A New Proof of the Girth - Chromatic Number Theorem

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Abstract

We give a new proof of the classical Erdös theorem on the existence of graphs with arbitrarily high chromatic number and girth. Rather than considering random graphs where the edges are chosen with some carefully adjusted probability, we use a simple counting argument on a set of graphs with bounded vertex degree.

1 Introduction

In 1959 Erdös [3], in a landmark paper, proved the existence of graphs with arbitrarily large chromatic number $\chi(G)$ and girth g. The central idea of Erdös' proof was as follows: we consider random graphs on n vertices where edges occur with some probability p. If p is small the graph will have few short cycles, and if p is large the graph will have no large independent sets. The right choice of p ensures that both happen at the same time in such a way that deleting a vertex from each short cycle leaves the graph with a low independence number, and hence high chromatic number. This technique has been extended in a number of ways, to uniquely k-colourable graphs by Bollobas and Sauer [1], and also to k-critical graphs. In the case of uniquely k-colorable graphs, it has been shown by Emden-Weinert, Hougardy and Kreuter [2] that such graphs exist satisfying order and maximal degree bounds of $k^{12(g+1)}$ and $5k^{13}$ respectively, and given a randomised algorithm which which outputs such a graph with probability $\frac{1}{2}$.

Another variation on the problem has been the replacement of the chromatic number $\chi(G)$ with the star chromatic number $\chi^*(G)$. This is defined as the infimum of the ratios $\frac{m}{d}$ for which G can be coloured with m colours in such a way that adjacent vertices have colours with m-circular distance at least d. (the m-circular distance of colours a, b with $a-b \in \{0, 1, \ldots, m-1\}$

is the minimum of $a - b$ and $m - a + b$.) Work of Steffen and Zhu [7, 8] has shown that there are graphs with arbitrarily high girth and any rational $\chi^*(G)$.

A number of constructive techniques have also been developed in work on this problem. Perhaps the most remarkable is due to Müller [5], who has proven the following. Let natural numbers k and g and a set A are given, and $P_1, \ldots, P_r$ be distinct partitions of A into at most k classes. Then there exists a k-chromatic graph of girth greater than g with A as a subset of its vertex set such that each P_i may be extended to a colouring of G, and these are the only possible colourings. It also follows from the work of Greenwell and Lovàsz [4] on colouring direct products of graphs that there exist graphs with $\chi(G) = k$ and odd girth at least g for any k and g. The results of Müller, and Greenwell and Lovàsz, have been extended to star colourings and star chromatic number by Nešetřil and Zhu [6]. The result we prove here is not new, rather it is the well known Erdös's

Theorem *For any g and k, there exists a graph of girth at least g which is not k-colourable.*

However we feel that the idea of the proof, which is just a simple counting argument on on a set of graphs with bounded vertex degrees, is sufficiently interesting to warrant attention.

2 A New Proof

For some n and d, denote by Γ the set of graphs on n vertices with at most nd edges and degree at most d^2, and whose girth is at least g. Similarly, denote by Ψ the set of graphs on n vertices with at most nd edges and degree at most d^2 which are k colourable. In both Γ and Ψ we consider the edges and vertices of our graphs to be ordered. Let Γ_m and Ψ_m be the subsets of Γ and Ψ consisting of graphs with exactly m edges.

The idea of the proof is as follows: because the degree of our graphs is bounded, the requirement that the girth is at least g becomes a local condition. By this we mean that if we go about constructing graphs in Γ by sucessively adding edges, only a bounded number of choices (as a function of n) are excluded at each step. On the other hand, a given vertex colouring excludes some constant proportion of the edges we may use in our graphs. Therefore for n sufficiently large the k-colourabilty requirement excludes more graphs than the girth requirement and we will have $|\Gamma_{nd}| > |\Psi_{nd}|$, i.e. there is a graph in our class which has girth at least g but is not k-colourable.

We begin by estimating $|\Psi_m|$ from above. Given a colouring C in which the number of vertices with colour i is denoted n_i, by Turàn's theorem the set of allowed edges E_C has maximal cardinality when all n_i are as close to being equal as possible, i.e.

$$\begin{aligned} |E_C| &\leq \binom{n}{2} - k\binom{n/k}{2} \\ |E_C| &\leq \frac{1}{2}n^2(1-\frac{1}{k}) \end{aligned}$$

Therefore for each m

$$|\Psi_m| \leq \frac{1}{2}n^2(1-\frac{1}{k})|\Psi_{m-1}|.$$

Also, $|\Psi_0| = k^n$ as there are this many initial vertex colourings. It then follows inductively that

$$|\Psi_m| \leq k^n(\frac{1}{2}n^2(1-\frac{1}{k}))^m$$

We now recursively estimate $|\Gamma_m|$ from below, by adding an edge to graphs in Γ_{m-1} and giving a lower bound on the proportion of the resulting graphs which are valid. At each step we have no more than nd edges in our graph, which means there exist at most $\frac{2n}{d}$ vertices which will exceed the degree bound of d^2 if we add another edge to them. Therefore there are at least $n(1-\frac{2}{d})$ vertices between which we may add an edge at each step of our graph construction. In order to ensure we do not create a cycle of length g or shorter when we add an edge, the edge may not end at any vertex within distance g of the initial vertex. Because all vertices have degree at most d^2, this excludes no more than $d^2 + d^4 + \ldots + d^{2g} \leq gd^{2g}$ choices. We then have at least

$$\frac{1}{2}n(1-\frac{2}{d})(n(1-\frac{2}{d}) - gd^{2g})$$

choices at each step of our construction, giving us the estimate

$$|\Gamma_m| \geq (\frac{1}{2}n(1-\frac{2}{d})(n(1-\frac{2}{d}) - gd^{2g}))^m.$$

Suppose now that the desired inequality fails, i.e. that

$$(\frac{1}{2}n(1-\frac{2}{d})(n(1-\frac{2}{d}) - gd^{2g}))^{nd} \leq (k^n\frac{1}{2}n^2(1-\frac{1}{k}))^{nd}$$

or

$$\frac{1}{2}n(1-\frac{2}{d})(n(1-\frac{2}{d}) - gd^{2g}) \leq k^{\frac{1}{d}}\frac{1}{2}n^2(1-\frac{1}{k})$$

for all n and d. Then the coefficient of n^2 on the left hand side must be less than or equal to the coefficient on the right hand side, so

$$(1 - \frac{2}{d})^2 \leq k^{\frac{1}{d}}(1 - \frac{1}{k})$$

for all d. However, as $d \to \infty$ the right hand side approaches $1 - \frac{1}{k}$ while the left hand side approaches 1. Therefore for some n and d we have $|\Gamma_{nd}| > |\Psi_{nd}|$, so there exists a graph with girth at least g which is not k-colourable. It may also be shown that a choice of $d = k^3$, $n = 9gk^{6g+1}$ suffices for all $k \geq 3$, while $d = k^2$, $n = 2gk^{4g+1}$ suffices for $k \geq 144$.

References

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