Fitting of the solvable model for scattering by Helmholtz resonator.

Boris Pavlov

Abstract

The analytic perturbation procedure is divergent on the continuous spectrum near the threshold of creation of resonances. For scattering on Helmholtz resonator with a small opening we suggest a modified procedure which is convergent and permits to observe the creation of the resonance. The role of the first step ("jump-start") in that perturbation procedure is played by a solvable model of the resonator, which is completely fitted based on the resonance parameters.

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1 Introduction

The classical problem on scattering frequencies of Helmholtz resonator, [1], was intensely studied during previous century. The pure mathematical study since 1970, see for instance [2, 3, 4, 5, 6],was aimed on proving existence of resonances and estimates of the resonance parameters. Another, more applied series of papers, was aimed on obtaining of explicit approximate formulae for the scattered waves. The demand for convenient approximate formulae for scattered waves was satisfied mainly thanks to advanced analytical approximation technique. In particular, the scattering problem for a resonator with a small opening or narrow channel was treated in [7, 8, 9] via matching of asymptotic expansion [10]. The advanced analytical approximation methods supply complete asymptotic expansions for the eigenfunctions or generalized resonance eigenfunctions, but they do not give any operator-theoretic information, e.g completeness of the set of the generalized resonance eigenfunctions on a certain sub-domain, factoring of the scattering matrix, convergence of expansion by resonance states. The alternative operator-theoretic approach based on Lax-Phillips theory, like [4, 5], which can be used to answer these questions, is hardly compatible with the complete asymptotic expansions obtained by advanced approximation methods, because of non-analyticity of the expansions with respect to the perturbation parameter.

In [11, 15] another approach to the scattering problem for a resonator with a small opening was suggested. This approach was based on a solvable model obtained as revisiting of the classical Kirchhoff model [12] where the perturbation of the field caused by the small opening is modelled by the Green function of the non-perturbed problem. Note in this connection the book [13], where another attempt to revisit the Kirchhoff method was done, on the physical level of rigor.

The model in [11, 15] is constructed in form of a self-adjoint extension of the Laplacian defined on a domain containing singular deficiency elements, submitted to some asymptotic boundary conditions. It gives explicit formulae for the relevant ("model") scattering matrix and scattered waves. The corresponding resonances are found as solutions of algebraic equations. These explicit formulae can help to answer the operator- theoretic questions mentioned above. On the other hand this model, being properly fitted, can exhibit also a *qualitatively sensible* physical behavior. First attempt to choose appropriate parameters to guarantee the *quantitative consistence* of the model was done in [15], see the further discussion and references in [16] . But the general question on fitting of all parameters of the model remained unsolved, because these parameters do not have any direct physical interpretation. Nevertheless the model based on operator extensions can emulate the resonance transmission across the quantum well and was conveniently used for mathematical design of nano-electronic devices, see [19, 20].

In this paper we suggest our version of analysis of a resonator with a small opening (section 4) which gives an approximate formula for the the scattering amplitude in a convenient form. Then we revisit the solvable model [11, 15] and suggest, in section 5, a fitting procedure. Our suggestion is based on local rational approximation of the Dirichlet-to-Neumann map on an essential interval of frequency. Note that the connection of the solvable model of a quantum network with rational approximation of the corresponding Dirichle-to-Neumann map was noticed in [22]. Similar observation for Friedrichs model is describrd in [23].

Our solvable model L_0^ε may serve, see discussion in section 6, as a first step of a modified analytic perturbation procedure on the continuous spectrum. The eigenfunctions of the the solvable model and the scattering matrix S_0^ε of the model are not analytic function of the perturbation parameter. But the analytic perturbation procedure from the solvable model to the perturbed problem with a narrow channel may be described in terms of analytic functions, if the *exact spectral data* are used for fitting of the solvable model. The corresponding two-steps perturbation procedure is described in section 6. We postpone to the following publication the derivation of the complete asymptotic expansion for the complementary analytic factor S_ε^0 of the scattering matrix $S = S_\varepsilon^0\, S_0^\varepsilon$.

In contrast to the purely-analytic advanced approximation technique, our approach can be based on numerical data of resonance eigenvalues and eigenfunctions of some intermediate operator on the *essential interval of energy*, see for instance [24, 25]. This permits to straighten the way to the derivation of the approximate formulae for the scattering matrix via avoiding some hard analysis of the inner problem near to the resonance, and replacing it by the direct computing. Note that the idea of using of the direct computing in combination with asymptotic analysis was effectively applied by S. Merkuriev to Faddeev equations in nuclear scattering problems, see [26] and references therein.

On the other hand, the suggested version of the analytic perturbation procedure preceded by "jump-start" has common features with some aspects of "geometric integration", see [27]. Straightforward application of numerical methods to differential equation is based on discrete reduction of the differential equation. Generally the approximate solution exhibits an exponential divergence from the exact solution. Modern geometric integration suggests special methods of discrete reduction to suppress the exponential divergence via submitting the discrete system to some geometric conditions. This conditions are selected such that the discrete model preserves essential features of the structure of the original system, see [28, 29].

In our case, for scattering system, we actually also suggest a "structure-preserving" model, which is completely fitted based on spectral data of a compact sub-system. We postpone to forthcoming paper the comparison of the evolution of the model with the evolution of the original system, but we aim on comparison of the corresponding scattering matrices. We show that the model scattering matrix S_ε has the same *local* structure on a certain interval of energy, as the scattering matrix S of the original system. If the model is fitted based on exact resonance data, then the ratio $S_\varepsilon^0 = S\,[S_0^\varepsilon]^{-1}$ is an analytic function of the perturbation parameter and the spectral parameter on an essential interval of the spectral parameter. Hence our "locally-structure-preserving" solvable model can play the role of a jump-start in analytic perturbation procedure: the analytic perturbation series for the scattering data of the perturbed operator with respect to the solvable model is convergent. The role of the geometric condition for our model is played by the condition of vanishing of the relevant hermitian symplectic boundary form (the "normal current" associated with the model Hamiltonian) on a Lagrangian plane defined by the specific boundary conditions. The corresponding boundary parameters are defined in course of fitting of the model.

2 Rational approximations of the DN map

Let Ω be a complement in R^3 of a compact domain with a smooth boundary $\partial\Omega$. We present Ω as a joining $\Omega_{in} \cup \Omega_{out}$ of a compact and non-compact parts respectively, which have a common piece $\Gamma = \partial\Omega_{in} \cup \partial\Omega_{out}$ of the boundary . We consider the self-adjoint operator L^N defined in $L_2(\Omega)$ by the Laplacian L with the homogeneous Neumann condition on $\partial\Omega$. Together with L we consider the split operator $L^N_{in} \oplus L^N_{out}$ defined in the orthogonal sum $L_2(\Omega_{in}) \oplus L_2(\Omega_{out})$, by Laplacian with additional homogeneous Neumann conditions on each side Γ_{in}, Γ_{out} of the common border Γ:

$$\left.\frac{\partial v^{in}}{\partial n}\right|_{\Gamma_{in}} \overset{W_2^{1/2}(\Gamma)}{=\!=} 0,\ \left.\frac{\partial v^{out}}{\partial n}\right|_{\Gamma_{out}} \overset{W_2^{1/2}(\Gamma)}{=\!=} 0.$$

Here the boundary values are taken in appropriate Sobolev Spaces, see [30]. Both L and $L^N_{in} \oplus L^N_{out}$ are self-adjoint extensions of the restricted operator L_{00} defined by the same differential expressions on the domain of functions which vanish in a neighborhood of Γ. The operators $L^D_{in,out}$ are defined in a similar way by the Friedrich extension, see [30, 31] on domains $D^D_{in,out}$ of W^2_2- functions satisfying the homogeneous Dirichlet condition on Γ, the Meixner condition in form $D^D_{in,out} \subset W^1_2$, in case on inner angles, and Neumann boundary conditions on $\partial\Omega_{in,out}\backslash\Gamma$. The operators $L^{N,D}_{in,\,out}$ are self-adjoint operators in $L_2(\Omega_{in,\,out})$ respectively. We denote by n the outer normal on Γ with respect to Ω_{in}. The corresponding resolvent kernels $G^{N,D}_{in,\,out}(x,\,y,\lambda)$ and the Poisson kernels

$$\mathcal{P}_{in,\,out}(x,\,y,\,\lambda) = \mp\frac{\partial G^D_{in,\,out}(x,\,y,\lambda)}{\partial n_y},\, y \in \Gamma_{in,\,out}, \tag{1}$$

for complex values of the spectral parameter λ are locally W^2_2-smooth and square integrable in $\Omega_{in,out}$ with boundary values $G^N_{in,\,out}(x,\,y,\lambda) \in W^{3/2}_2(\Gamma)$, $x \in \Gamma$, $y \in \Omega_{in,out}$ and $\mathcal{P}_{in,\,out}(x,\,y,\,\lambda) \in W^{1/2}_2(\Gamma)$, $y \in \Gamma, x \in \Omega_{in,out}$. The behavior of $G^N_{in,\,out}(x,\,y,\,\lambda)$ when both x, y are near Γ is described by the following asymptotic which may be derived from the integral equations of potential theory :

$$G^N_{in,out}(x, x_\Gamma, \lambda) =$$

$$\frac{1}{2\pi|x - x_\Gamma|} + g^{log}_{in,\,out} + g_{in,out}(\lambda) + o(1), \tag{2}$$

here $g^{log}_{in,\,out}$ admits a logarithmic estimate and is independent of the spectral parameter, see [14],[15], and the third term $g_{in,out}(\lambda)$ contains the spectral information, see [32]. The spectra of the operators $L^{N,D}_{in}$ are discrete, and the spectra of $L^{N,D}_{out}$ are absolutely continuous on $[0,\infty)$ with no embedded eigenvalues, see [17, 18]. Solutions of classical boundary problems for the corresponding equation

$$L_{in,out}u = \lambda u,\, \Im\lambda \neq 0,\, u \in W^2_2(\Omega_{in,\,out}),\ \frac{\partial u}{\partial n}|_\Gamma = \rho^{in,\,out} \in W^{1/2}_2(\Gamma),$$

can be represented by the "re-normalized" simple layer potentials for Neumann problems:

$$u_{in,out}(x) = \pm \int_\Gamma G^N_{in,\,out}(x,\,y,\,\lambda)\rho_{in,\,out}(y)d\Gamma. \tag{3}$$

For the Dirichet problem with non-homogeneous data on Γ :

$$L_{in,out}u = \lambda u,\ \Im\lambda \neq 0,\ u \in W_2^2(\Omega_{in,\,out}),\ \ u|_\Gamma = \hat{u}_{in,\,out} \in W_2^{3/2}(\Gamma),$$

and homogeneous Neumann condition on $\partial\Omega_{in,out}\backslash\Gamma$ the solution is presented by the re-normalized double-layer potential supported by Γ:

$$u_{in,out}(x) = \int_\Gamma \mathcal{P}^D_{in,\,out}(x,\,y,\,\lambda)\hat{u}_{in,\,out}(y)d\Gamma. \tag{4}$$

The Dirichlet-to-Neumann map (DN-map) is represented for regular points λ of the operators $L^D_{in,out}$ as

$$\Big(\Lambda_{in,out}(\lambda)\hat{u}_{in,\,out}\Big)(x_\Gamma) = \frac{\partial}{\partial n}\Big|_{x=x_\Gamma}\int_\Gamma \mathcal{P}_{in,\,out}(x,\,y,\,\lambda)\hat{u}_{in,\,out}(y)d\Gamma. \tag{5}$$

The inverse maps $\pm\Lambda^{-1}_{in,out} : W_2^{1/2}(\Gamma) \to W_2^{3/2}(\Gamma)$ are presented at regular points of the operators $L^N_{in,out}$ as restrictions onto Γ of the integral transformation (Neumann-to-Dirichlet map, ND map):

$$\Big(\mathcal{Q}_{in,out}(\lambda)\rho_{in,\,out}\Big)(x_\Gamma) = \pm\int_\Gamma G^N_{in,\,out}(x_\Gamma,\,y,\,\lambda)\rho_{in,\,out}(y)d\Gamma. \tag{6}$$

For general properties of solutions of second order partial differential equations, and, in particular, Laplace equation see [31], for discussion of connections between DN-map and inverse problems see [33, 34, 35].

In scattering problems we need to evaluate DN - map on real axis of the spectral parameter λ. One can see from the straightforward integration by parts with W_2^2-solutions of the boundary problem that the DN-map is an analytic function of the spectral parameter λ with a positive imaginary part for exterior boundary problem and with a negative imaginary part for interior one. We use hereafter the dot-product in $L_2(\Gamma)$ defined as $\langle u,\,v\rangle = \int_\Gamma \bar{u}\,vd\Gamma$. Then we have:

$$\Im\langle u_\Gamma,\,\Lambda_{out}u_\Gamma\rangle|_\Gamma = \Im\langle u_\Gamma,\,\frac{\partial u}{\partial n}|_\Gamma\rangle|_\Gamma = \Im\int_\Gamma \frac{\partial G^D_{out}(x_\Gamma, y_\Gamma, \lambda)}{\partial n(x_\Gamma)\partial n(y_\Gamma)}u(x_\Gamma)\bar{u}(y_\Gamma)dx_\Gamma dy_\Gamma > 0,$$

$$\Im\langle,\,u_\Gamma,\,\Lambda_{in}u_\Gamma\rangle|_\Gamma = \Im\langle u_\Gamma,\,\frac{\partial u}{\partial n}|_\Gamma\rangle|_\Gamma = -\Im\int_\Gamma \frac{\partial G^D_{in}(x_\Gamma, y_\Gamma, \lambda)}{\partial n(x_\Gamma)\partial n(y_\Gamma)}u(x_\Gamma)\bar{u}(y_\Gamma)dx_\Gamma dy_\Gamma < 0,$$

for $u_\Gamma \in W_2^{3/2}(\Gamma)$, $\Im\lambda \neq 0$, and acts from $W_2^{3/2}(\Gamma)$ onto $W_2^{1/2}(\Gamma)$. It can be derived from [37], that it has weak non-tangential boundary values on real axis from the upper and the lower half-planes, which we denote by $\Lambda_{out}(\lambda \pm i0)$ respectively. Similarly the weak limits of $\Lambda_{in}(\lambda)$ exist on the complement of the discrete spectrum σ^D of the inner Dirichlet problem in Ω_{in} and are bounded operators from $W_2^{3/2}(\Gamma)$ onto $W_2^{1/2}(\Gamma)$. The following statement, see [38], shows, that the singularities of the DN-map $\Lambda^D_{in}(\lambda)$ as an unbounded operator in $W_2^{3/2}(\Gamma)$, and the poles at the eigenvalues of the inner Dirichlet problems may be separated :

Theorem 2.1 *Consider the Dirichlet Laplacian L^D_{in} in $L_2(\Omega_{in})$ on a compact domain $\Omega_{in} \subset R_3$ with a smooth boundary. The DN-map Λ_{in} of L^D_{in} has the following representation on the complement of the corresponding spectrum σ^D_{in} in complex plane λ, with $M > 0$:*

$$\Lambda_{in}(\lambda) = \Lambda_{in}(-M) - (\lambda + M)\mathcal{P}^+_{-M}\mathcal{P}_{-M} - (\lambda + M)^2\mathcal{P}^+_{-M}R_\lambda\mathcal{P}_{-M}, \tag{7}$$

where R_λ is the resolvent of L^D_{in}, and $\mathcal{P}_\lambda$ is the corresponding Poisson kernel. The operators

$$\Lambda_{in}(-M),\ \ \mathcal{P}^+_{-M}\mathcal{P}_{-M}$$

are respectively bounded from $W^{3/2}_2(\Gamma)$ onto $W^{1/2}_2(\Gamma)$ and compact in $W^{3/2}_2(\Gamma)$, and the operator-function

$$\left(\mathcal{P}^+_M R_\lambda \mathcal{P}_M\right)(x_\Gamma, y_\Gamma) = \sum_{\lambda_s\in\Sigma_L} \frac{\frac{\partial\varphi_s}{\partial n}(x_\Gamma)\frac{\partial\varphi_s}{\partial n}(y_\Gamma)}{(\lambda_s + M)^2(\lambda_s - \lambda)} \tag{8}$$

is compact in $W^{3/2}_2(\Gamma)$.

Similar statement is true for the local DN-map when the Dirichlet boundary conditions are required only of the part Γ of the boundary. Then the Poisson map is defined only on Γ. The theorem 2.3 gives a convenient approach to construction of rational approximations of the DN- map, because the spectral series (8) is convergent in appropriate sense on the complement of spectrum of the operator the operator L^D_{in}. The rational approximation of the Dirichlet-to-Neumann map serves a tool for constructing solvable models. In this paper we use a local few - approximation on an "essential interval of the spectral parameter".

Dirichlet-to Neumann map for the Laplacian on the non-compact domain is defined as the boundary current $\frac{\partial u_\lambda}{\partial n}$ of the square integrable solution of the Helmholtz equation $-\triangle u_\lambda = \lambda u_\lambda$, if λ is regular. On the continuous spectrum, $\lambda \geq 0$, the Dirichlet-to-Neumann map is defined as the boundary current of the outgoing solution , which is obtained as $\lim_{\varepsilon\to 0} u_{\lambda+i\varepsilon}$. In this paper the non-compact Ω_{out} domain is obtained as a complement of a compact domain Ω_{in}. We calculate the boundary currents for both inner and outer domain via differentiation in outgoing direction with respect to Ω_{in}. A statement similar to theorem 2.3 is true also for DN-map in the exterior domain,

$$\Lambda_{out}(\lambda) = \Lambda_{out}(-M) + (\lambda + M)\mathcal{P}^+_{-M}\mathcal{P}_{-M} + (\lambda + M)^2\mathcal{P}^+_{-M}R_\lambda\mathcal{P}_{-M}, \tag{9}$$

with only difference that first terms of the decomposition contain the DN-map and Poisson kernel for the exterior domain and the generalized kernel in the last term is represented via the integral over the absolutely continuous spectrum $\sigma^a_L = [0, \infty)$. The integrand is combined of the normal derivatives of the scattered waves $\psi(x, |k|, \nu)$, $k = |k|\nu$, $|\nu| = 1$:

$$\mathcal{P}^+_{-M}R_\lambda\mathcal{P}_{-M}(x_\Gamma,\, y_\Gamma) =$$

$$\frac{1}{(2\pi)^3}\int_{|k|^2\in\Sigma^a_L} \frac{\frac{\partial\bar{\psi}}{\partial n}(x_\Gamma,\, |k|, \nu)\frac{\partial\psi_s}{\partial n}(y_\Gamma\, |k|, \nu)}{(|k|^2 + M)^2(|k|^2 - \lambda)} d^3k.$$

The absolutely-continuous spectra $\sigma^{D,N}_{out}$ of of both operators $L^{D,N}_{out}$ fill the positive semi-axis $0 \leq \lambda < \infty$ with infinite multiplicity and the scattered waves $\psi(x, k)$- are parametrized by the energy $\lambda > 0$, $|k| = \sqrt{\lambda}$, and the direction $\nu, |\nu| = 1$, or just by the *momentum* $k = |k|\nu \in R^3$.

3 DN-map in Scattering problems

The asymptotic of the scattered wave at infinity in the direction ω , $x \to \omega, \infty, \ |\omega| = 1$, involves so-called *scattering amplitude* $a(|k|, \omega, \nu) = a(k, l)$for $k = \omega |k|, \ l = \nu |k|$:

$$\psi(x,k) = e^{i|k|<x,\nu>} - 2\pi^2 \frac{e^{-i|k||x|}}{|x|} a(k,l) + o(\frac{1}{|x|}). \tag{10}$$

The scattering matrix is defined on the space $Ł_2(\Sigma_2)$ of all square-integrable functions on the unit sphere Σ_2 by the unitary operator

$$(Sh)(\omega) = h(\omega) - i\pi|k| \int_{\Sigma^2} a_\nu(|k|\omega, |k|\nu) h(\nu) d\Sigma_2 = (I + T)\, h,$$

where the integral operator T in the right side is called T-matrix. The kernel of T-matrix is the scattering amplitude.

The Scattering matrix is connected with DN-maps of the operators $L^{D}_{in,out}$ by the formula (22). To derive that formula we need to introduce the notion of *matching*. Recall that the split operators $L^{D,N}_{in} \oplus L^{D,N}_{out}$ were constructed as self-adjoint extensions of the restricted operator L_{00}. Practically all self-adjoint extensions can be obtained via restrictions of the adjoint operator L^{+}_{00} onto Lagrangian planes where the boundary form of L^{+}_{00} vanishes. The deficiency indices of L_{00} are infinite and the corresponding *boundary form* of the adjoint operator $L^+_{00} := L^+_{in} \oplus L^+_{out}$ on elements $v^{in} \oplus v^{out}$ from $W^2_2(\Omega_{in}) \oplus W^2_2(\Omega_{out})$ can be calculated in terms of boundary values of elements from the domain of L^{+}_{00} and the boundary values of their normal derivatives on Γ, for instance

$$\mathcal{J}_{out}(u,v) = \langle L^{+}_{out} u, v\rangle - \langle u, L^{+}_{out} v\rangle = \frac{\partial \bar u}{\partial n} v - \bar u \, \frac{\partial u}{\partial n}\Big|_{\Gamma_{out}}.$$

We introduce the symplectic coordinates - the boundary values - of the deficiency elements to represent the boundary form. Denote by

$$[v]|_\Gamma := v^{out}|_\Gamma - v^{in}|_\Gamma \in W^{3/2}_2(\Gamma), \tag{11}$$

the jump of the element $u := v^{in} \oplus u^{out}$ at the boundary and by

$$\{v\}\Big|_\Gamma := \frac{1}{2}\left[v^{out}\Big|_\Gamma + v^{in}\Big|_\Gamma \right] \in W^{3/2}_2(\Gamma), \tag{12}$$

the mean value of it and by

$$\left[\frac{\partial v}{\partial n}\right]\Big|_\Gamma, \{\frac{\partial v}{\partial n}\}\Big|_\Gamma \in W^{1/2}_2(\Gamma) \tag{13}$$

respectively the jump and the mean value of it's normal derivatives. Then we represent the boundary form of the adjoint operator L^+_{00} as an hermitian bilinear form of the boundary values integrated on the bilateral surface $\Gamma = \Gamma_{in} \cup \Gamma_{out}$

$$\langle L^+_{00} u, \, v\rangle - \langle u, \, L^+_{00} v\rangle = \int_{\Gamma_{in}} \left(u \frac{\partial \bar v}{\partial n} - \frac{\partial u}{\partial n} \bar v \right) d\Gamma \; - \int_{\Gamma_{out}} \left(u \frac{\partial \bar v}{\partial n} - \frac{\partial u}{\partial n} \bar v \right) d\Gamma,$$

or as a linear combination of bilinear forms of jumps/mean values integrated over Γ:

$$\int_\Gamma \left(- \left[\frac{\partial u}{\partial n}\right] \{\bar v\} + \{u\} \left[\frac{\partial \bar v}{\partial n}\right] \right) d\Gamma \; - \int_\Gamma \left(- \{\frac{\partial u}{\partial n}\} [\bar v] + [u] \{\frac{\partial \bar v}{\partial n}\} \right) d\Gamma := \mathcal{J}_c(u,v) + \mathcal{J}_{dc}(u,v). \tag{14}$$

The second integral in the last line vanishes for all non-jumping elements $u : [u]|_{\partial\Omega} = 0$, $\{u\}|_{\partial\Omega} = u|_{\partial\Omega}$ from the domain $W_2^2(\Omega_{in}) \oplus W_2^2(\Omega_{out})$ of L_{00}^+. Together with the restriction L_{00} we consider the restriction L_0 to the domain of all elements u vanishing on the surface Γ with zero jump of the normal derivative :

$$u|_\Gamma \overset{W_2^{3/2}(\Gamma)}{==} 0, \ \left[\frac{\partial u}{\partial n}\right]\Big|_\Gamma \overset{W_2^{1/2}(\Gamma)}{==} 0.$$

The adjoint operator L_0^+ is an extension of L_{00}. It is defined by the same differential expression on all elements from $D(L_{00}^+)$ which are continuous (have zero jump at the surface Γ): $[u]|_\Gamma = 0$. The boundary form of the operator L_0^+ coincides with the first addendum $\mathcal{J}_c(u,v)$ in (14) which is represented, due to $u - \{u\}\Big|_\Gamma = 0$ for continuous functions, as

$$\mathcal{J}_c(u,v) = \int_\Gamma \left(\left[-\frac{\partial u}{\partial n}\right]\bar{v} + u\left[\frac{\partial \bar{v}}{\partial n}\right]\right) d\Gamma. \tag{15}$$

Both forms (14,15) are *hermitian symplectic forms* in the infinitely-dimensional space of corresponding complex symplectic variables - the boundary data. In particular the boundary data for (15) are $\{u, [\frac{\partial u}{\partial n}]\}$. Any self-adjoint extension of the restricted operator L_{00}, or L_0 may be described by some boundary condition connecting the symplectic variables by hermitian operator thus defining the corresponding *Lagrangian plane*, see [32]. In particular the underlying operator L can be considered as an extension of L_0 which corresponds to the *matching boundary conditions*

$$[u]|_\Gamma \overset{W_2^{3/2}(\Gamma)}{==} 0, \ \left[\frac{\partial \bar{v}}{\partial n}\right]\Big|_\Gamma \overset{W_2^{3/2}(\Gamma)}{==} 0. \tag{16}$$

We will connect the resolvent of the self-adjoint operator L with the resolvent of the orthogonal sum of the self-adjoint operators $L_{in}^N \oplus L_{out}^N$ defined in $L_2(\Omega_{in}) \oplus L_2(\Omega_{out})$ by the homogeneous Neumann boundary conditions. We will re-write the above boundary conditions on Γ in terms of deficiency elements. The deficiency elements of the operator L_0 have *non jumping* boundary values from $W_2^{3/2}(\Gamma)$ and satisfy the adjoint homogeneous equation

$$L_0^+ u_\rho := -\triangle u_\rho = \lambda u_\rho.$$

They may be represented in form of a *re-normalized simple-layers* formed of re-normalized potentials as:

$$u_{in,\,out}(x.\bar{\lambda}) = \int_\Gamma G_{in,\,out}(x,\,s,\,\lambda)\rho(s)d\Gamma, \ \Im\lambda \neq 0. \tag{17}$$

with densities $\rho \in W_2^{1/2}(\Gamma)$. They have the normal boundary values $\hat{u}_{in,out} \in W_2^{3/2}(\Gamma)$, $\frac{\partial \hat{u}_{in,out}}{\partial n} \in W_2^{1/2}(\Gamma)$, which are evaluated via integration by parts:

$$\hat{u}_{in,out}(x) = \int_\Gamma G_{in,out}(x,s,\lambda)\rho_{in,out}(s)d\Gamma_s :=$$

$$\left(\mathcal{Q}_{in,\,out} * \rho_{in,out}\right)(x), \ x \in \Gamma,$$

$$\frac{\partial \hat{u}_{in,out}}{\partial n}(x) = \pm\rho_{in,out}(x), \ x \in \Gamma. \tag{18}$$

From the last formula (18) we see, that the boundary values of the non-jumping deficiency elements $u : [u]|_\Gamma = 0$ of the operators $l_0^{in,\,out}$ are connected by the infinite-dimensional analog of the Krein's Q-function [36]:

$$\hat{u}^{in,out} = \pm\mathcal{Q}_{in,\,out} * \frac{\partial \hat{u}_{in,out}}{\partial n}, \tag{19}$$

where the integral operators $\mathcal{Q}_{in,\,out}*$ transform $W_2^{1/2}(\Gamma)$ into $W_2^{3/2}(\Gamma)$. We see, that these operators, if exist for given λ, are inverse of the corresponding Dirichlet-to-Neumann maps $\Lambda_{in,\,out}$. Taking into account the continuity of the piece-wise defined solution $\hat{u}$ and non-jumping condition for its normal derivatives

$$\hat{u}_{in}(x_\Gamma) = \hat{u}_{out}(x_\Gamma),\ \rho_{in}(x_\Gamma) = \frac{\partial \hat{u}_{in}}{\partial n}(x_\Gamma) = \frac{\partial \hat{u}_{out}}{\partial n}(x_\Gamma) = -\rho_{out}(x_\Gamma)$$

we obtain, see [38], the following statement:

Theorem 3.1 *The resolvent kernel $G_\lambda(x,y)$ of the operator L^N for x,y in Ω_{out} is represented by the Krein formula*

$$G^N(x,y,\lambda) = G^N_{out}(x,y,\lambda) -$$
$$G^N_{out}(x,*,\lambda)\left[\mathcal{Q}_{in}(\lambda) + \mathcal{Q}_{out(\lambda)}\right]^{-1} G^N_{out}(*,y,\lambda), \tag{20}$$

where the starlets stay for variables on Γ. The corresponding formula for the scattered waves ψ_ν in outer domain has the form

$$\psi_\nu(x,\,\lambda) = \psi^{out}_\nu(x,\,\lambda) -$$
$$G^N_{out}(x,*,\lambda)\left[\mathcal{Q}_{in}(\lambda) + \mathcal{Q}_{out}(\lambda)\right]^{-1} \psi^{out}_\nu(*,\,\lambda), \tag{21}$$

where all operator-functions for real λ are calculated as limits from the upper half-plane. The expression for the scattering amplitude of the operator L^N is given by the formula:

$$a(\omega,\nu,\lambda) = a^{out}(\omega,\nu,\lambda) -$$
$$\frac{1}{8\pi^3}\psi^{out}_\omega(*,\,\lambda)\left[\mathcal{Q}_{in}(\lambda) + \mathcal{Q}_{out}(\lambda)\right]^{-1} \psi^{out}_\nu(*,\,\lambda). \tag{22}$$

where all operator-functions are calculated as weak limits from the upper half-plane.

Proof It is sufficient to prove that the formal expressions which stay in the formulae for the resolvent kernel and the for the scattering matrix, are correctly defined, at least almost everywhere. We will verify the fact for the scattering matrix. Really, the scattered wave ψ^{out}_ν is locally W^2_2- smooth, hence the restriction of it onto Γ at least belongs to $W^{3/2}_2(\Gamma)$. The inverse of the operator $\left[\mathcal{Q}_{in}(\lambda) + \mathcal{Q}_{out}\right]^{-1}$ exists almost everywhere of real axis lambda and acts from $W^{3/2}_2(\Gamma)$ into $W^{1/2}_2(\Gamma)$. Hence the result $\left[\mathcal{Q}_{in}(\lambda) + \mathcal{Q}_{out}\right]^{-1}\psi^{out}_\nu(*,\,\lambda)$ belongs to $W^{1/2}_2(\Gamma)$ and hence may serve as a functional on $\psi^{out}_\omega(*,\,\lambda) \in W^{3/2}_2(\Gamma)$. This is sufficient to define correctly the expression for the amplitude in (22) almost everywhere.

4 Transport properties of a narrow channel

Assume that the inner and outer domains $\Omega_{in,out}$ are connected by the narrow channel Ω_δ, which is included into $\Omega_{in,out}$. We are able to derive an approximate expression for the amplitude under certain assumptions about the geometry of the channel. We assume that the channel Ω_δ has a form of a circular cylinder, hight H, $0 < x < H$, radius δ, $0 < \rho < \delta$, with the bottom section $\Gamma_0 = \Gamma$, and the upper section Γ_H. Denote by $\nu^2_{n,s}(\delta) = \delta^{-2}\left[\nu^1_{n,s}\right]^2$ the eigenvalues of the Laplacian on the section of the cylinder, with homogeneous Neumann boundary conditions at $\rho = \delta$, and by $P_{n,s}$ the projections onto the corresponding normalized eigenfunctions $Y_n J_n(\nu_{n,s}\,\rho)$ with $Y_n(\varphi) =$ Const $e^{\pm\, in\,\varphi}$ and the eigenvalues $\nu^2_s = \delta^{-2}\,(\nu^1)^2_s$ defined by $J'_n(\nu_{n,s}\delta) = 0$. Denote by P_0 the projection onto the constant eigenfunction $Y_{0,0} = (\sqrt{\pi}\,\delta)^{-1}$ corresponding to the eigenvalue $\nu_0^2 = 0$ and by $P_{n,s}$, $(n,s) \neq (0,0)$ other eigen-projections which correspond to eigenvalues $\nu_{n,s}^2(\delta) = \nu_{n,s}^2(1)\,\delta^{-2} \neq 0$. Denote $\sum_{(n,s)\neq(0,0)} P_{n,s} = P^{\perp}$. It is convenient to consider the Neumann Laplacian $-\triangle_1$ with respect to the scaled variables $\delta^{-1}x$, with eigenvalues $(\nu_s^1)^2$. We define the Dirichlet-to-Neumann map Λ^{δ} of the channel as an algorithm of calculation of the normal current in the outgoing direction with respect to Ω_{in} for the solution of the Helmholtz equation in Ω_δ with the boundary data u_H, u_Γ on the upper and lower sections Γ, Γ_H. By separation of variables we obtain:

$$\Lambda^{\delta} = \begin{pmatrix} \Lambda^{HH} & \Lambda^{H\Gamma} \\ \Lambda^{\Gamma H} & \Lambda^{\Gamma\Gamma} \end{pmatrix} = \frac{\sqrt{\lambda}}{\sin\sqrt{\lambda}\,H} \begin{pmatrix} \cos\sqrt{\lambda}H & -1 \\ 1 & -\cos\sqrt{\lambda}H \end{pmatrix} P_0 +$$

$$\sum_{n,s=1}^{\infty} \frac{\sqrt{\nu_{n,s}^2 - \lambda}}{\sinh\sqrt{\nu_{n,s}^2 - \lambda}\,H} \begin{pmatrix} \cosh\sqrt{\nu_{n,s}^2 - \lambda}H & -1 \\ 1 & -\cosh\sqrt{\nu_{n,s}^2 - \lambda}H \end{pmatrix} P_{n,s} := \Lambda_0^{\delta} + \Lambda_{\delta}^{\perp}. \tag{23}$$

We say that the channel Ω_δ is *narrow*, if

$$\delta H^{-1} << 1,\ \lambda\delta^2 << 1.$$

If the following additional condition is fulfilled:

$$\lambda H^2 << 1,$$

we say that the channel is both *short* and *narrow*. In particular, for narrow channel the DN-map of the channel is defined mainly by the constant eigenfunction of the cross-section:

$$\Lambda_\delta \approx \Lambda_0^{\delta} + \begin{pmatrix} \delta^{-1}\sqrt{-\triangle_1}\,P_H^{\perp} & 0 \\ 0 & -\delta^{-1}\sqrt{-\triangle_1}\,P_\Gamma^{\perp} \end{pmatrix}, \tag{24}$$

where $-\triangle_1$ is the Neumann Laplacians in the orthogonal complement of constants on the sections presented in terms of the scaled variables (on the corresponding scaled section radius 1). In case of narrow channel we can neglect, compared with 1, the exponentially small terms $e^{-H/\delta}$ and the terms containing factors $\lambda\delta^2$. For short narrow channels further cancellations are possible, see below. One can see from the asymptotic (24), that the non-trivial transmission through narrow channel and/or short narrow channel is observed only for zero modes. This observation is crucial for construction and fitting of the solvable model below. But if the analytic perturbation procedure is developed, then the complete formula (23) must be used.

We represent the exterior domain in form of a joining of the channel Ω_δ and the non-compact complement Ω_H

$$\Omega_{out} = \Omega_\delta \cup \Omega_H.$$

Consider the Laplacian L_H on Ω_H with Neumann homogeneous boundary condition on $\partial(\Omega_H \backslash \Omega_\delta) \backslash \Gamma_H$ and Dirichlet boundary condition on Γ_H. Denote by Λ^H the Dirichlet-to-Neumann map of L_H:

$$\Lambda^H : W_2^{3/2}(\Gamma_H) \to W_2^{1/2}(\Gamma_H)$$

and the inverse ND-map

$$\mathcal{Q}_H v(x) = -\int_{\Gamma_H} G_H^N(x,y) v(y) d\sigma_y \Big|_{\Gamma_H}, \tag{25}$$

where G^N_H is the Green function of the Neumann problem in Ω_H. The operator $\mathcal{Q}_H$ is dissipative, for complex λ, hence $\mathcal{Q}^{1/2}_H$ is defined as an operator acting from $W^{1/2}_2(\Gamma_H)$ to $W^1_2(\Gamma_H)$. Recall that $\Lambda^H : W^{3/2}_2(\Gamma_H) \to W^{1/2}_2(\Gamma_H)$. Then the one-dimensional operators $a_H = P_0 \Lambda^H P_0$, $b_H = P_\perp \Lambda^H P_0$ act respectively from $W^{3/2}_2(\Gamma_H)$ to $W^{3/2}_2(\Gamma_H)$ and from $W^{3/2}_2(\Gamma_H)$ to $W^{1/2}_2(\Gamma_H)$.

Transport properties of the narrow and short narrow channel are revealed via comparison of the inverse ND-maps Q_H, Q_Γ associated with Neumann Laplacean L^N_H, L^N_Γ in Ω_H and $\Omega_\Gamma = \Omega_H \cup \Omega_\delta$. Though the calculation are done in general narrow channel case, the final result is formulated for most practical short narrow channel. We postpone general analysis of the narrow channel of constant length H to forthcoming publication, see also analysis of elliptic equations in domain with narrow channel in [8, 39] and references therein.

Lemma 4.1 *For short narrow channel the ND-map Q_Γ on the bottom section is connected to the ND map on the upper section by the approximate formula:*

$$Q_\Gamma v_\Gamma \approx -[Q^{00}_H + Q^{0\perp}_H \sqrt{-\triangle}\, Q^{\perp 0}_H] v_\Gamma + \frac{I}{\sqrt{-\triangle}} v^\perp_\Gamma .$$

Proof Assuming that the Green function $G^N_H(x,s)$ of L^N_H is known, we calculate the Green function $G^N_G(x,s)$ of L^N_Γ on the bottom section Γ. In fact, it is sufficient to find the solution $u(x)$ of the Neumann problem for Laplacian $Lu = \lambda u$ with boundary data $u'_\Gamma = \frac{\partial u}{\partial n}(x_\Gamma) = v_\Gamma$ on the bottom section. On the upper section this solution should match smoothly the solution of the exterior Neumann problem represented as

$$u_H(x) = -\int_{\Gamma_H} G^N_H(x,s) v_H(s)ds,\ x \in \Omega_H, \tag{26}$$

where v_H is the normal derivative in the outward direction on Γ_H. Denoting by u_Γ, v_Γ and u_H, v_H respectively the Cauchy data of u on the bottom and upper sections of the channel, and denoting by z the variable along the channel, $0 < z < H$, we can present u as:

$$\begin{gathered} u(x) = -\frac{\cos\sqrt{\lambda}z}{\sqrt{\lambda}\sin\sqrt{\lambda}H} v^0_H - \frac{\cos\sqrt{\lambda}(z-H)}{\sqrt{\lambda}\sin\sqrt{\lambda}H} v^0_\Gamma + \\ \frac{\cosh\sqrt{-\triangle-\lambda}\, z}{\sqrt{-\triangle-\lambda}\sinh\sqrt{-\triangle-\lambda}H} v^\perp_H + \frac{\cosh\sqrt{-\triangle-\lambda}\,(z-H)}{\sqrt{-\triangle-\lambda}\sinh\sqrt{-\triangle-\lambda}H} v^\perp_\Gamma . \end{gathered} \tag{27}$$

Here $v^0 = P_0 v$ and $v^\perp = P^\perp v$ - projections onto the spectral subspaces of Neumann Laplacian $\triangle$ on the section of the channel: P_0 is the spectral projection onto the constant eigenfunction corresponding to zero eigenvalue and $P^\perp = I \ominus P_0$. Then

$$u_\Gamma = -\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}H} v^0_H - \frac{\cot\sqrt{\lambda}H}{\sqrt{\lambda}} v^0_\Gamma + \frac{I}{\sqrt{-\triangle-\lambda}\cosh\sqrt{-\triangle-\lambda}H} v^\perp_H + \frac{\coth\sqrt{-\triangle-\lambda}H}{\sqrt{-\triangle-\lambda}} v^\perp_\Gamma ,$$

$$u_H = -\frac{\cot\sqrt{\lambda}H}{\sqrt{\lambda}} v^0_H - \frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}H} v^0_\Gamma + \frac{\coth\sqrt{-\triangle-\lambda}H}{\sqrt{-\triangle-\lambda}} v^\perp_H + \frac{1}{\sqrt{-\triangle-\lambda}\sinh\sqrt{-\triangle-\lambda}H} v^\perp_\Gamma . \tag{28}$$

Then, from the equations

$$u_H = Q_H v_H,\ u_\Gamma = Q_\Gamma v_\Gamma \tag{29}$$

for narrow channel $\delta \to 0$ we find

$$Q_H v_H = -\frac{\cot\sqrt{\lambda}H}{\sqrt{\lambda}} v^0_H - \frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}H} v^0_\Gamma + \frac{I}{\sqrt{-\triangle-\lambda}} v^\perp_H + O(e^{-H/\delta}),$$

$$Q_\Gamma v_\Gamma = -\frac{\cot\sqrt{\lambda}H}{\sqrt{\lambda}} \overset{0}{v}_\Gamma - \frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}H} \overset{0}{v}_H + \frac{1}{\sqrt{-\triangle-\lambda}} v_\Gamma^\perp + O(e^{-H/\delta}), \tag{30}$$

where by $O(e^{-H/\delta})$ are denoted terms admitting an exponentially small estimate in the Sobolev's norm $\| * \|_{W_2^{3/2}}$ on the section of the channel. Neglecting the exponentially small terms and denoting $P^\perp Q_H P^\perp := Q_H^{\perp\perp}$, $P^\perp Q_H P_0 := Q_H^{\perp 0} \dots$, we obtain:

$$v_H^\perp = -\frac{I}{Q_H^{\perp\perp} - \frac{1}{\sqrt{-\triangle-\lambda}}} Q_H^{\perp 0} \overset{0}{v}_H = [I + O(\delta^2)] \sqrt{-\triangle-\lambda}\, Q_H^{\perp 0} \overset{0}{v}_H,$$

because $\| \sqrt{-\triangle-\lambda}\, Q_H^{\perp\perp} \| \leq C\delta^2$ due to embedding theorem $\| u - P_0 u \|_{L_2(\Gamma)} \leq C\ \delta\ \| u \|_{W_2^1(\Gamma)}$. Then

$$\left(Q_H^{0\perp} \sqrt{-\triangle-\lambda} Q_H^{\perp 0} + Q_H^{00} + \frac{\cot\sqrt{\lambda}H}{\lambda} \right) \overset{0}{v}_H = -\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}H} \overset{0}{v}_\Gamma. \tag{31}$$

Substituting the found expression for $\overset{0}{v}_H$ into the second equation (30) and cancelling the exponential small terms we obtain an approximate formula for Q_Γ:

$$Q_\Gamma v_\Gamma \approx \left[-\frac{\cot\sqrt{\lambda}H}{\sqrt{\lambda}} + \frac{1}{[\sqrt{\lambda}\sin\sqrt{\lambda}H]^2} \frac{1}{Q_H^{00} + Q_H^{0\perp}\sqrt{-\triangle-\lambda} Q_H^{\perp 0}} + \frac{\cot\sqrt{\lambda}H}{\sqrt{\lambda}} \right] \overset{0}{v}_\Gamma + \frac{I}{\sqrt{-\triangle-\lambda}} v_\Gamma^\perp.$$

Now we proceed assuming that the channel is both narrow and short, for given λ:

$$\delta^{-1} H << 1,\ \sqrt{\lambda}H << 1,\ \sqrt{-\triangle-\lambda} \approx \sqrt{-\triangle},$$

which allows us to substitute $\sin\sqrt{\lambda}H$ by $\sqrt{\lambda}H$ and assume $\cos\sqrt{\lambda}H \approx 1$. Then we obtain further simplification of the expression for Q_Γ

$$Q_\Gamma v_\Gamma \approx -\frac{Q_H^{00} + Q_H^{0\perp}\sqrt{-\triangle}\, Q_H^{\perp 0}}{1 + \lambda H \left(Q_H^{00} + Q_H^{0\perp}\sqrt{-\triangle}\, Q_H^{\perp 0} \right)} \overset{0}{v}_\Gamma + \frac{I}{\sqrt{-\triangle}} v_\Gamma^\perp \approx [-Q_H^{00} + Q_H^{0\perp}\sqrt{-\triangle}\, Q_H^{\perp 0}] \overset{0}{v} + \frac{I}{\sqrt{-\triangle}} v_\Gamma^\perp.$$

End of the proof.

Later we will calculate asymptotically the first term in the square bracket and prove that the non-constant part of the second term is dominated by the first term.

The formula (22) fore the scattering amplitude contains the values of the scattered waves ψ_{out} on the bottom section of the channel, while normally the values of the scattered waves on the upper section are more convenient for observation. Another aspect of the transport properties of the narrow/ short narrow channel is: the trivial transmission the boundary data of the scattered wave ψ_{out} from the upper section to the bottom section.

Lemma 4.2 *In case of narrow channel the values $\psi_\omega^N|_\Gamma$ on the bottom section of the channel are connected with the values $\psi_\omega^N|_H$ of the scattered wave on the upper section of the of the channel by the formula:*

$$\psi_\omega^N|_\Gamma \approx P_0\, \psi_\omega^N|_\Gamma \approx \cos\sqrt{\lambda}H P_H \psi_\omega^N|_H - (\lambda)^{1/2} P_0 \psi_\omega^N|_H.$$

Proof is based on the connection between the values on the sections of the channel:

$$\begin{pmatrix} \Lambda^{HH} & \Lambda^{H\Gamma} \\ \Lambda^{\Gamma H} & \Lambda^{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \psi_\omega^N|_H \\ \psi_\omega^N|_\Gamma \end{pmatrix} = \begin{pmatrix} \Lambda_H\, \psi_\omega^N|_H \\ 0 \end{pmatrix},$$

the explicit formula DN_δ and the asymptotic (24). The projection P_0 should be applied to both left and right side of both equations. End of the proof.

Note that for short channel, due to Neumann boundary condition on Γ $\Lambda_H \psi_\omega^N|_H = \frac{\partial \psi_\omega}{\partial n}|_H \to 0$ when $H \to 0$, hence

$$\psi_\omega|_\Gamma \approx P_0 \psi_\omega|_\Gamma \approx \cos\sqrt{\lambda} H P_0\, \psi_\omega|_H \approx P_0\, \psi_\omega|_H. \tag{32}$$

5 Scattering amplitude for Helmholtz resonator with the narrow channel

Now we are able to derive the approximate formula for the scattering amplitude of the Helmholtz resonator with short narrow channel. We will base on the exact expression (22) for the amplitude, but substitute into it the asymptotic expressions for $\mathcal{Q}_{out} = Q_\Gamma$ derived in lemma 3.1 and the above asymptotic formula for $\psi_\omega|_\Gamma$ in terms of $\psi_\omega|_H$. This gives the following approximate equation:

$$a(\omega,\nu,\lambda) \approx a^{out}(\omega,\nu,\lambda) -$$

$$\frac{1}{8\pi^3} P_0 \psi_\omega^{out}|_H \left[\mathcal{Q}_{in}(\lambda) + [-Q_H^{00} + Q_H^{0\perp} \sqrt{-\triangle}\, Q_H^{\perp 0}] + \frac{I}{\sqrt{-\triangle}} \right]^{-1} P_0 \psi_\nu^{out}|_H. \tag{33}$$

This asymptotic formula allows major simplifications. Fix a a regular value of the spectral parameter $M < 0$ and introduce the spectral characteristic of the inner problem constructed in form of the spectral series over eigenfunctions ψ_s of L_{in}^N at the center $x_{0\Gamma}$ of the section Γ

$$\sum_{s=1}^{\infty} \frac{\psi_s(x_{0\Gamma})\, \psi_s(x_{0\Gamma})}{(\lambda_s - \lambda)(\lambda_s - M)} := \mathcal{M}_\Gamma(\lambda). \tag{34}$$

The similar spectral characteristics of the outer problem is presented in form of the spectral integral over scattered waves of L_H^N at the center x_{0H} of the upper section:

$$\frac{1}{8\pi^2} \int_0^\infty |k|^2 d|k| \int_{\Sigma_1} d\omega \frac{\bar{\psi}_\omega(x_{0H},|k|)\, \psi_\omega(x_{0H},|k|)}{(|k|^2 - \lambda)(|k|^2 - M)} := \mathcal{M}_H(\lambda) \tag{35}$$

Theorem 5.1 *There exist a real function with asymptotic behavior near origin,* $\gamma(\delta, M) = \gamma_1 \delta^{-1} + \gamma_{ln} \ln \delta^{-1} + \gamma_0$ *defined by the coefficients* $\gamma_1 > 0, \gamma_{ln}, \gamma_0$*, such that the scattering amplitude of the Helmholtz resonator is defined by the approximate formula*

$$a(\omega,\nu,\lambda) \approx a^{out}(\omega,\nu,\lambda) - \frac{1}{8\pi^3} \frac{P_0 \psi_\omega^{out}|_H\ P_0 \psi_\nu^{out}|_H}{\mathcal{M}_H + \mathcal{M}_\Gamma + \gamma(\delta)} \tag{36}$$

Proof We just need to solve the equation:

$$\left[\mathcal{Q}_{in}(\lambda) + [-Q_H^{00} + Q_H^{0\perp} \sqrt{-\triangle}\, Q_H^{\perp 0}] + \frac{I}{\sqrt{-\triangle}} \right] u = P_0 \psi_\nu^{out}|_H.$$

Applying the projections P_0, $P^\perp$ to both parts of this equation we obtain the linear system for $u = (u_0,\, u^\perp)$:

$$P_0 \mathcal{Q}_{in}(\lambda) u_0 + P_0 \mathcal{Q}_{in}(\lambda) P^\perp u^\perp + [-Q_H^{00} + Q_H^{0\perp} \sqrt{-\triangle}\, Q_H^{\perp 0}] u_0 = P_0 \psi_\nu^{out}|_H$$

$$P^{\perp}\mathcal{Q}_{in}(\lambda)u_0 + P^{\perp}\mathcal{Q}_{in}(\lambda)P^{\perp}u^{\perp} + \frac{I}{\sqrt{-\triangle}}u^{\perp} = 0.$$

From the second equation we see that

$$u^{\perp} = -\left[P^{\perp}\mathcal{Q}_{in}(\lambda)P^{\perp} + \frac{I}{\sqrt{-\triangle}}\right]^{-1} P^{\perp}\mathcal{Q}_{in}(\lambda)u_0.$$

Similarly to the corresponding reasoning in lemma 4.1 we conclude that on the selected essential interval Δ of the spectral parameter the operator in the square brackets is invertible if $\lambda \neq \lambda_s$ and $\delta << 1$, so that

$$\| \sqrt{-\triangle}P^{\perp}\mathcal{Q}_{in}(\lambda)P^{\perp} \| \leq C^2 \min_{\lambda_s \in \Delta} |\lambda - \lambda_s|^{-1}\delta^2 < 1,$$

and

$$u^{\perp} = -(1+\delta^2)\sqrt{-\triangle}P^{\perp}\mathcal{Q}_{in}(\lambda)u_0.$$

We also need the estimates on the essential interval Δ:

$$\| \sqrt{-\triangle}P^{\perp}\mathcal{Q}_{in}(\lambda) \| < C \min_{\lambda_s \in \Delta} |\lambda - \lambda_s|^{-1}\delta, \ \| \mathcal{Q}_{in}(\lambda)\sqrt{-\triangle}P^{\perp}\mathcal{Q}_{in}(\lambda) \| < C^2 \min_{\lambda_s \in \Delta} |\lambda - \lambda_s|^{-1}\delta^2.$$

One can see from the second estimate that on the essential interval Δ outside a small neighborhood of the eigenvalues of the inner operator L^N_{in} the term $P_0\mathcal{Q}P_0$ dominates $\| \mathcal{Q}_{in}(\lambda)\sqrt{-\triangle}P^{\perp}\mathcal{Q}_{in}(\lambda)$. Hence, outside the small neighborhood of the eigenvalues the first equation can be rewritten approximately after neglecting the term containing the factor δ^2 as

$$\left[P_0\mathcal{Q}_{in}(\lambda)P_0 - Q^{00}_H + Q^{0\perp}_H\sqrt{-\triangle}\,Q^{\perp 0}_H\right]u_0 = P_0\psi^{out}_{\nu}|_H.$$

All terms of the last express depend essentially on δ. We calculate the asymptotics of the for $\delta \to 0$. Using the symptotics (2) for Green functions $G^N_{in}(x,y)$ on Γ and for $G^N_H(x,y)$ on Γ_H and Hilbert formula we obtain for some regular M in form of spectral series/ integrals:

$$G^N_{in}(x,y) = \frac{1}{2\pi|x-y|} + \alpha_{\Gamma}\ln\frac{1}{|x-y|} + C_{\Gamma} + \sum_{s=1}^{\infty}\frac{\psi_s(x)\,\psi_s(y)}{(\lambda_s - \lambda)(\lambda_s - M)},$$

$$G^N_H(x,y) = \frac{1}{2\pi|x-y|} + \alpha_H\ln\frac{1}{|x-y|} + C_H + \frac{1}{8\pi^2}\int_0^{\infty}|k|^2 d|k|\int_{\Sigma_1} d\omega\frac{\bar{\psi}_{\omega}(x,|k|)\,\psi_{\omega}(y,|k|)}{(|k|^2-\lambda)(|k|^2-M)}. \quad (37)$$

Then the calculation of $P_0 G^N_{in} P_0$, $P_0 G^N_H P_0$, is reduced to integrals over Γ with constant cross-section eigenfunctions $e_0 = (\sqrt{\pi}\delta)^{-1}$

$$\frac{1}{\pi\delta^2}\int_{\Gamma}\int_{\Gamma} G^N_H(x,y)dxdy = \frac{1}{\pi\delta^2}\int_{\Gamma}\int_{\Gamma}\frac{dxdy}{2\pi|x-y|} +$$
$$\frac{1}{\pi\delta^2}\int_{\Gamma}\ln\frac{1}{|x-y|}dxdy + C_H + \frac{1}{8\pi^2}\int_0^{\infty}|k|^2 d|k|\int_{\Sigma_1} d\omega\frac{\bar{\psi}_{\omega}(x,|k|)\,\psi_{\omega}(y,|k|)}{(|k|^2-\lambda)(|k|^2-M)} +$$
$$\frac{1}{8\pi^2}\int_0^{\infty}|k|^2 d|k|\int_{\Sigma_1} d\omega\frac{\bar{\psi}_{\omega}(x_{0H},|k|)\,\psi_{\omega}(x_{0H},|k|)}{(|k|^2-\lambda)(|k|^2-M)} + O(\delta). \quad (38)$$

Here x_{0H} is the center of Γ_H, and the integrals can be reduced to the integrals on the unit disc D_1:

$$\frac{1}{\pi\delta^2}\int_{\Gamma}\int_{\Gamma}\frac{dxdy}{2\pi|x-y|} = \delta^{-1}\frac{1}{\pi}\int_{D_1}\int_{\Gamma}\frac{dxdy}{2\pi|x-y|},$$

$$\frac{1}{\pi\delta^2}\int_{\Gamma}\ln\frac{1}{|x-y|}dxdy = \ln\delta^{-1} + \frac{1}{\pi}\int_{D_1}\ln\frac{1}{|x-y|}dxdy.$$

The integral $(\pi\delta^2)^{-1}\int_\Gamma G^N_{in}(x,y)dxdy$ is calculated in a similar way, but includes the spectral term

$$\sum_{s=1}^{\infty}\frac{\psi_s(x_{0\Gamma})\,\psi_s(x_{0\Gamma})}{(\lambda_s-\lambda)(\lambda_s-M)} := \mathcal{M}_\Gamma(\lambda).$$

The spectral integral in the formula (38) will be denoted by $\mathcal{M}_H(\lambda)$. It remains to calculate the asymptotic of the additional term $Q^{0\perp}_H\sqrt{-\triangle}\,Q^{\perp 0}_H$. Non-trivial contribution to to it is defined only by the polar terms $\frac{1}{2\pi|x-y|}$. Consider the function $\varphi(x) = \int_\Gamma\frac{dy}{2\pi|x-y|}\frac{1}{\sqrt{\pi\delta}} \in W^1_2(\Gamma)$. Due to embedding result quoted just above the formula (31) we have:

$$\mathcal{I}(\delta) = \|\, Q^{0\perp}_H\sqrt{-\triangle}\,Q^{\perp 0}_H\,\| = \langle\sqrt{-\triangle}(\varphi - P_0\varphi),\,(\varphi-P_0\varphi)\rangle \leq \|\,\varphi\,\|^2_{W^1_2}\ \delta. \tag{39}$$

Due to homogeneity the integral can be presented as $\delta\,\mathcal{I}(1)$, where $\mathcal{I}(1)$ is the corresponding integral for $\delta=1$. Terms $(\pi\delta^2)^{-1}\langle(-\triangle)^{1/2}\int_\Gamma\frac{ds}{2\pi|*-s|}, P^\perp\int_\Gamma\ln\frac{1}{|*-s|}ds\rangle := L(\delta)$ containing a combination of the pole and the logarithmic singularity are estimated by $\ln\delta^{-1}$. Contributions from all terms which depend on λ are estimated by δ^β, $\beta>0$ and hence can be neglected. Summarizing obtained estimates we see

$$a(\omega,\nu,\lambda) \approx a^{out}(\omega,\nu,\lambda) - \frac{1}{8\pi^3}\frac{P_0\psi^{out}_\omega|_H\ P_0\psi^{out}_\nu|_H}{\mathcal{M}_H + \mathcal{M}_\Gamma + \gamma(\delta)} \tag{40}$$

where $\gamma(\delta)$ is a real function with prescribed asymptotic behavior near $\delta=0$, $\gamma(\delta) = \gamma_1\delta^{-1} + \gamma_{ln}\ln\delta^{-1} + \gamma_0$. One can see from above reasoning that the first coefficient is standard, the second depends on local geometrical characteristics of the resonator at the entrance and exit of the channel, and the last depends on M accumulates spectral characteristics of the outer and inner problem. We postpone exact calculation of the coefficients $\gamma_1,\gamma_l,\gamma_0$ to the forthcoming publication. End of the proof.

6 Solvable model

In case when the opening is *small*, $\delta\sqrt{\lambda} << 1$, one can assume that both Γ,Γ_H degenerate into points γ, γ_H. Assuming that the channel Ω_δ is *short* $H\sqrt{\lambda}$, we may ignore the wave-process in Ω_δ. Then one can expect that the operator L may be substituted by a solvable model suggested in [11]. The model is constructed based on restriction of the Laplacians L^N_{in}, L^N_H in the inner and outer domains Ω_{in}, Ω_H onto smooth functions vanishing near γ, γ_H. The role of deficiency elements is played by the the Green function $G^N_{in,out}(x,\,y,\,\lambda)$ of L^N_{in}, L^N_H at the poles γ, γ_H. Due to Hilbert identity we have

$$G^N_\lambda(x,\gamma) = G^N_{-M}(x,\gamma) + (\lambda+M)G^N_\lambda(x,*)G^N_{-M}(*,\gamma). \tag{41}$$

Due to (2) the first term of (41) in the right-hand side has the asymptotic for $x\to\gamma$

$$G^N_{-M}(x,\gamma) = \frac{1}{2\pi|x-\gamma|} + g^{\log}_{in}(\gamma).$$

The second term of (41) is continuous at $x=\gamma$

$$g_{in}(\gamma,\lambda) = (\lambda+M)G^N_\lambda(\gamma,*)G^N_{-M}(*,\gamma)$$

and contains the spectral information on the operator L_H^N. Thus we have:

$$G_{in}^N(x, \gamma\,\lambda) = \frac{1}{2\pi|x-\gamma|} + g_{in}^{\log}(\gamma) + g_{in}(\gamma, \lambda) + o(1),\ \ x \to \gamma, \tag{42}$$

see [11, 14, 15, 20]. Similar asymptotic is valid for the Green-function of the outer problem

$$G_H^N(x, \gamma\,\lambda) = \frac{1}{2\pi|x-\gamma|} + g_{out}^{\log}(\gamma) + g_{out}(\gamma, \lambda) + o(1),\ \ x \to \gamma. \tag{43}$$

It is worth to notice that the functions $g_{in}(\gamma, \lambda)$, $g_H(\gamma, \lambda)$ which appear in the asymptotics coincide with the functions $\mathcal{M}_\Gamma$, $\mathcal{M}_H$ introduced in (34,35). The solvable model is constructed as a self-adjoint extension of the orthogonal sum $(L_{in}^N)_0 \oplus (L_H^N)_0$ of the restricted operators. The domain of the extension is obtained via imposing a special boundary conditions, see (44), onto the asymptotic boundary values of elements from the domain of the corresponding adjoint operators $(L_{in}^N)_0^+ \oplus (L_H^N)_0^+$. The boundary forms of the adjoint operators are calculated in terms of the boundary values A, B for instance : $u_{out}(x) = A_{out}^u \left[\frac{1}{2\pi|x-\gamma|} + g_{in}^{\log}(\gamma)\right] + B_{out}^u$, $u_{out} \sim (A_{out}^u,\, B_{out}^u)$ and $v_{out} \sim (A_{out}^v,\, B_{out}^v)$, then

$$\mathcal{J}_{out}(u, v) = \bar{B}_{out}^u\, A_{out}^v - \bar{A}_{out}^u\, B_{out}^v.$$

Similarly the inner boundary form is calculated. The self-adjoint extension is obtained via restriction of the direct sum of the inner and outer adjoint operators onto the Lagrangian plane $\mathcal{L}$ where the sum $\mathcal{J}_{out}(u, v) + \mathcal{J}_{in}(u, v)$ vanishes. This plane is obtained via imposing appropriate conditions on the asymptotic boundary values, for instance

$$\begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix} \begin{pmatrix} B_{out} \\ A_{in} \end{pmatrix} = \begin{pmatrix} A_{out} \\ -B_{in} \end{pmatrix}. \tag{44}$$

The operator L_β^N defined by this boundary condition is self-adjoint. The difficult part of the problem is fitting of the parameters of the model, in particular, the physically reasonable choice of the matrix β_{ik}. It is not clear a-priori if elements of that matrix have any physical meaning. But we guess that the choice of the matrix may depend on "essential" interval Δ of the spectral parameter and is defined by the properties of the resonance eigen-functions which correspond to the resonance eigenvalue(s) λ_0... of the inner operator L_{in} situated inside the interval. We will fit the boundary parameters based on comparison of the explicit expression for the model scattering amplitude and an approximate expression (40).

We choose the Ansatz for the scattered waves of the model operator in form:

$$\psi_\omega^\beta(x) = \psi_\omega(x) + A_H G_H^N(x, \gamma_H, \lambda)$$

in Ω_H, and

$$\psi_\omega^\beta(x) = A_\Gamma G_\Gamma^N(x, \gamma, \lambda),$$

where $G_H^N(x, \lambda)$, $G_\Gamma^N(x, \lambda)$ are, respectively the limit values of the Green functions of L_H^N, L_{in}^N from the upper half-plane: $\lambda = \Lambda + i0$. The asymptotic boundary values of the Ansatz are calculated as

$$A_{out} = A_H,\, B_{out} = \psi_\omega(\gamma_H) + A_H\, g_{out}(\gamma_H,\, \lambda);\ \ A_{in} = A_\Gamma,\, B_{in} = A_\Gamma\, g_{in}(\gamma,\, \lambda).$$

Then, inserting the asymptotic boundary values into the boundary condition (refbconditions) we obtain, after an elementary calculation the expression for A_H:

$$A_H = \frac{\beta_{00}(\beta_{11} + g_{in}) - |\beta_{01}|^2}{g_{in} + \beta_{11} - \beta_{00} g_{out}(\beta_{11+g_{in}}) + |\beta_{01}|^2\, g_{out}}.$$

Comparing with approximate expression (40) for the scattering amplitude, with regard of $g_{in} = \mathcal{M}_\Gamma$, $g_{out} = \mathcal{M}_H$, we conclude that $\beta_{00} = 0$, $|\beta_{01}|^2 = 1$, $\beta_{11} = \gamma(\delta) = \gamma_1 \delta^{-1} + \gamma_{ln} \ln \delta^{-1} + \gamma_0$, where the coefficient in the leading term is defined as $\frac{1}{2\pi^2} \int_{D_1} \int_{D_1} \frac{dxdy}{|x-y|}$, and other coefficients are defined by the local structure of the boundaries $\partial\Omega_H$, $\partial\Omega_{in}$ near the points γ_H, γ.

7 Analytic perturbation procedure with jump-start

Connection between the resonances and convergence of analytic perturbation procedure in celestial mechanics was noticed by Poincare, [40] and intensely discussed by Prigogine and his collaborators, see for instance [41, 42, 43]. We suggested to use the solvable model as the first step in analytic perturbation procedure, see [22, 23]. It can be interpreted as a realization of the dream of Prigogine about the intermediate operator in the perturbation problem for the pair $A_0, A_\varepsilon = A_0 + \varepsilon V$. Prigogine assumed that there exists a function B_0 of the non-perturbed operator A_0 for which the spectral analysis can be done in explicit form, and such that the perturbation series for the pair B_0, A_ε is convergent. In our case the role of the intermediate operator is played by the solvable model - the one-dimensional perturbation of A_0. But the perturbation is chosen locally - for scattering processes in given interval of the spectral parameter - near to the resonance eigenvalue. The essence of our suggestion in simplest case of Lax-Phillips scattering [44] is the following. Assume that A_0, A_ε is a Lax-Phillips pair and $S(p, \varepsilon)$ is the corresponding scattering matrix with the spectral parameter $p = \sqrt{\lambda}$, $\Im p \geq 0$. The scattering matrix in [44] is presented in the upper half-plane $\Im p \geq 0$ in form $S = \Theta\ B$ of a product of a singular factor Θ and the Blaschke-Potapov product B, see [45]. In simplest case of perturbation problem each elementary factor of the product is defined by the position of the corresponding zero (the "resonance") $p_s(\varepsilon)$ and the corresponding orthogonal projection $P_s(\varepsilon)$ which depend on ε analytically:

$$B_s(p) = \left[\frac{p - p_s(\varepsilon)}{p - \bar{p}_s(\varepsilon)}\right] P_s + P_s^\perp.$$

If the order of Blaschke-factors is fixed, then the projections P_s are uniquely defined:

$$B(p) = \prod_s \left\{ \left[\frac{p - p_s(\varepsilon)}{p - \bar{p}_s(\varepsilon)}\right] P_s + P_s^\perp \right\} =: \prod_s B_s. \tag{45}$$

We assume now that for $\varepsilon \to 0$ the resonance $p_1(\varepsilon)$ is approaching the real point κ_1, such that the real and imaginary parts of the resonance are real analytic functions of ε, but other resonances are separated from κ_1. Then the product of them is an analytic function $B^1(p, \varepsilon)$ in a certain vicinity $\omega_1^\varepsilon(0) \times \omega_1^\varepsilon(\kappa_1)$ of $(0, \kappa_1)$ in the appropriate product space. we assume that the singular factor $\Theta = \Theta(p, \varepsilon)$ is also an analytic in the same neighborhood. The important fact is that the Blaschke-factor b_1 *is not analytic* there, because the zero p_1 and the pole $\bar{(p)}_1$ shrink to κ_1 when $\varepsilon \to 0$. Hence we have the decomposition of the scattering matrix into the product

$$S(p, \varepsilon) = \Theta(p, \varepsilon) B^1(p, \varepsilon) \times b_1(p, \varepsilon) := S_1(p, \varepsilon) \times S^1(p, \varepsilon) \tag{46}$$

of analytic and non-analytic factors. The non-analyticity of the factor $S^1(p, \varepsilon) = b_1(p, \varepsilon)$ causes the non-analyticity of the whole product $S(p, \varepsilon)$ and corresponds to the fact of the non-analyticity of the Scattering matrix with respect to the perturbation parameter at the "threshold of creation of resonances". We suggest to modify the perturbation procedure *locally*, eliminating, for instance, the non-analytic factor $S_0^\varepsilon := B_0$ via the "jump-start": by introducing of the *intermediate operator* A_ε^1, which is selected such that $S^1(p, \varepsilon) = b_1(p, \varepsilon)$ is the Scattering matrix for the pair A_ε^1, A_0. It is important that the the operator A_{ε}^{1} can be constructed as a *solvable model*. The complementary factor scattering matrix $S_1(p,\varepsilon)$ of the scattering matrix coincides with the scattering matrix of the pair A_{ε}, A_{ε}^{1}. It is an analytic function in $\omega_1^{\varepsilon}(0)\times\omega_1^{\varepsilon}(\kappa_1)$. In particular, the corresponding scattering amplitude can be decomposed into the power series of on ε converging in $\omega_1^{\varepsilon}(0)\times\omega_1^{\varepsilon}(\kappa_1)$. This is what we call *jump-start of the analytic perturbation procedure* (the term suggested by L. Faddeev). The solvable model we constructed in previous section is one of simplest realization of this construction. Note that the statement concerning analyticity of the complementary factor is true under condition that the jump-start scattering matrix $S_1(p,\varepsilon)$ coincides exactly with the factor $b_1(p,\varepsilon)$ of the scattering matrix. Hence the construction of the solvable model requires exact spectral data κ_1, P_1 of the perturbed operator. Recovering of them may be a difficult problem, comparable with the original spectral problem. But if we do not attempt to construct the convergence of the series of successive approximations for the spectral data of the perturbed operator, but just a reasonably convenient approximation for the spectral data in a " small, but not too small" neighborhood of κ, then some good asymptotic for κ_1, P_1 will serve perfectly. Practical corollaries from this observation will be discussed elsewhere.

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