Long simulations of the Solar System: Brouwer's Law and Chaos

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Abstract

The numerical integration of Newton's equations of motion for self-gravitating systems, particularly in the context of our Solar System's evolution, remains a paradigm for complex dynamics. We implement Störmer's multistep method in backward difference, summed form and perform arithmetic according to a technique we call "significance ordered computation." We achieve results where the truncation error of our 13^{th} order integrator resides well below machine (double) precision and roundoff error accumulation is random, not systematic. In a previous study, we achieved this "Brouwer's Law" in integrations of the 2-D Kepler Problem. Here we show that such error growth can be attained in 3-D Solar System integrations. Our integrations are such that the positions of the major planets are known with an estimated error of no more than 2° after 10^9 years, a precision unmatched by earlier investigations. Further, we show the outer Solar System is not chaotic, as has previously been reported, but rather computational

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errors in positions grow no faster than $t^{3/2}$ conforming with existing models for stochastic error growth in an otherwise well-behaved ordinary differential equation system.

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1 Introduction

Celestial mechanics in general, and Solar System dynamics in particular, offers profound challenges to both applied and computational mathematicians. As the Sun and planets formed, planetesimals, solid bodies rich in metals, silicates and possibly ice, underwent a violent dynamical birth, often resulting in their amalgamation or their accretion into the planets, or being gravitationally ejected from the Solar System. This winnowing process depended upon the planetesimals' energy and angular momentum. Asteroids are found primarily between the orbits of Mars and Jupiter, and at the leading and trailing Lagrange points of Jupiter. Comets reside in both the Edgeworth-Kuiper Belt and the Oort Cloud—beyond the realm of the planets. Planetesimals that survived this sorting process represent a class of objects whose trajectories reside near separatrices that distinguish them from objects that ultimately fell into the Sun and planets, or were catapulted away, and hold essential clues to the Solar System's origin and evolution. Planetesimal trajectories can reside, therefore on the edge on chaos and the Solar System's dynamics requires very special care.

Several studies, for example [4], [5], [6], [3] and [10], have reported the results of billion-year integrations of planets and planetesimals in the Solar System. Our ability to track the complex five billion year history of the Solar System is highly contingent upon our resourcefulness in following accurately the trajectories of planetesimals living on the edge of chaos. We [7] reported test results for a modified version of the Störmer method (for example [8]

p. 462, [9] p. 291) integrating the 2-D Kepler Problem. The integrator used a backward difference, summed form implementation and a technique we call "significance-ordered computation." We performed two sets of 10 million orbit integrations, with orbital eccentricities of 0.05 and 0.5. For both sets of integrations, sixteen sets of initial conditions with different initial orbital longitudes were chosen, and the RMS errors in energy and longitude calculated. We showed the error in the system energy grew approximately as $n^{1/2}$ and the error in the longitude as $n^{3/2}$, where n is the number of steps and is directly proportional to the CPU time. In other words, the accumulated error was as would be expected in the absence of *systematic* error in the integration scheme.

When an integration technique is applied to Solar System problems, its accuracy is typically examined using the error growth in the total system energy and other conserved quantities (e.g., in an action-angle formulation) such as the longitude or phase difference of the planets; see, for example, [12], [13], [15], [17]. However, it is often insufficient to minimize only the error in the energy or other conserved quantities. It is also important to minimize the phase error. For example, while the orbits of the planets Neptune and Pluto cross, the planets are in a 3:2 orbital resonance that keeps them from approaching each other. If the numerical error in the angular position of the planets is too large, Neptune would eventually have a close encounter and either scatter or collide with Pluto.

2 Solar System Integration Results

We performed sixteen different integrations of the planets, except Mercury whose mass was absorbed into that of the sun. For each of these sixteen integrations, we integrated for a time interval equivalent to 2^i Venus orbits, where i is an integer from 0 to 18. We used the final positions and velocities of the Sun and planets from the forward integrations as starting conditions to integrate *backwards* in time. At the end of each backward integration, we calculated the relative error in the energy and the angular position of the entire system. These forward and backward integrations represent excellent method-independent tests of an integrator's performance in a nonlinear regime. Since the errors in an integrated are continually being added, both deterministic (from truncation error) and stochastic (from roundoff error), and since these error sources are statistically independent, no cancellation of error occurs. The initial planetary positions and velocities for these integration were taken from the Jet Propulsion Laboratory ephemeris DE245 which ranges over approximately 300 years. We extracted sixteen sets of initial conditions at random multiples of ten years.

The total energy of a gravitationally-interacting N body system is

$$E = \frac{1}{2}\sum_{i=1}^{n} M_i v_i^2 - \sum_{j=1}^{n-1}\sum_{i=j+1}^{n} G\frac{M_i M_j}{r_{ij}} \tag{1}$$

where M_i, $i = 1, \ldots, n$, is the mass of the i-th body, v_i its center-of-mass speed, and G the universal gravitational constant. We are interested in the relative energy error of our integration method, or $\Delta E/E$, where

$$\frac{\Delta E}{E} = \frac{E - E_0}{E_0} = \frac{GE - GE_0}{GE_0} \tag{2}$$

We employ the latter form since we know the product of G and the masses with much greater precision than we know G or the masses separately.

For our integrations of the planets, we also examine the angular position error of the planets. Given the initial position for a planet $\mathbf{r}_i$, and its final position $\mathbf{r}_f$, we define the angular position error λ as:

$$\lambda = \arcsin\left(\frac{\|\mathbf{r}_i \times \mathbf{r}_f\|}{\|\mathbf{r}_i\|\|\mathbf{r}_f\|}\right) \tag{3}$$

where $\|\cdot\|$ denotes the L_2 norm. If our computations had no truncation or roundoff error, we would expect that these forward and backward integrations would yield $\mathbf{r}_i = \mathbf{r}_f$. Thus, λ is a useful measure of the accumulated error present in these calculations.

The relative energy error for our integration of the planets and Sun is plotted in Fig. 1. The error, consistent with our 2-D Kepler problem runs, grows at a rate less than $n^{1/2}$ and after 524,288 orbits of Venus (approximately 322,000 years) we have a total relative energy error of less than 2.44×10^{-12}. The quantitative value of the difference between our observed error growth and $n^{1/2}$ is likely due to the finite number of integrations we have performed; our previous paper showed this difference was to be expected.

The angular position errors for the terrestrial planets are plotted in Fig. 2, and those for the jovian planets in Fig. 3. The angular position error for Pluto, while not shown, grew as $n^{1.45}$, and at the end of the run had an angular position error of less than 10^{-5} radians. In fact, the angular position errors for all planets grow at rates less than $n^{3/2}$. At the end of the integrations, the angular position error for Jupiter is approximately 2.1×10^{-7} radians. Extrapolated to one billion years, this yields an error of less than one degree. By comparison, [12] reported an error in Jupiter's longitude of 0.83 degrees over five million years; [1] list their error in Jupiter's position as 100 degrees in 100 Million years.

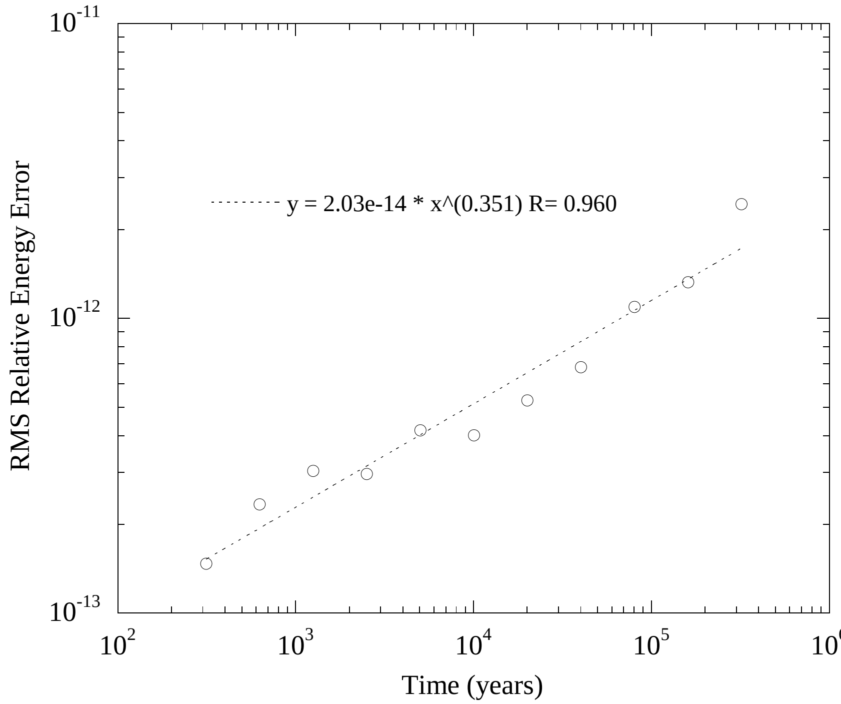

Figure 1: Energy error for the entire Solar System.

Further tests were based upon integrations of the outer Solar System, adding the mass of the terrestrial planets to that of the Sun. For 16 different sets of initial conditions generated from the JPL DE245 ephemeris we integrated the trajectories of the jovian planets for a time interval equivalent to 2^i Jupiter orbits, where i is an integer between 0 and 25. At the end of each integration, we used the positions and velocities of the Sun and planets as starting conditions to integrate backwards in time.

Figure 4 shows the relative RMS energy error for the entire system. We observe the energy error grows as $n^{0.48}$, very nearly $n^{1/2}$, indicating the absence of systematic error growth. Given that we are using a finite number of

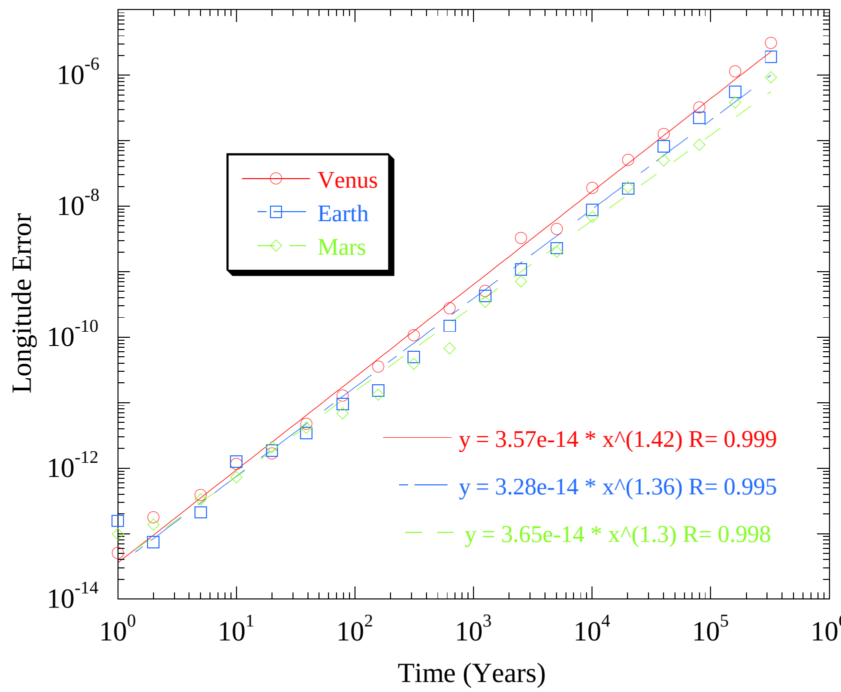

Figure 2: Angular position error for the terrestrial planets.

integrations, our results are in good agreement with theory.

Figure 5 shows the RMS angular position errors for all of the jovian planets. The errors planets grow at rates nearly $n^{3/2}$ and after 2^{26} Jupiter orbits (2^{25} orbits forward and 2^{25} orbits backwards, nearly 800 million years), the error for all planets is less than 1.9×10^{-2} radians ($\approx 1.09°$). Extrapolated, the angular position error after 10^9 years is less than two degrees for all planets

Apart from the fact that these results show Brouwer's Law can be achieved in a practical simulation, they point to another clear implication. Previous studies such as in [16] have reported the outer Solar System is chaotic. Chaotic systems are characterized by the exponential divergence of nearby trajectories and have both stable and unstable manifolds. In particular, suppose we select different initial conditions and compute the orbit for some length of time and then reverse the direction of the integration in order to identify how close we are upon return to our initial conditions. During the course of the computation, we expect that truncation error and roundoff error will cause the computed solution to depart from the exact solution associated

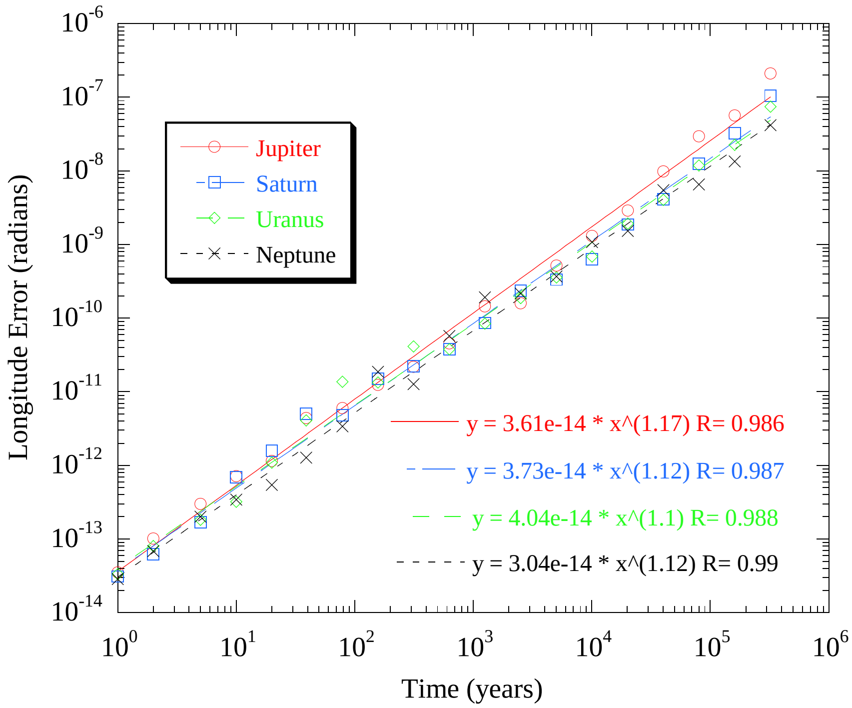

Figure 3: Angular position error for the jovian planets.

with the initial conditions. If chaotic behavior is to be expected for that set of initial conditions —i.e. an unstable manifold is present— the orbit will show exponential divergence from the exact solution going both forward and backward in time and will return to the starting point with an error that has grown exponentially in time. The rate the error grows with time will be characterized by the same Lyapunov time which characterizes the chaotic character of the problem. If chaotic behavior is *not* to be expected— i.e. no unstable manifold exists—the divergence of the computed solution upon return from its initial conditions will not depend upon the elapsed time in an exponential fashion, but will vary as a power-law in elapsed time owing to the cumulative effects of truncation and roundoff error. If an unstable manifold was present, only a set of measure zero of initial conditions (in the presence of computational error) would not show exponential divergence. None of our sixteen randomly selected initial conditions showed chaotic behaviour. Our results agree with those of [11] who performed long integrations and found that planetary motions were stable.

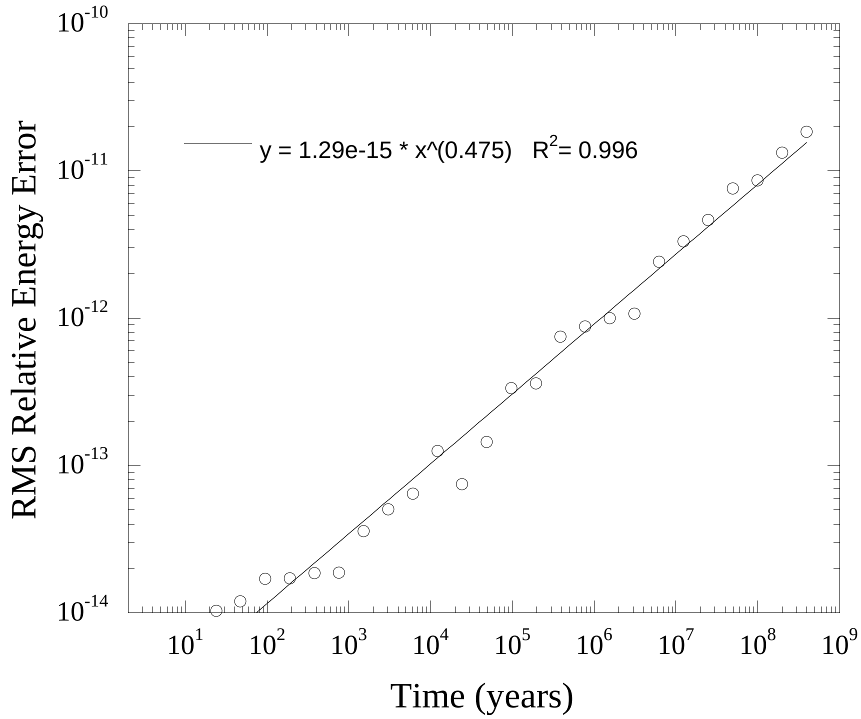

Figure 4: Energy error for the outer Solar System.

3 Conclusion

All of our N-body tests manifested a quantitatively similar behavior in error growth as the energy and longitude error obtained in the 2-D Kepler problem, and all manifested error growth diagnostic of an integrator whose primary source of error is stochastic roundoff error. In addition to showing that Brouwer's Law is attainable in a 3-D integration of the Solar System, we have shown that previous claims of outer Solar System chaos are questionable.

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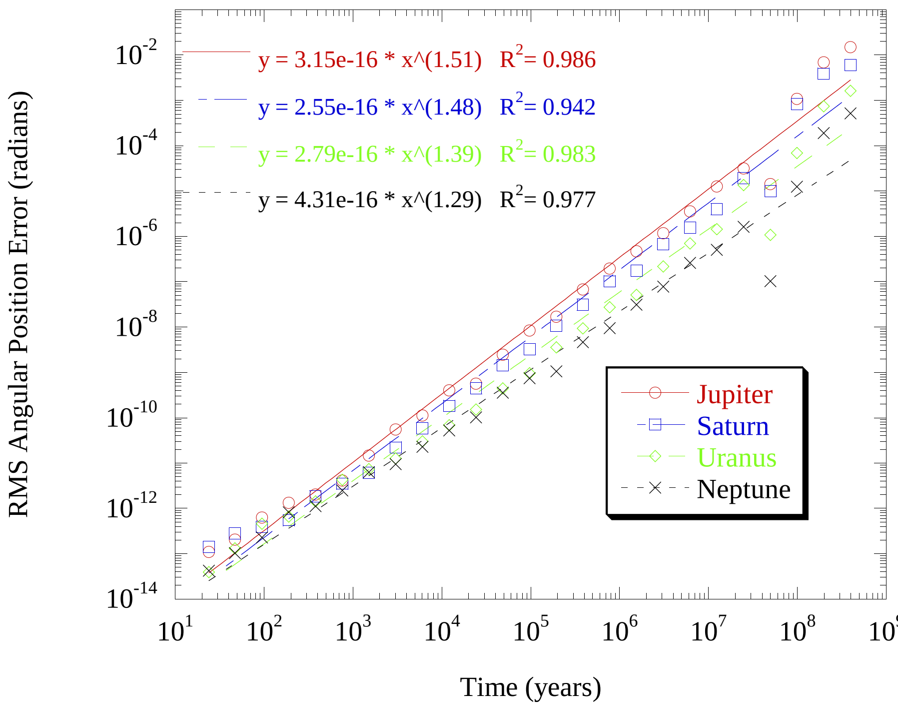

Figure 5: Angular position error for the outer Solar System.

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