

$$\frac{l}{z} = \frac{1}{2}\left[u + \frac{1}{u}\right],$$

$$u = \frac{l - \sqrt{l^2 - z^2}}{z},$$

with the branch fixed by the condition of regularity $u(z)$ at the origin. This map transforms the domain Ω_l into the exterior of the unit disc $|u| > 1$

Note that the exterior Dirichlet problem for Laplacian in the complement of the unit disk, $|u| > 1$, and the boundary conditions:

$$\Phi(u) = 1 \text{ if } u = e^{i\theta},\ 0 < \theta < \pi, \quad \Phi(u) = -1 \text{ if } u = e^{i\theta},\ -\pi < \theta < 0 \tag{57}$$

has the solution $\Phi = \frac{1}{2\pi}\Re \ln \frac{1+u}{1-u}$. Then inserting the above expression for u in terms of z, we obtain the formula for Φ:

$$\Phi(z) = \frac{1}{2\pi}\Re \ln \frac{1 - \frac{l-\sqrt{l^2-z^2}}{z}}{1 + \frac{l-\sqrt{l^2-z^2}}{z}}, \tag{58}$$

which may be verified also via direct calculation, together with the corresponding Meixner condition

$$\int_{\Omega_l} |\bigtriangledown \Phi(z)|^2 dm < \infty.$$

Summarizing above results and noticing that Φ depends on non-dimensional coordinate $\frac{z}{l}$ we obtain the expression for the solution of the basic Dirichlet problem in case on infinite plates:

$$V_\infty(x,y) = \frac{V}{2\pi}\Re\ \ln \frac{1 - \frac{1-\sqrt{1-(z/l)^2}}{z/l}}{1 + \frac{1-\sqrt{1-(z/l)^2}}{z/l}}. \tag{59}$$

We denoted here the constructed solution by $V_\infty(x,y)$

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