THE IDEAL GENERATED BY σ-NOWHERE DENSE SETS

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ABSTRACT. In this paper, we consider the ideal $\mathscr{I}_\sigma$ generated by all σ-nowhere dense sets in a topological space. Properties of this ideal and its relations with the Volterra property are explored. We show that $\mathscr{I}_\sigma$ is compatible with the topology for any given space, an analogue to the Banach category theorem. Some applications of this result and the Banach category theorem are also given.

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1. INTRODUCTION

Let (X, τ) be a topological space. An *ideal* $\mathscr{I}$ on X is a family of subsets of X such that (i) $B \in \mathscr{I}$, if $B \subseteq A$ and $A \in \mathscr{I}$; (ii) $A \cup B \in \mathscr{I}$, if $A, B \in \mathscr{I}$. If (ii) is replaced by (ii)' $\bigcup_{n<\omega} A_n \in \mathscr{I}$ for any sequence $\langle A_n : n < \omega \rangle$ in $\mathscr{I}$, then $\mathscr{I}$ is called a *σ-ideal*. For any given ideal $\mathscr{I}$ on X, the minimal σ-ideal containing $\mathscr{I}$ shall be called the *σ-extension of $\mathscr{I}$*. An ideal is said to be *proper* if it is not equal to the power set $\mathscr{P}(X)$ of X. All these notions come from the algebra of $\mathscr{P}(X)$ if some appropriate operations are introduced. Ideals in general topological spaces were considered in [K], and a more modern study can be found in [JH].

One connection between an ideal and the topology on a given space arises through the concept of the local function of a subset with respect to the ideal.

Definition 1.1 ([DGR], [JH]). Let (X, τ) be a space, $\mathscr{I}$ an ideal on X.

$$A^*(\mathscr{I}) = \{x \in X : A \cap N \notin \mathscr{I} \text{ for every } N \in \mathscr{N}(x)\}$$

is called the *local function of $A \subset X$ with respect to $\mathscr{I}$*, where $\mathscr{N}(x)$ denotes the collection of all neighbourhoods of x in (X, τ).

The local function operator was used in [DGR] in the investigation of ideal resolvability. Observe that $\mathrm{cl}^*(A) = A \cup A^*(\mathscr{I})$ defines a Kuratowski closure operator, which generates a new topology $\tau^*(\mathscr{I})$ on X finer than τ. It can be easily checked that

$$\mathscr{B}(\mathscr{I}) = \{U \setminus I : U \in \tau \text{ and } I \in \mathscr{I}\}$$

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is a base for the topology $\tau^*(\mathscr{I})$. For general properties of the local function operator and $\tau^*(\mathscr{I})$, we refer readers to [JH].

Ideals have frequently been used in fields closely related to topology, such as, real analysis, measure theory, and descriptive set theory. The following ideals have been of particular interest:

$\mathscr{I}_n$ – the ideal of all nowhere dense sets in (X,τ),
$\mathscr{I}_m$ – the σ-ideal of all meager sets in (X,τ),
$\mathscr{I}_b$ – some σ-ideal consisting of boundary sets in (X,τ),
$\mathscr{I}_0$ – the σ-ideal of all Lebesgue measure zero sets in $\mathbb{R}^n$.

For example, by using $\mathscr{I}_b$, Semadeni [S] established a purely topological generalization of the Carathéodory characterization of functions which are equal to Riemann-integrable functions almost everywhere. The crucial fact Semadeni required is the following: If a set A is locally in $\mathscr{I}_b$ (for each $x \in A$, there is a neighbourhood of x in the subspace A which is a member of $\mathscr{I}_b$) then A is a member of $\mathscr{I}_b$. Requirements similar to this one have been used in many other places. For instance, in Bourbaki's integration theory in locally compact spaces, it is required that a set which is locally negligible is of measure zero. Another interesting result is that if a set is locally in $\mathscr{I}_m$ then it is a member of $\mathscr{I}_m$. This is the Banach category theorem, first proved by Banach for metric spaces in [B].

In [CG] and [GGP], Cao, Gauld, Greenwood and Piotrowski studied the Volterra property. The class of Volterra spaces is closely related to the class of Baire spaces. Cao and Gauld [CG] proved an analogue to the Oxtoby's Banach category theorem stated in [HM], namely, the union of any family of non-weakly Volterra open subspaces is still non-weakly Volterra. Since the ideal $\mathscr{I}_m$ plays an important role in the study of Baire and and other related properties, and in particular the Banach category theorem can be formulated by using the σ-ideal $\mathscr{I}_m$, it is natural to consider which ideal might play a similar role in the study of the Volterra property, and if there exists an ideal analogue to the Banach category theorem for the Volterra property.

In the following we show the ideal $\mathscr{I}_\sigma$ is such an ideal. After discussing some basic properties of $\mathscr{I}_\sigma$ in Section 2, relations between the Volterra property and $\mathscr{I}_\sigma$ are investigated in Section 3, and the compatibility of $\mathscr{I}_\sigma$ with the topology of any given space is established in Section 4. The last section considers applications of the Banach category theorem and its analogue.

2. Preliminaries

In this section, we discuss some basic properties of the ideal generated by all σ-nowhere dense sets in a topological space, where σ-nowhere dense sets are defined as follows:

Definition 2.1. A subset of a space (X,τ) is called *σ-nowhere dense* if it is an F_σ-set with empty interior.

Note that any subset with empty interior is also called a *boundary set*.

Let $\mathscr{I}_\sigma$ be the ideal generated by all σ-nowhere dense sets in (X,τ). It is clear that $\mathscr{I}_n \subset \mathscr{I}_\sigma \subset \mathscr{I}_m$, and that the σ-extension of $\mathscr{I}_\sigma$ is precisely $\mathscr{I}_m$. The following examples show that in general $\mathscr{I}_\sigma$ is distinct from $\mathscr{I}_n$ and $\mathscr{I}_m$ in a space (X,τ), and the topology $\tau^*(\mathscr{I}_\sigma)$ may be strictly between τ and the discrete topology on X.

Example 2.2. Let (X,τ) be the real line $\mathbb{R}$ with the usual topology. Since $\mathbb{Q} \in \mathscr{I}_\sigma \setminus \mathscr{I}_n$, we have $\mathscr{I}_n \neq \mathscr{I}_\sigma$. It follows that $\tau^*(\mathscr{I}_\sigma) \neq \tau$, since $\mathbb{Q}$ is $\tau^*(\mathscr{I}_\sigma)$-closed, but not τ-closed. Suppose that $\{x\}$ is $\tau^*(\mathscr{I}_\sigma)$-open for some $x \in X$, then there exists an open set $U \in \tau$ and an $I \in \mathscr{I}_\sigma$ such that $\{x\} = U \setminus I$. But I is a subset of a countable union of nowhere dense sets, and since X is Baire, this gives a contradiction. Hence $\tau^*(\mathscr{I}_\sigma)$ is not the discrete topology on X. □

Example 2.3. If X is any countably infinite set with the cofinite topology, then any open set is meagre but not in $\mathscr{I}_\sigma$. □

Example 3.9 below gives a Tychonoff space (X,τ) in which $\mathscr{I}_\sigma \neq \mathscr{I}_m$, and in which $\tau^*(\mathscr{I}_\sigma)$ is neither τ nor discrete.

Lemma 2.4. *Let (X,τ) be a topological space, and $m \in \mathbb{N}$. For each family $\{E_i : i < m\}$ of σ-nowhere dense sets, there is another family $\{G_i : i < m\}$ of σ-nowhere dense sets such that $G_i \subset E_i$ for each $i < m$, $\bigcup_{i<m} E_i = \bigcup_{i<m} G_i$, and for each $k < m$, $\operatorname{int}_\tau(\bigcup_{i\le k} G_i) \cap \operatorname{int}_\tau(\bigcup_{i\ge k} G_i) = \varnothing$.*

Proof. Let $G_0 = E_0$. For each $k < m$, let $N_k = \operatorname{int}_\tau(\bigcup_{i\le k} E_i) \cap \operatorname{int}_\tau(\bigcup_{i\ge k} E_i)$, and $G_i = E_i \setminus \bigcup_{k<i} N_k$ for each $0 < i < m$ (Note that $N_0 = \varnothing$, and so $G_1 = E_1$). Observe that for each $i < m$, $\operatorname{int}_\tau G_i \subset \operatorname{int}_\tau E_i = \varnothing$, and since $\bigcup_{k<i} N_k$ is open, G_i is an F_σ-set in X.

Suppose that $x \in \bigcup_{i<m} E_i \setminus \bigcup_{i<m} G_i$. Let $n = \min\{i : x \in E_i\}$. Then $x \in E_n$ and hence $x \in \bigcup_{k<n} N_k$. Suppose $x \in N_j$ and $j < n$, then $x \in \operatorname{int}_\tau \bigcup_{i\le j} E_i$ contradicting the minimality of n. Thus $\bigcup_{i<m} E_i = \bigcup_{i<m} G_i$. Then, it remains to establish that for each $k < m$,

$$\operatorname{int}_\tau\left(\bigcup_{i\le k} G_i\right) \cap \operatorname{int}_\tau\left(\bigcup_{i\ge k} G_i\right) = \varnothing.$$

To this end, we shall show that

$$\operatorname{int}_\tau\left(\bigcup_{i\le k} G_i\right) \cap \operatorname{int}_\tau\left(\bigcup_{i\ge k} G_i\right) \subset G_k$$

for all $k < m$. Suppose that $x \in \operatorname{int}_\tau(\bigcup_{i\le k} G_i) \cap \operatorname{int}_\tau(\bigcup_{i\ge k} G_i)$. Observe that $x \notin \bigcup_{i>k} G_i$ because for each $i > k$,

$$G_i \subset E_i \setminus N_k \subset E_i \setminus \left(\operatorname{int}_\tau\left(\bigcup_{j\le k} G_j\right) \cap \operatorname{int}_\tau\left(\bigcup_{j\ge k} G_j\right)\right).$$

Hence, if $x \in \operatorname{int}_\tau(\bigcup_{i\ge k} G_i)$ then $x \in G_k$. □

The following theorem and its corollaries are useful in the sequel.

Theorem 2.5. *In a topological space (X, τ), any subset of X, that is the union of finitely many σ-nowhere dense sets, can be expressed as the union of exactly two σ-nowhere dense sets.*

Proof. Suppose that $A = \bigcup_{i \leq m}(\bigcup_{n<\omega} C(n,i))$, where each $C(n,i)$ is closed, and $\mathrm{int}_\tau(\bigcup_{n<\omega} C(n,i)) = \varnothing$. For each $i < m$, let $E_i = \bigcup_{n<\omega} C(n,i)$. We will define two σ-nowhere dense sets, D_0 and D_1, such that $A = D_0 \cup D_1$. By Lemma 2.4, we may assume that for each $k < m$,

$$N_k = \mathrm{int}_\tau\left(\bigcup_{i\leq k} E_i\right) \cap \mathrm{int}_\tau\left(\bigcup_{i\geq k} E_i\right) = \varnothing.$$

Now, define $D_0 \subset X$ and $D_1 \subset X$ by

$$D_0 = E_0 \cup (E_1 \setminus \mathrm{int}_\tau(E_0 \cup E_1)) \cup ... \cup \left(E_m \setminus \mathrm{int}_\tau\left(\bigcup_{i\leq m} E_i\right)\right),$$

and

$$D_1 = E_m \cup (E_{m-1} \setminus \mathrm{int}_\tau(E_m \cup E_{m-1})) \cup ... \cup \left(E_0 \setminus \mathrm{int}_\tau\left(\bigcup_{i\leq m} E_i\right)\right).$$

Suppose that $x \in A$. If $x \in E_m$, then $x \in D_1$. Now, suppose that $x \in (\bigcup_{i<m} E_i) \setminus D_0$. Then there exists some $k < m$ such that $x \in E_k$, and hence $x \in \mathrm{int}_\tau\left(\bigcup_{i\leq k} E_i\right)$. Thus $x \notin \mathrm{int}_\tau\left(\bigcup_{i\geq k} E_i\right)$, and so $x \in D_1$. We have shown that $A = D_0 \cup D_1$. Observe that

$$\begin{aligned} D_0 &= \bigcup_{n<\omega}\Bigg\{(E_0 \cap C(n,0)) \cup ((E_1 \setminus \mathrm{int}_\tau(E_0 \cup E_1)) \cap C(n,1)) \\ &\quad \cup ... \cup \left(\left(E_m \setminus \mathrm{int}_\tau\left(\bigcup_{i<m} E_i\right)\right) \cap C(n,m)\right)\Bigg\} \\ &= \bigcup_{n<\omega}\Bigg\{C(n,0) \cup (C(n,1) \setminus \mathrm{int}_\tau(E_0 \cup E_1)) \\ &\quad \cup ... \cup \left(C(n,m-1) \setminus \mathrm{int}_\tau\left(\bigcup_{i<m} E_i\right)\right)\Bigg\}, \end{aligned}$$

and hence D_0, and similarly D_1, is an F_σ-set of (X, τ). In addition, suppose that $W = \mathrm{int}_\tau D_0 \neq \varnothing$. Then, we let

$$k = \min\left\{j \leq m : W \subset \mathrm{int}_\tau\left(\bigcup_{i\leq j} E_i\right)\right\}.$$

For each $y \in W$ and $l \geq k$, $y \notin E_l \setminus \mathrm{int}_\tau(\bigcup_{i\leq l} E_i)$ and hence there exists $j_y < k$ such that $y \in E_{j_y} \setminus \mathrm{int}_\tau(\bigcup_{i\leq j_y} E_i)$. Hence $W \subset \mathrm{int}_\tau(\bigcup_{i<k} E_i)$,

contradicting the minimality of k. This implies that $\mathrm{int}_\tau D_0 = \varnothing$. In a similar way, one can show $\mathrm{int}_\tau D_1 = \varnothing$. Therefore, D_0 and D_1 are σ-nowhere dense sets of (X, τ). □

Corollary 2.6. *For any space (X, τ), $I \in \mathscr{I}_\sigma$ if and only if there are two σ-nowhere dense sets E and F in X such that $I \subset E \cup F$.*

Given a space (X, τ), and a subspace $Y \subset X$, let τ_Y denote the subspace topology on Y, and let $\mathscr{I}_\sigma(Y)$ denote the ideal generated by all σ-nowhere dense sets in the subspace (Y, τ_Y).

Lemma 2.7. *Let (X, τ) be a space, and let $O \subset X$ be an open subspace. Then $\mathscr{I}_\sigma(O) = \mathscr{I}_\sigma \cap \mathscr{P}(O)$.*

Proof. If $I \in \mathscr{I}_\sigma(O)$, then by Corollary 2.6, there are two σ-nowhere dense subsets E and F in the subspace O such that $I \subset E \cup F$. Suppose that $E = \bigcup_{n<\omega} E_n$ and $F = \bigcup_{n<\omega} F_n$, where $\mathrm{int}_O E = \mathrm{int}_O F = \varnothing$, and for each $n < \omega$, E_n and F_n are closed subsets of the subspace (O, τ_O). Then, it is easy to check that $\bigcup_{n<\omega} \mathrm{cl}_\tau E_n \in \mathscr{I}_\sigma$ and $\bigcup_{n<\omega} \mathrm{cl}_\tau F_n \in \mathscr{I}_\sigma$, and hence $I \in \mathscr{I}_\sigma \cap \mathscr{P}(O)$, and so $\mathscr{I}_\sigma(O) \subset \mathscr{I}_\sigma \cap \mathscr{P}(O)$.

Conversely, if $I \in \mathscr{I}_\sigma \cap \mathscr{P}(O)$, then by Corollary 2.6, $I \subset O$ and $I \subset E \cup F$ for two σ-nowhere dense sets in (X, τ). Furthermore, $E \cap O$ and $F \cap O$ are two σ-nowhere dense sets in the subspace (O, τ_O). Hence, $I \in \mathscr{I}_\sigma(O)$, and thus $\mathscr{I}_\sigma \cap \mathscr{P}(O) \subset \mathscr{I}_\sigma(O)$. □

3. The Volterra property and $\mathscr{I}_\sigma$

In this section we explore relationships between the Volterra property and the ideal $\mathscr{I}_\sigma$ of a topological space (X, τ).

Definition 3.1 ([GGP])**.** A space is called *Volterra* (resp. *weakly Volterra*) if the intersection of every two dense G_δ-sets is dense (resp. non-empty).

As a consequence of Theorem 2.5, we obtain Theorem 2.3 of [CG].

Theorem 3.2 ([CG])**.** *A space (X, τ) is weakly Volterra if and only if the intersection of any finitely many dense G_δ-sets is non-empty.*

Proposition 3.3. *Let (X, τ) be a space. Then $\mathscr{I}_\sigma$ is proper if and only if (X, τ) is weakly Volterra.*

Proof. By Corollary 2.6, $X \in \mathscr{I}_\sigma$ if and only if there exist σ-nowhere dense sets D and E such that $X = D \cup E$, if and only if $(X \setminus D) \cap (X \setminus E) = \varnothing$, if and only if X is not weakly Volterra. □

Lemma 3.4 ([GGP])**.** *A space (X, τ) is Volterra if and only if $\mathscr{I}_\sigma$ does not contain a non-empty open subset of (X, τ).*

Corollary 3.5. *Let (X, τ) be a space. If $A \subset X$ and $A^*(\mathscr{I}_\sigma) = X$, then A is a Volterra subspace of (X, τ).*

Corollary 3.6. *Let (X,τ) be a space. If there exists a subset $A\subset X$ such that $A^*(\mathscr{I}_\sigma)=X$ then (X,τ) is Volterra.*

Recall that a space is *resolvable* if it contains two disjoint dense subsets, and a space is *irresolvable* if it is not resolvable. A *strongly irresolvable* space is a space all of whose open subspaces are irresolvable.

The notion of $\mathscr{I}$-resolvability, was introduced in [DGR], and is defined as follows: suppose $\mathscr{I}$ is an ideal on X, a space (X,τ) is *$\mathscr{I}$-resolvable* if there are two disjoint subsets A,B of X such that $A^*(\mathscr{I})=B^*(\mathscr{I})=X$. Observe that by definition, if a space is $\mathscr{I}$-resolvable for some ideal $\mathscr{I}$ then it is resolvable, and if (X,τ) is $\mathscr{I}$-resolvable then it is $\mathscr{J}$-resolvable for any ideal $\mathscr{J}\subset\mathscr{I}$.

In general, there is no relationship between the Volterra property and resolvability, but it turns out that the stronger property, $\mathscr{I}$-resolvability or strong irresolvability has an interesting relationship with the Volterra property.

Proposition 3.7. *If (X,τ) is strongly irresolvable then it is Volterra.*

The proof of this proposition is easy, so we omit it. Also note that the converse is not true. Just consider $\mathbb{R}$ with the usual topology.

Proposition 3.8. *If (X,τ) is $\mathscr{I}_\sigma$-resolvable then it is Volterra.*

Proof. If (X,τ) is $\mathscr{I}_\sigma$-resolvable, by Theorem 3.6 of [DGR], X contains two disjoint $\tau^*(\mathscr{I}_\sigma)$-dense subsets A and B. Suppose $U\in\tau\cap\mathscr{I}_\sigma$. Since $U\in\mathscr{I}_\sigma$, $U\smallsetminus A\in\mathscr{I}_\sigma$ and hence $U\smallsetminus(U\smallsetminus A)=U\cap A\in\tau^*(\mathscr{I}_\sigma)$. Since $(U\cap A)\cap B=\varnothing$, $U\cap A=\varnothing$. Similarly, $U\cap B=\varnothing$ so that $U=\varnothing$, and therefore, by Lemma 3.4, (X,τ) is Volterra. □

The converse of Proposition 3.8 is not in general true as the following example shows.

Example 3.9. Let X be any set with $|X|=\aleph_0$. Van Douwen showed, [D, Example 1.9], that there exists a Tychonoff topology τ on X such that:

- (X,τ) is dense-in-itself;
- (X,τ) is hereditarily irresolvable, that is, every subspace of (X,τ) is irresolvable; and
- there exists a nowhere dense set which is not closed in (X,τ).

Thus (X,τ) is not $\mathscr{I}_\sigma$-resolvable, and by Proposition 3.7, (X,τ) is a Volterra space.

Note also that $\mathscr{I}_\sigma\neq\mathscr{I}_m$ since by Proposition 3.3, $\mathscr{I}_\sigma$ is proper, but since X is Tychonoff, dense-in-itself and countable, $\mathscr{I}_m$ is not proper. Furthermore, $\tau^*(\mathscr{I}_m)\neq\tau^*(\mathscr{I}_\sigma)$. It is clear $\tau^*(\mathscr{I}_m)$ is the discrete topology on X, whereas $\tau^*(\mathscr{I}_\sigma)$ is not. We show in Corollary 5.4 in Section 5 that $\tau^*(\mathscr{I}_\sigma)$ is not even regular. □

It was shown in [DGR, Theorem 3.3] that a space is $\mathscr{I}_m$-resolvable if and only if it has two disjoint dense Baire subspaces. In the light of this result, it is interesting to consider the relationship between $\mathscr{I}_\sigma$-resolvability and the Volterra proerty. Since the relationship between Baire spaces and $\mathscr{I}_m$ resembles that between Volterra and $\mathscr{I}_\sigma$, we might expect an analogous result. However:

Proposition 3.10. *If (X,τ) is $\mathscr{I}_\sigma$-resolvable then it contains two disjoint dense Volterra subspaces.*

Proof. Suppose that A and B are two subsets of X such that $A^*(\mathscr{I}_\sigma) = B^*(\mathscr{I}_\sigma) = X$. It is clear that A and B are dense in X. By Corollary 3.5, A and B are two Volterra subspaces of (X,τ). □

The converse of Proposition 3.10 is not in general true.

Example 3.11 ([GGP])**.** Let E and O be the sets of all even positive integers and all odd positive integers respectively, and let $X = E \cup O$. For any pair $m,n \in X$, we define $U_{mn} \subset X$ by letting

$$U_{mn} = \{x : x \in E \text{ and } x \geq 2m;\ \text{or } x \in O \text{ and } x \geq 2n-1\}.$$

It can easily be checked that $\tau = \{\varnothing\} \cup \{U_{mn} : m,n \in X\}$ is a topology on X. Now, the space (X,τ) enjoys the following three properties:
- E and O are two disjoint dense Volterra subspaces of X.
- The space X is not weakly Volterra.
- The space X is not $\mathscr{I}_\sigma$-resolvable. Since X itself is not a weakly Volterra space, by Proposition 3.3, $\mathscr{I}_\sigma$ is not proper. It follows that for any subset $A \subset X$, $A^*(\mathscr{I}_\sigma) = \varnothing$. Thus, (X,τ) is not $\mathscr{I}_\sigma$-resolvable. □

4. Compatibility of $\mathscr{I}_\sigma$ and its topology

In this section we establish the compatibility of $\mathscr{I}_\sigma$ and τ for any topological space (X,τ).

Definition 4.1. Let (X,τ) be a space. An ideal $\mathscr{I}$ on X is said to be *compatible with* τ, denoted by $\mathscr{I} \sim \tau$, if for each $A \subset X$ and for every point $x \in A$ there is a neighborhood U of x such that $U \cap A \in \mathscr{I}$, then $A \in \mathscr{I}$.

It is easy to see that $\mathscr{I}_n \sim \tau$ for any topological space (X,τ). The next theorem is called the Banach category theorem in the literature, see [K].

Theorem 4.2 ([K])**.** *For any topological space (X,τ), $\mathscr{I}_m \sim \tau$.*

Several equivalent versions and generalizations via ideals of Theorem 4.2 are given in [KPR]. Also, in [KPR, Example 26], it is shown that there is a space (X,τ) and an ideal between $\mathscr{I}_n$ and $\mathscr{I}_m$ on X which is not compatible with τ. However, $\mathscr{I}_\sigma$ is a compatible ideal as our next theorem shows.

Theorem 4.3. *Let (X,τ) be a topological space. Then $\mathscr{I}_\sigma \sim \tau$.*

Proof. Suppose $A \subset X$, and for each $x \in A$ there is an open neighborhood U_x of x such that $A \cap U_x \in \mathscr{I}_\sigma$. By Zorn's lemma, there is a maximal collection $\mathscr{V}$ of pairwise disjoint non-empty open subsets of X such that if $V \in \mathscr{V}$ then $V \subset U_x$ for some $x \in A$. Decompose A as

$$A = \left(A \cap (\bigcup \mathscr{V})\right) \cup \left(A \setminus \bigcup \mathscr{V}\right).$$

Since $\{U_x : x \in A\}$ is an open cover of A, by the maximality of $\mathscr{V}$, we have $A \setminus \bigcup \mathscr{V} \subset \overline{\bigcup \mathscr{V}} \setminus \bigcup \mathscr{V}$. Furthermore, following from the fact

$$\overline{\bigcup \mathscr{V}} \setminus \bigcup \mathscr{V} \in \mathscr{I}_n \subseteq \mathscr{I}_\sigma,$$

we have $A \setminus \bigcup \mathscr{V} \in \mathscr{I}_\sigma$. Thus, in order to show $A \in \mathscr{I}_\sigma$, it will suffice to show $A \cap (\bigcup \mathscr{V}) \in \mathscr{I}_\sigma$.

For each $V \in \mathscr{V}$, $A \cap V \subset A \cap U_x$ for some $x \in A$. Since $A \cap U_x \in \mathscr{I}_\sigma$, then $A \cap V \in \mathscr{I}_\sigma$. By Lemma 2.7, $A \cap V \in \mathscr{I}_\sigma(V) \cap \mathscr{I}_\sigma(\bigcup \mathscr{V})$. Now, for each $V \in \mathscr{V}$, by Corollary 2.6, there are closed sets $D(V, n, i)$ in the subspace (V, τ_V) such that for each $i < 2$, $\operatorname{int}_V \left(\bigcup_{n<\omega} D(V, n, i)\right) = \varnothing$ and

$$A \cap V \subset \bigcup_{i<2} \left(\bigcup_{n<\omega} D(V, n, i)\right).$$

Since $\mathscr{V}$ is a pairwise disjoint family of non-empty open sets, each $D(V, n, i)$ is closed in the subspace $\bigcup \mathscr{V}$, the set $D(n, i) = \bigcup\{D(V, n, i) : V \in \mathscr{V}\}$ is also closed in the subspace $\bigcup \mathscr{V}$. Thus, for each $i < 2$, $\bigcup_{n<\omega} D(n, i)$ is an F_σ-set in $\bigcup \mathscr{V}$. For any fixed $i < 2$, suppose that there is some non-empty open subset W of $\bigcup \mathscr{V}$ such that $W \subset \bigcup_{n<\omega} D(n, i)$. Then there exists some $V \in \mathscr{V}$ such that $W \cap V \neq \varnothing$ and $W \cap V \subset \bigcup_{n<\omega} D(V, n, i)$. This is a contradiction, because $\bigcup_{n<\omega} D(V, n, i)$ has empty interior in the subspace V. Therefore, for each $i < 2$, $\bigcup_{n<\omega} D(n, i)$ is σ-nowhere dense in $\bigcup \mathscr{V}$. Furthermore, following from the fact

$$A \cap \left(\bigcup \mathscr{V}\right) \subset \left(\bigcup_{n<\omega} D(n, 0)\right) \cup \left(\bigcup_{n<\omega} D(n, 1)\right),$$

we obtain $A \cap (\bigcup \mathscr{V}) \in \mathscr{I}_\sigma(\bigcup \mathscr{V})$. Finally, by Lemma 2.7, $A \cap (\bigcup \mathscr{V}) \in \mathscr{I}_\sigma$. Therefore, $A \in \mathscr{I}_\sigma$. □

Corollary 4.4 ([CG]). *In any topological space (X, τ), the union of any family of open non-weakly Volterra subspaces is not weakly Volterra.*

Proof. Let $\mathscr{V}$ be a family of open non-weakly Volterra subspaces of (X, τ). For any $x \in \bigcup \mathscr{V}$, select $V_x \in \mathscr{V}$ such that $x \in V_x$. Moreover, it can shown easily that an open set G in not a weakly Volterra subspace if and only if $G \in \mathscr{I}_\sigma$. So, V_x is an open neighborhood of x such that $V_x \cap (\bigcup \mathscr{V}) \in \mathscr{I}_\sigma$. Therefore, by Theorem 4.3, $\bigcup \mathscr{V} \in \mathscr{I}_\sigma$. This implies that $\bigcup \mathscr{V}$ is not a weakly Volterra subspace of (X, τ). □

5. Applications

In this section, we shall give some applications of the Banach Category Theorem and Theorem 4.3.

5.1. **Feeble continuity of the inversion of a para-topological group.** Recall that a group with a topology is called a *paratopological group* if its multiplication is jointly continuous. In addition, if its inversion is also continuous, then it is called a *topological group.* The Sorgenfrey line is an example of a paratopological group which is not a topological group. Bouziad observed that if the inversion of a paratopological group is quasi-continuous, then it is a topological group (see [KKM, Lemma 4]). The problem arising here is whether the inversion of a paratopological group can have some weak form of continuity under certain circumstances. For example, Guran [G] asked the following question:

Question 5.1 ([G])**.** *Let $(G,\cdot,\tau)$ be a Baire regular paratopological group. Must the inversion of $(G,\cdot,\tau)$ be feebly continuous?*

Recall that a mapping $f:(X,\tau)\to(Y,\mu)$ is *feebly continuous* if for every non-empty open set $V\subset Y$, $\mathrm{int}_\tau[f^{-1}(V)]\neq\varnothing$. The general answer to Question 5.1 is negative as shown by Ravsky in [R]. Next, we shall show that the answer is affirmative under some mild restriction.

Proposition 5.2. *Let $(G,\cdot,\tau)$ be a Baire paratopological group. If (G,τ) has a countable π-network, the inversion of $(G,\cdot,\tau)$ is feebly contiuous.*

Proof. Let $U\subset G$ be a non-empty open set. Pick a point $y\in U$. Since the multiplication is jointly continuous at (e,y), one can choose open sets V and W in (G,τ) such that $e\in V$, $y\in W$ and $V\cdot W\subset U$. It follows that $W^{-1}\cdot V^{-1}\subset U^{-1}$. Now, observe that $(W^{-1})^*(\mathscr{I}_m)$ is a closed set of G. Let $g\in(W^{-1})^*(\mathscr{I}_m)$. Since $g\cdot V$ is an open set containing g, by the construction of $(W^{-1})^*(\mathscr{I}_m)$, $(g\cdot V)\cap W^{-1}\notin\mathscr{I}_m$. In particular, $(g\cdot V)\cap W^{-1}\neq\varnothing$, which implies $g\in W^{-1}\cdot V^{-1}$. Thus,

$$(W^{-1})^*(\mathscr{I}_m)\subset W^{-1}\cdot V^{-1}\subset U^{-1}.$$

Let $\mathscr{N}=\{N_n:n<\omega\}$ be a countable π-network for (G,τ), and let

$$G_n=\{h\in G:N_n\subset h\cdot W\}$$

for each $n<\omega$. It can easily be checked that $G=\bigcup_{n<\omega}G_n$. Since G is Baire, there exists $n_0<\omega$ such that $G_{n_0}\notin\mathscr{I}_m$. Select any point $g_0\in N_{n_0}$. Then, $G_{n_0}\subset g_0\cdot W^{-1}$. It follows that $g_0\cdot W^{-1}\notin\mathscr{I}_m$ and hence, $W^{-1}\notin\mathscr{I}_m$. On the other hand, by the Banach category theorem, $W^{-1}\cap[G\smallsetminus(W^{-1})^*(\mathscr{I}_m)]\in\mathscr{I}_m$. Thus, we can conclude that $(W^{-1})^*(\mathscr{I}_m)\notin\mathscr{I}_m$, otherwise, $W^{-1}\in\mathscr{I}_m$, which gives a contradiction. Consequently,

$$\varnothing\neq\mathrm{int}_\tau[(W^{-1})^*(\mathscr{I}_m)]\subset\mathrm{int}_\tau(W^{-1}\cdot V^{-1})\subset\mathrm{int}_\tau(U^{-1}),$$

and therefore the inversion of G is feebly continuous. □

5.2. **Pointwise $\mathscr{I}_\sigma$-continuity.** Let X,Y be topological spaces, and $\mathscr{I}$ an ideal on X. According to [KP], a mapping $f:X\to Y$ is called *pointwise $\mathscr{I}$-continuous*, if for every $x\in X$ and every neighborhood V of $f(x)$, there exists a neighborhood U of x in X such that $U\smallsetminus f^{-1}(V)\in\mathscr{I}$.

Proposition 5.3. *Let X be a space, and Y a regular space. Then a mapping $f : X \to Y$ is pointwise $\mathscr{I}_\sigma$-continuous if and only if $f \restriction X_0 : X_0 \to Y$ is continuous for some closed $X_0 \subseteq X$ with $X \smallsetminus X_0 \in \mathscr{I}_\sigma$.*

Proof. Sufficiency is in [KP, Theorem 2]. For necessity, let

$$X_0 = X \smallsetminus \bigcup\{G : G \text{ is open in } X \text{ and } G \in \mathscr{I}_\sigma\}.$$

By Theorem 4.3, $X_0 \subset X$ is closed and $X \smallsetminus X_0 \in \mathscr{I}_\sigma$. Furthermore, by [KP, Lemma 4], $f \restriction X_0 : X_0 \to Y$ is continuous. □

Corollary 5.4. *Let (X, τ) be a Volterra space. If the topology $\tau^*(\mathscr{I}_\sigma)$ is regular then $\tau^*(\mathscr{I}_\sigma) = \tau$.*

Proof. Let $f : (X, \tau) \to (X, \tau^*(\mathscr{I}_\sigma))$ be the identity mapping. It is clear that f is pointwise $\mathscr{I}_\sigma$-continuous. By Proposition 5.3, there is a closed set $X_0 \subset X$ such that $X \smallsetminus X_0 \in \mathscr{I}_\sigma$. Since (X, τ) is Volterra, by Lemma 3.4, $X \smallsetminus X_0 = \varnothing$. Thus f is continuous, which implies that $\tau^*(\mathscr{I}_\sigma) = \tau$. □

5.3. **Continuity of additive mappings.** Let (X, τ) be a topological space. We shall say that a subset $A \subset X$ is *$\mathscr{I}_\sigma$-modulo to* a subset $B \subset X$ if $(A \smallsetminus B) \cup (B \smallsetminus A) \in \mathscr{I}_\sigma$. Let

$$\mathscr{M}(\mathscr{I}_\sigma) = \{A \subset X : A \text{ is } \mathscr{I}_\sigma\text{-modulo to some open set of } X\}.$$

It can be easily checked that $\mathscr{M}(\mathscr{I}_\sigma)$ satisfies the following properties:
- $\tau \subset \mathscr{M}(\mathscr{I}_\sigma)$;
- if $A \in \mathscr{M}(\mathscr{I}_\sigma)$, then $X \smallsetminus A \in \mathscr{M}(\mathscr{I}_\sigma)$;
- if $A \subset X$ is closed, then $A \in \mathscr{M}(\mathscr{I}_\sigma)$; and
- if $A, B \in \mathscr{M}(\mathscr{I}_\sigma)$, then $A \cup B \in \mathscr{M}(\mathscr{I}_\sigma)$.

Proposition 5.5. *Let $f : X \to Y$ be an additive mapping from a linear topological space X into a linear topological space Y, and let $A \subset X$. If $A \in \mathscr{M}(\mathscr{I}_\sigma) \smallsetminus \mathscr{I}_\sigma$ and $f \restriction A : A \to Y$ is continuous, then $f : X \to Y$ is continuous.*

Proof. By the additivity of f, it will suffice to show that f is continuous at 0 in X. To do this, let V be an arbitrary neighborhood of 0 in Y. Choose an open neighborhood W of 0 in Y such that $W - W \subset V$. It is clear that $f^{-1}(W) - f^{-1}(W) \subset f^{-1}(V)$. Since $A \notin \mathscr{I}_\sigma$, from Theorem 4.3, there must be a point $x_0 \in A$ such that $A \cap N \notin \mathscr{I}_\sigma$ for every open neighborhood N of x_0. Without loss of generality, we may assume that $x_0 = 0$. Now, as $f \restriction A : A \to Y$ is continuous at 0, there is an open neighborhood O of 0 in X such that $f(A \cap O) \subset W$. Put $B = A \cap O$. Then $B \in \mathscr{M}(\mathscr{I}_\sigma) \smallsetminus \mathscr{I}_\sigma$.

Next, we show that $B - B$ is a neighborhood of 0 in X. To this end, let $U \subset X$ be an open subset with $(B \smallsetminus U) \cup (U \smallsetminus B) \in \mathscr{I}_\sigma$. Since $B \notin \mathscr{I}_\sigma$, U must be non-empty, and thus $U - U$ is a non-empty open neighborhood of 0. Now, take an arbitrary point $x \in U - U$. Then $(x + U) \cap U$ is a non-empty open subset of X. Moreover, we claim that $(x+U)\cap U \notin \mathscr{I}_\sigma$, otherwise $(x+U)\cap U$ would be a weakly Volterra subspace of X. As

$$X = \bigcup\{n \cdot ((x+U)\cap U) : n \in \mathbb{N}\},$$

by Corollary 4.4, the space X itself is not weakly Volterra. From Proposition 3.3, $\mathscr{I}_\sigma$ is not proper, otherwise $B \in \mathscr{I}_\sigma$. This is a contradiction, verifying the claim. In addition, from the previous claim and since

$$(U \setminus B) \cup ((x+U) \setminus (x+B)) \in \mathscr{I}_\sigma,$$

and

$$((x+U)\cap U) \setminus ((x+B)\cap B) \subset (U \setminus B) \cup ((x+U) \setminus (x+B)),$$

we conclude that $(x+B)\cap B \neq \varnothing$, and thus $x \in B-B$. Hence, $U-U \subset B-B$, which implies that $B-B$ is a neighborhood of 0 in X.

Finally, since $B - B \subset f^{-1}(W) - f^{-1}(W) \subset f^{-1}(V)$, we conclude that $f^{-1}(V)$ is a neighborhood of 0 in X. Hence, f is continuous at 0. □

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