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On Complexity of Lobbying in Multiple Referenda

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Summary: In this paper we show that lobbying in conditions of "direct democracy" is virtually impossible, even in conditions of complete information about voters preferences, since it would require solving a very computationally hard problem. We use the apparatus of parametrized complexity for this purpose.

Key words: lobbying, referendum, parametrized complexity

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1 Direct and Representative Democracy

Countrywide votes on a specific issue are an accepted way of resolving political issues in many countries around the world. Such votes are usually termed "referenda." A referendum gives the people the chance to vote directly on a specific issue. Although people can also make choices at general elections, these elections are usually fought on a number of issues and often no clear verdict on any one issue is delivered. So instead of voting for only representatives, referenda allow citizens to vote directly on some federal matters. In Switzerland and California, for example, referenda are very common.

It is a commonplace that an ideal democratic political system should combine both referenda and representative government. A key issue is the relative weightings of these two ingredients. Referenda are costly. However, in the fully computerized society, to which we are gradually moving, referenda could be cheap and fast. Hence the relative weightings of the two ingredients may be expected to change.

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Another development that might drive this change is the relative simplicity of lobbying such legislative bodies as the American Congress and House of Representatives. In his book, Phillips observes that Washington has become increasingly dominated by an interest-group elite which is now so deeply entrenched and so resistant to change that the proper functioning of government is impossible [15]. He suggests that representative democracy be restored to Athenian direct democracy through the use of referenda.

In this paper we show that lobbying in conditions of "direct democracy" is computationally virtually impossible, even in conditions of complete information about voters' preferences. We use the apparatus of parametrized complexity for this purpose. We envision that computational complexity may play a positive role in voting, protecting the integrity of social choice. Such a role would resemble the situation in public-key cryptography [7] where computational complexity protects the privacy of communication. As far as we know, this is the first paper which considers applications of parametrized complexity to social choice. Previously, complexity issues in social choice were considered in [1–6, 9–11, 13, 14].

2 Parametrized Complexity

For those not familiar with computational complexity, we provide a quick sketch of concepts and terminology. The reader should consult [8, 12] for more details.

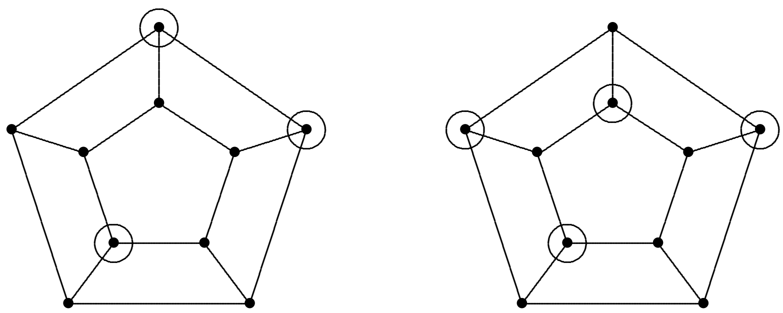
The standard paradigm of complexity theory is embodied in the contrast between P and NP problems. Problems in P are those which admit an algorithm that, given any input x of size n, produces the output $Output(x)$ required by the problem specification in time $O(n^{\alpha})$, that is in time bounded by Cn^{α}, where α and C are constants. The notation P designates the class of problems solvable in polynomial time. Such algorithms are generally considered to be tractable. NP denotes the class of non-deterministic polynomial time solvable problems. For such problems, for each input x, there is a polynomial time algorithm that justifies that $Output(x)$ is indeed the output required by the specifications of the problem. NP contains P and it is believed that $P \neq NP$. The hardest problems in NP are called NP-complete. They are all equivalent in a sense that any such problem can be reduced to an instance of any other NP-complete problem and such reduction can be made in polynomial time. So, if one NP-complete problem can be solved in polynomial time, then all of them can be solved in this way and it would follow that $NP = P$. NP-completeness is therefore taken as evidence of inherent intractability.

However, in reality we are often interested in the tractability of problems when values of a certain parameter k (representing some aspect of the input) are small. In this case we need to undertake the parametrized comlexity analysis as developed by Downey and Fellows in [8]. A problem is said to be in the class FPT (Fixed Parameter Tractable) if there exists an algorithm solving the problem and running in time $f(k)n^c$, where c is a fixed constant and f is an arbitrary computable function. If our problem belongs to this class, then it is tractable for small values of k. Unlike the P versus NP paradigm, here we obtain a hierarchy of parametrized complexity classes

$$FPT = W[0] \subseteq W[1] \subseteq W[2] \subseteq \ldots$$

(see [8] for exact definition of these classes). Being $W[2]$-complete is considered strong evidence that the problem is not tractable even for small values of the parameter. Two $W[2]$-complete problems that will be important later in this paper are described below.

Given a graph $G = (V, E)$ with a set of vertices V and the set of edges E, we say that a subset of the set of vertices $V' \subseteq V$ is a *dominating set* if every vertex in V is adjacent to at least one vertex in V'. If V' is dominating and consists of k vertices we will say that it is a k*-dominating* set. The set V' is called *independent* if no two vertices of V' are adjacent. The picture below shows a 3-dominating set which is not independent and an independent 4-dominating set.



3-dominating set　　Independent 4-dominating set

The k-DOMINATING SET problem takes as input a graph G and a positive integer k, which is considered as parameter. The question asks whether there exists a k-dominating set in G. The k-DOMINATING SET problem has been shown to be $W[2]$-complete by Downey and Fellows (1999). They consider that "k-DOMINATING SET problem represents some fundamental "wall of intractability" where there is no significant alternative to trying all k-subsets for solving the problem." [8], p.15.

The INDEPENDENT k-DOMINATING SET problem is also $W[2]$-complete. The input is the same as for the k-DOMINATING SET, and the question asks whether G has an independent dominating set of size k.

3 Lobbying on a Restricted Budget

We consider the problem faced by an actor that wishes to influence the vote of a certain legislative body or a referendum on a number of issues by trying to exert influence on particular agents. We will refer to this actor as "The Lobby". It is assumed that The Lobby has complete information about agents' preferences. The

Lobby has a fixed budget and has to be selective in choosing agents to distribute the limited budget among them. It is reasonable to assume that the number of agents k that can realistically be influenced is relatively small, and hence this aspect of the input is appropriate as a parameter for the complexity analysis. Hence the use of parametrized complexity developed by Downey and Fellows (1999) is completely appropriate for this problem. This is the first time that parametrized complexity is used in application to social choice studies. Our formal model of the problem is as follows:

The problem: OPTIMAL LOBBYING (OL)

> *Instance:* An n by m 0/1 matrix $\mathcal{E}$, a positive integer k, and a length m 0/1 vector x. (Each row of $\mathcal{E}$ represents an agents. Each column represents a referendum in the election or a certain issue to be voted on by the legislative body. The 0/1 values in a given row represent the natural inclination of the agent with respect to the referendum questions put to a vote in the election. The vector x represents the outcomes preferred by The Lobby.)
>
> *Parameter:* k (representing the number of agents to be influenced)
>
> *Question:* Is there a choice of k rows of the matrix, such that these rows can be edited so that in each column of the resulting matrix, a majority vote in that column yields the outcome targeted by The Lobby?

Proposition 1 OPTIMAL LOBBYING *is* $W[2]$*-hard.*

Proof One of the standard techniques of proving a problem is $W[2]$-hard is to reduce a problem that is already known to be $W[2]$-hard to our problem. We reduce from the $W[2]$-complete k-DOMINATING SET problem. Given a graph $G = (V, E)$, and a positive integer k for which we wish to determine whether G has a k-element dominating set, we produce the following set of inputs to the OPTIMAL LOBBYING problem. (We will assume that the number of vertices n is odd, and that the minimum degree of G is at least k, since k-DOMINATING SET remains $W[2]$-complete under these restrictions.)

- The 0/1 matrix $\mathcal{E}$ consists of two sets of rows, the *top set,* indexed by $V = \{1, ..., n\}$, and the *bottom set,* consisting of $n - 2k + 1$ additional rows. The matrix $\mathcal{E}$ has $n + 1$ columns, with the first column being the *template column*, and the remaining n columns indexed by V.
- The template column has 0's in all of the top set row entries, and 1's in all of the bottom set row entries.
- A column indexed by a vertex v, in the top row positions, has 0's in those rows that are indexed by vertices $u \in N[v]$. In the bottom row positions, the entries can be computed by first setting all of these entries to 1, and then changing (arbitrarily) $n - k - |N[v]| + 1$ of these entries to 0. (This insures that in every column indexed by a vertex the total number of 0's is one more than the total number of 1's.)
- The vector $x = (1, 1, \ldots, 1)$ of length $n + 1$ has a 1 in each position.
- The parameter k remains the same.

We claim that this is a yes-instance of OL if and only if G has a k-dominating set.

One direction is easy. If G has a k-dominating set, then The Lobby corrupts the corresponding agents, or formally, we edit the corresponding rows. With respect to the first (template) column, we thus have the opportunity to change k of the 0's to 1's. Since in the first column, initially, the "1" outcome was losing by $2k - 1$ votes, and since each of these k edit operations decreases the *difference* by 2 (as there is one more 1 and one less 0), the outcome in the first (template) column is a victory for the "1" outcome, by 1. Since the chosen rows for editing represent a dominating set in G, we are similarly able to advantage each vertex column contest by at least 2, and since each of these was losing by one vote, we are able to secure majorities of 1 in every column.

Conversely, suppose the described instance of OL has a solution. Necessarily, the rows chosen to be edited must be in the top set of rows (indexed by vertices of G), since otherwise obtaining a majority of 1's in the first column will not be possible. Any solution that consists of rows in the top set of rows must therefore provide at least one opportunity, for each vertex column (indexed by v), of editing in a row that is indexed by a vertex $u \in N[v]$. Thus, any such solution corresponds to a k-dominating set in G.

Proposition 2 OPTIMAL LOBBYING (OL) *is in* $W[2]$.

Proof One of the standard techniques of proving that a problem is in the class $W[2]$ is to reduce our problem to another problem which is already known to be in $W[2]$. We reduce to the $W[2]$-complete INDEPENDENT k-DOMINATING SET problem [8], page 464. Given an n by m 0/1 matrix $\mathcal{E} = (e_{ij})$, a positive integer k, and a length m 0/1 vector x, proceed as follows:

1. Calculate $w = \lfloor n/2 \rfloor + 1$, which is the number of votes required to pass any particular referendum question.
2. For $1 \leq j \leq m$, let
$$\delta(j) = \begin{cases} \max(0, w - \sum_i e_{ij}), & x_j = 1, \\ \max(0, \sum_i e_{ij} - w + 1), & x_j = 0. \end{cases}$$
3. Since $\delta(j)$ is the number of votes that The Lobby is away from the desired outcome in the jth referendum, when $\delta(j) > k$, for at least one j, we have a trivial negative instance.
4. For each $J = 1, \ldots, m$, let $C_j = \{i \mid e_{ij} \neq x_j,\ 1 \leq i \leq n\}$. Then C_j is the set of voters who are naturally inclined to vote against the interests of The Lobby in the jth referendum.

An OL solution of size k will be any set $K \subseteq \{1, \ldots, n\}$ such that the cardinality of K is k and $|K \cap C_j| \geq \delta(j)$ for every $j = 1, \ldots, m$.

Let us construct the graph G as specified below. The vertex set of G consists of the following vertices:

- x_{ab} is a vertex, for $1 \leq a \leq k$, $1 \leq b \leq n$.
- $x_{a\infty}$ is a vertex, for $1 \leq a \leq k$.
- y_{cd} is a vertex, for $1 \leq c \leq m$, $1 \leq d \leq \binom{k}{k-\delta(c)+1}$.

The edges of G are as follows:

- For every $1 \le a \le k$, the subgraph induced on $\{x_{ab} \mid 1 \le b \le n \text{ or } b = \infty\}$ is complete.
- For every $1 \le b \le n$ (but not $b = \infty$) the subgraph induced on $\{x_{ab} \mid 1 \le a \le k\}$ is complete.
- For every $1 \le c \le m$, let f_c be a bijection from $\{1, 2, \dots, \binom{k}{k-\delta(c)+1}\}$ to the set of all subsets of $\{1, \dots, k\}$ of cardinality $k - \delta(c) + 1$. Then the vertex y_{cd} is connected by an edge to each member of $\{x_{ab} \mid a \in f_c(d),\, b \in C_c\}$.

We will show now that G has a k-Independent Dominating Set S if and only if $(\mathcal{E}, k, x)$ is a positive instance of OL. First, assume that G has a k-Independent Dominating Set S. Then each $x_{a\infty}$ is dominated, and, since it is connected only to vertices x_{ab}, where $1 \le b \le n$, at least one vertex x_{ab} must be in S for each $1 \le a \le k$.As S is of size k, it includes exactly one of the x_{ab} for each a. As S is independent, it cannot include x_{sb} and x_{tb} for $s \ne t$.

Now, let $K = \{b \mid x_{ab} \in S \text{ for some } a\}$. The cardinality of K is k, so, if $|K \cap C_j| \ge \delta(j)$ for every j, then K is an OL solution of size k.

For every j, consider the set $Y_j = \{y_{jd} \mid 1 \le d \le \binom{k}{k-\delta(j)+1}\}$. Since each of these vertices is dominated, some member of $\{x_{ab} \mid a \in f_j(d),\, b \in C_j\}$ is in S for each d. Since $f_j(d)$ ranges over all subsets of $\{1, \dots, k\}$ of cardinality k, at least $\delta(j)$ members of $\{x_{ab} \mid a \in \{1, \dots, k\}, b \in C_j\}$ are in S and therefore at least $\delta(j)$ members of C_j are in K. Thus K is an OL solution.

Conversely, imagine that K is an OL solution of size k. Choose an arbitrary enumeration θ of elements of K and denote $S = \{x_{i\theta(i)} \mid 1 \le i \le k\}$. S is independent, because there is no edge between $x_{i\theta(i)}$ and $x_{j\theta(j)}$ unless $i = j$. Since i ranges over $1, \dots, k$, each vertex x_{ab} is dominated. Since K is an OL solution, for each j at least $\delta(j)$ members of C_j are in K. Thus, by the construction of S, at least $\delta(j)$ members of $\{x_{ab} \mid a \in \{1, \dots, k\}, b \in C_j\}$ are in S, so that some member of $\{x_{ab} \mid a \in f_j(d),\, b \in C_j\}$ is in S for each d, and y_{jd} is dominated for each j and each d. Thus S is an Independent Dominating Set of size k.

Together, the two propositions above give the following complete classification of the parametrized complexity of the problem.

Theorem 1 OPTIMAL LOBBYING *is $W[2]$-complete.*

4 Conclusion

This paper shows that parameterized complexity is a very appropriate tool for analyzing the computational difficulty of problems in social choice. We believe that the methods of parameterized complexity will be especially useful when dealing with problems regarding voting. Indeed, any voting situation stipulates the existence of two parameters: the number of voters n and the number of alternatives m. The sizes of these two parameters are very different. While the number of voters can be, and usually is, very large, the number of alternatives is small, seldom exceeding 20. Hence, the contribution of the relatively small number of alternatives to the complexity of the problem is limited, and this should be reflected in the method of investigation. We believe the best way to do so is to use the conceptual framework of parameterized complexity.

Some 15 years ago, Bartholdi, Tovey and Trick [1] pioneered the study of voting procedures from the viewpoint of complexity theory. In particular, they proved that DODGSON SCORE and KEMENY SCORE are NP-complete and DODGSON WINNER and KEMENY WINNER are NP-hard. The latter two problems were proved to be complete for parallel access to NP [13, 14]. The problems KEMENY SCORE and KEMENY WINNER are Fixed Parameter Tractable. However, the parametrized complexity of DODGSON SCORE and DODGSON WINNER remains open.

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