The product of a Baire space with a hereditarily Baire metric space is Baire

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Abstract. In this paper we prove that the product of a Baire space with a metrizable hereditarily Baire space is again a Baire space. This answers a recent question of J. Chaber and R. Pol.

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A topological space X is called a *Baire* space if for each sequence $(O_n : n \in \mathbb{N})$ of dense open subsets of X, $\bigcap_{n\in\mathbb{N}} O_n$ is dense in X and a Baire space Y is called *barely Baire* if there exists a Baire space Z such that $Y \times Z$ is **not** Baire. It is well known that there exist metrizable barely Baire spaces (see, [2]). On the other hand, it has recently been shown in [1] that the arbitrary product of hereditarily Baire (i.e., each closed subspace of them is Baire) metrizable spaces is again Baire. However, in this same paper the authors lament that they do not know whether there exists a metrizable Baire space X and a metrizable hereditarily Baire space Y such that $X \times Y$ is not Baire. In this paper we resolve this situation. Specifically, we show that if X is Baire and Y is hereditarily Baire and metrizable then $X \times Y$ is also Baire. In order to simplify the appearance of the proof of our main theorem we shall invoke the machinery of topological games. In particular, we shall use the game characterisation of Baireness due to Saint Raymond, [3].

Let (X, τ) be a topological space. On X we consider the *Choquet* game $\mathscr{G}_X$ played between two players α and β. The player β goes first (always!) and chooses a non-empty open subset $B_1 \subseteq X$. Player α must then respond by choosing a non-empty open subset $A_1 \subseteq B_1$. Following this, player β must select another non-empty open subset $B_2 \subseteq A_1 \subseteq B_1$ and in turn player α must again respond by selecting a non-empty open subset $A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1$. Continuing this procedure indefinitely the players α and β produce a sequence $((A_n, B_n) : n \in \mathbb{N})$ of pairs of open subsets called a *play* of the $\mathscr{G}_X$-game. We shall declare that α *wins* a play $((A_n, B_n) : n \in \mathbb{N})$ of the $\mathscr{G}_X$-game if, $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$, otherwise the player β is said to have won. By a *strategy* t for the player β we mean a "rule" that specifies each move of the player β in every possible situation that can occur. More precisely, a strategy $t := (t_n : n \in \mathbb{N})$ for β is a sequence of τ-valued functions such that $\emptyset \neq t_{n+1}(A_1, \ldots A_n) \subseteq A_n$ for each $n \in \mathbb{N}$. The domain of each function t_n is precisely the set of all finite sequences $(A_1, \ldots A_{n-1})$ of length $n-1$ in $\tau \setminus \{\emptyset\}$ with $A_j \subseteq t_j(A_1, \ldots A_{j-1})$ for all $1 \leq j \leq n-1$. (Note: the sequence of length 0 will be denoted by $\emptyset$.) Such a finite sequence $(A_1, A_2, \ldots A_{n-1})$ or infinite sequence $(A_n : n \in \mathbb{N})$ is called a *t-sequence.* A strategy $t := (t_n : n \in \mathbb{N})$ for the player β is called a *winning strategy* if each infinite t-sequence is won by β.

Theorem 1 [3] *Let X be a topological space. Then X is a Baire space if, and only if, β does not have a winning strategy in the Choquet game played on X.*

The following result is proved by a straight-forward induction (on the number of elements of Z) argument.

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Lemma 1 *Let X be a topological space, (Y,d) be a metric space and O be a dense open subset of $X \times Y$. Then given any finite subset Z of Y, $\varepsilon > 0$ and non-empty open subset U of X there exists a finite subset Y' of Y and a non-empty open subset V of U such that $V \times Y' \subseteq O$ and for each $z \in Z$ there exists a $y \in Y'$ such that $d(y,z) < \varepsilon$.*

Theorem 2 *Let X be a Baire space and (Y,d) be a hereditarily Baire metric space. Then $X \times Y$ is a Baire space.*

Proof: Let $(O_n : n \in \mathbb{N})$ be a decreasing sequence of dense open subsets of $X \times Y$. We need to show that $\bigcap_{n\in\mathbb{N}} O_n$ is dense in $X \times Y$. To this end, let U be a non-empty open subset of X and V be a non-empty open subset of Y; we will show that $(\bigcap_{n\in\mathbb{N}} O_n) \cap (U \times V) \neq \emptyset$. To achieve this we will appeal to the game characterisation of Baireness given in Theorem 1. Thus, we shall inductively define a (necessarily non-winning) strategy $t := (t_n : n \in \mathbb{N})$ for the player β in the Choquet game played on X.

Step 1. Choose $y \in V$ and a non-empty open subset $U_\emptyset \subseteq U$ such that $U_\emptyset \times \{y\} \subseteq O_1$. Note this choice is possible since $(U \times V) \cap O_1 \neq \emptyset$. Let $Y_\emptyset := \{y\}$, $Z_\emptyset := Y_\emptyset$ and define $t_1(\emptyset) := U_\emptyset$.

Step 2. For each t-sequence (A_1) of length 1 we can apply Lemma 1 to get a finite subset $Y_{(A_1)}$ of V and a non-empty open subset $U_{(A_1)}$ of A_1 so that:

(i) for each $z \in Z_\emptyset$ there exists a $y \in Y_{(A_1)}$ such that $d(y,z) < 1/2$;

(ii) $U_{(A_1)} \times Y_{(A_1)} \subseteq O_2$.

Then we define

(iii) $Z_{(A_1)} := Z_\emptyset \cup Y_{(A_1)}$;

(iv) $t_2(A_1) := U_{(A_1)}$.

Now suppose that the finite subsets $Y_{(A_1,\ldots A_j)}$, $Z_{(A_1,\ldots A_j)}$ of V, the non-empty open subset $U_{(A_1,\ldots A_j)}$ of A_j and the strategy t_{j+1} have been defined for each t-sequence $(A_1,\ldots A_j)$ of length j, $1 \leq j \leq (n-1)$ so that:

(i) for each $z \in Z_{(A_1,\ldots A_{j-1})}$ there exists a $y \in Y_{(A_1,\ldots A_j)}$ such that $d(y,z) < 1/(j+1)$;

(ii) $U_{(A_1,\ldots A_j)} \times Y_{(A_1,\ldots A_j)} \subseteq O_{j+1}$;

(iii) $Z_{(A_1,\ldots A_j)} := Z_{(A_1,\ldots A_{j-1})} \cup Y_{(A_1,\ldots A_j)}$;

(iv) $t_{j+1}(A_1,\ldots A_j) := U_{(A_1,\ldots A_j)}$.

Step $(n+1)$. For each t-sequence $(A_1,\ldots A_n)$ of length n we can apply Lemma 1 to get a finite subset $Y_{(A_1,\ldots A_n)}$ of V and a non-empty open subset $U_{(A_1,\ldots A_n)}$ of A_n so that:

(i) for each $z \in Z_{(A_1,\ldots A_{n-1})}$ there exists a $y \in Y_{(A_1,\ldots A_n)}$ such that $d(y,z) < 1/(n+1)$;

(ii) $U_{(A_1,\ldots A_n)} \times Y_{(A_1,\ldots A_n)} \subseteq O_{n+1}$.

Then we define

(iii) $Z_{(A_1,\ldots A_n)} := Z_{(A_1,\ldots A_{n-1})} \cup Y_{(A_1,\ldots A_n)}$;

(iv) $t_{n+1}(A_1,\ldots A_n) := U_{(A_1,\ldots A_n)}$.

This completes the definition of $t := (t_n : n \in \mathbb{N})$. Now since X is a Baire space t is not a winning strategy for the player β in the $\mathscr{G}_X$-game. Therefore there exists a t-sequence $(A_n : n \in \mathbb{N})$ where $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$ (i.e., α wins). Choose $x \in \bigcap_{n\in\mathbb{N}} A_n$ and define, for each $n \in \mathbb{N}$, the open sets $(W_n : n \in \mathbb{N})$ so that $\{x\} \times W_n := (\{x\} \times Y) \cap O_n$. Let $Z := \bigcup\{Z_{(A_1,\ldots A_{n-1})} : n \in \mathbb{N}\} \subseteq V$. Then $\overline{Z}$ is a Baire space and by construction $W_n \cap Z$ is dense in Z for each $n \in \mathbb{N}$. Indeed, if $z \in Z$, $n \in \mathbb{N}$ and $\varepsilon > 0$ then we can choose $k \in \mathbb{N}$ sufficiently large so that $n < k$, $1/k < \varepsilon$ and $z \in Z_{(A_1,\ldots A_{k-1})}$. Then there exists a $y \in Y_{(A_1,\ldots A_k)}$ such that $d(y, z) < 1/(k + 1) < \varepsilon$ and $(x, y) \in O_{k+1} \cap (\{x\} \times Y)$, which implies that $y \in W_{k+1} \subseteq W_n$. Thus, $y \in B(z, \varepsilon) \cap (W_n \cap Z) \neq \emptyset$. Next, choose $y \in \bigcap_{n\in\mathbb{N}} W_n \cap V \cap \overline{Z}$, then $(x, y) \in (\bigcap_{n\in\mathbb{N}} O_n) \cap (U \times V) \neq \emptyset$; which completes the proof. ☺

Remark In the previous theorem it is possible to weaken the hypothesis on Y while not affecting the conclusion (i.e., that $X \times Y$ is Baire). For example, if Y is hereditarily Baire and first countable then it is not difficult to see how to modify the proof in order to retain the same conclusion. If one is willing invest a bit more effort then it can be shown that if Y is expressible as a product of hereditarily Baire metric spaces then $X \times Y$ is Baire, despite the fact, that in this particular case, Y is not necessarily obliged to be hereditarily Baire (see [1] for the idea behind this).

References

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