Condition for the Discreteness of the Laplacean on a Manifold

Mark Harmer and Boris Pavlov
Department of Mathematics
University of Auckland
New Zealand

Abstract

Here we propose a simple condition for the compactness of the resolvent of the Laplace-Beltrami operator on a class of smooth Riemannian manifolds.

1 Introduction

Some work has been done on finding conditions for the discreteness of operators on manifolds, see for instance [5, 2] and references therein. Related results may be found in [3]. Here we prove a simple condition for the compactness of the resolvent of the Laplace-Beltrami operator on a class of smooth manifolds of arbitrary dimension. Our proof takes a lot from the work in [1] and at the centre of the proof is a weighted Hardy inequality, independently discovered by the authors but which is a specific case of the inequality in [4].

2 Condition for Discreteness

Consider a smooth n-dimensional manifold $\mathcal{M}$ which is both non-compact and has finite volume. The cases where $\mathcal{M}$ is compact or has infinite volume are easily disposed of: for compact $\mathcal{M}$ we immediately have that the Laplacean is discrete and for infinite volume we immediately have that it has a branch of continuous spectrum.

We consider the class of manifolds for which the following assumption holds: the non-compact part of $\mathcal{M}$ can be written as a finite union of disjoint sets $\mathcal{M}_{\text{nonc}} = \bigcup_i U_i$ and furthermore we assume that on each U_i we can find coordinates so that the metric is expressed

$$|dl|^2 = ds^2 + \sigma^2(s) \, |d\xi|^2 \ .$$

Here $|d\xi|^2$ is the metric on a compact $n-1$-dimensional manifold $\mathcal{K}$ and the non-compact region is parameterised by

$$U_i = \{(s, \xi) \in (0, \infty) \times \mathcal{K}\} \ .$$

It is clear from the definition that the coordinate s measures arc-length along geodesics. The condition for finiteness of the volume is equivalent to the integrablity of σ^{n-1} on the semi axis. The form of volume may be written

$$d\mu = \sigma^{n-1}(s)\, ds\, d\xi^{n-1} \tag{1}$$

where $d\xi^{n-1}$ is the form of volume on $\mathcal{K}$. The Laplace-Beltrami operator is easily seen to have the form

$$\mathcal{L} \equiv \sigma^{1-n} \frac{\partial}{\partial s} \sigma^{n-1} \frac{\partial}{\partial s} + \Delta_\xi$$

where again Δ_ξ is the Laplacean on $\mathcal{K}$.
Now consider the Hilbert space

$$H = L^2\left(\mathcal{M}, d\mu\right)$$

of square integrable functions on $\mathcal{M}$ with respect to the volume form with inner product

$$\langle f, g \rangle_{\mathcal{M}} = \int_{\mathcal{M}} f \bar{g}\, d\mu \, .$$

It is well known that for a suitable domain $\mathcal{L}$ is a selfadjoint operator in H. The Dirichlet integral of $\mathcal{L}$ is $\langle \mathcal{L} f, f \rangle_{\mathcal{M}}$. Using the coordinates defined above it is clear that on each non-compact U_i the Dirichlet integral takes the form

$$\int_{U_i} \left(|f_s|^2 + |\nabla_\xi f|^2 \right) d\mu$$

where the subscript indicates differentiation and ∇_ξ is the divergence on $\mathcal{K}$. To state our condition for discreteness we define a new measure ν on each of the U_i. Really for subsets

$$V_{i,a,\mathcal{A}} = \{(s, \xi) \in (a, \infty) \times \mathcal{A}\} \subset U_i \, ,$$

where $\mathcal{A} \subset \mathcal{K}$, we define

$$\nu\left(V_{i,a,\mathcal{A}}\right) = |\mathcal{A}| \left(\int_0^a \frac{ds}{\sigma^{n-1}(s)} \right)^{-1} \, . \tag{2}$$

Here $|\mathcal{A}|$ is the measure of $\mathcal{A}$ in $\mathcal{K}$. Let us denote $V_{i,s} = V_{i,s,\mathcal{K}}$. We note that ν, like μ, is smooth with respect to Lebesque measure in the coordinates and finite on subsets $V_{i,s,\mathcal{A}}$ for $s > 0$ (however, unlike μ, it is not finite in neighbourhoods of $s = 0$). Our theorem is written in terms of these measures.

Theorem 2.1 *The Laplace-Beltrami operator on $\mathcal{M}$ has compact resolvent iff in each non-compact neighbourhood U_i*

$$\lim_{s\to\infty} \frac{\mu\left(V_{i,s}\right)}{\nu\left(V_{i,s}\right)} = 0\,.$$

For its proof we need the following weighted Hardy inequality which is written in terms of the measures μ and ν with the dependence on the coordinates ξ on the sphere dropped.

Lemma 2.1 *Given $d\mu$ and defining the associated measure $d\nu$ as above we have the following inequality*

$$\int_0^\infty f^2(s)\,d\nu(s) \le 4\int_0^\infty \left(f_s(s)\right)^2\,d\mu(s)\,. \tag{3}$$

for functions vanishing in a neighbourhood of the origin.

The proof of this lemma may be found using standard techniques. For details and a more general formulation see the paper by Muckenhoupt [4].

Theorem 2.2 *The set*

$$N = \left\{f;\ |f|^2_{\mathcal{M}} = \langle f,f\rangle_{\mathcal{M}} < c, \langle \mathcal{L}f,f\rangle_{\mathcal{M}} < c\right\}$$

is compact in H iff in each non-compact neighbourhood U_i of $\mathcal{M}$

$$\lim_{s\to\infty} \frac{\mu\left(V_{i,s}\right)}{\nu\left(V_{i,s}\right)} = 0\,. \tag{4}$$

Proof: ($\Leftarrow$) It is not difficult to see that the set

$$N_a = \left\{f;\ |f|^2_{\mathcal{M}_a} < c, \langle \mathcal{L}f,f\rangle_{\mathcal{M}_a} < c\right\}$$

is compact in $L^2(\mathcal{M}_a, d\mu)$ where

$$\mathcal{M}_a = \mathcal{M}\setminus\bigcup_i V_{i,a}$$

is a compact set. Consequently we just need to show that the 'tails'

$$\int_{V_{i,a}} |f|^2 d\mu$$

of $f \in N$ go to zero uniformly as $a \to \infty$.
Without loss of generality we may assume that $f \in N$ is real and smooth and, since we are only interested in the behaviour of the tails, that f is zero in a neighbourhood of $s = 0$. Writing

$$f^2(s) = 2\int_0^s f(r)f_s(r)\,dr\,,$$

where for now we ignore the dependence on ξ, we get

$$\begin{aligned}
\int_a^\infty f^2(s)\,d\mu(s) &= 2\int_a^\infty \int_0^s f(r)f_s(r)\,dr\,d\mu(s) \\
&= 2\left[-\int_s^\infty d\mu(r)\int_0^s f(t)f_s(t)\,dt\Big|_{s=a}^{s=\infty} + \int_a^\infty\int_s^\infty d\mu(r)\, f(s)f_s(s)\,ds\right] \\
&= 2\left[\int_a^\infty d\mu(r)\int_0^a \frac{\nu(t,\infty)}{\nu(t,\infty)} f(t)f_s(t)\,dt + \right.\\
&\qquad \left. + \int_a^\infty \nu^{-1}(s,\infty)\int_s^\infty d\mu(r)\, f(s)f_s(s)\nu(s,\infty)\,ds\right] \\
&\le 2\left[\nu^{-1}(a,\infty)\int_a^\infty d\mu(r)\int_0^a |f(t)f_s(t)|\nu(t,\infty)\,dt + \right.\\
&\qquad \left. + \sup_{s\ge a}\left(\nu^{-1}(s,\infty)\int_s^\infty d\mu(r)\right)\int_a^\infty |f(s)f_s(s)|\nu(s,\infty)\,ds\right] \\
&\le 2\sup_{s\ge a}\left(\nu^{-1}(s,\infty)\int_s^\infty d\mu(r)\right)\int_0^\infty |f(s)f_s(s)|\nu(s,\infty)\,ds\,.
\end{aligned}$$

Here we have used the notation

$$\nu(s,\infty) = \left(\int_0^s \frac{dr}{\sigma^{n-1}(r)}\right)^{-1}.$$

We note from (2) that

$$d\nu(s) = \frac{\nu^2(s,\infty)}{\sigma^{n-1}(s)}\,ds\,,$$

where we emphasise that we have ignored the dependence on ξ. Then the right hand side of the inequality becomes

$$\begin{aligned}
\int_0^\infty |f(s)f_s(s)|\nu(s,\infty)\,ds &= \int_0^\infty \frac{|f(s)|\nu(s,\infty)}{\sigma^{\frac{n-1}{2}}(s)}\,|f_s(s)|\sigma^{\frac{n-1}{2}}(s)\,ds \\
&\le \left[\int_0^\infty |f(s)|^2\,d\nu(s)\right]^{\frac{1}{2}}\left[\int_0^\infty |f_s(s)|^2\,d\mu(s)\right]^{\frac{1}{2}} \\
&\le 2\int_0^\infty |f_s(s)|^2\,d\mu(s)
\end{aligned}$$

thanks to the Hardy inequality (3). Integrating over the angular coordinates the inequality becomes

$$\int_{V_{i,a}} f^2(s)d\mu \le 4\sup_{s\ge a}\frac{\mu\left(V_{i,s}\right)}{\nu\left(V_{i,s}\right)}\langle\mathcal{L}f,f\rangle_{\mathcal{M}}\,.$$

Since $f \in N$ the Dirichlet integral is bounded and by the hypothesis the supremum can be made as small as desired by making a large so we are done.
($\Rightarrow$) We proceed by contradiction assuming that the limit (4) is strictly positive. We claim that this implies the existence of a sequence $\{s_l\}$ such that

$$\int_0^{s_l} \frac{ds}{\sigma^{n-1}} \int_{s_l}^{\hat{s}_l} \sigma^{n-1} ds \geq c_0 > 0 \tag{5}$$

where $\hat{s}$ is the unique solution of $z(\hat{s}) = 2z(s)$ and

$$z(s) = \int_0^s \frac{dr}{\sigma^{n-1}} \, .$$

Really by contradiction we suppose that given $\epsilon > 0$ we can find S such that for all $s > S$

$$\int_0^s \frac{ds}{\sigma^{n-1}} \int_s^{\hat{s}} \sigma^{n-1} ds < \epsilon \, .$$

Then

$$\begin{aligned}
\frac{\mu\left(V_{i,s}\right)}{\nu\left(V_{i,s}\right)} &= \int_0^s \frac{ds}{\sigma^{n-1}} \int_s^{\infty} \sigma^{n-1} ds \\
&= \int_0^s \frac{ds}{\sigma^{n-1}} \left(\int_s^{\hat{s}} \sigma^{n-1} ds + \int_{\hat{s}}^{\hat{\hat{s}}} \sigma^{n-1} ds + \cdots \right) \\
&= z(s) \int_s^{\hat{s}} \sigma^{n-1} ds + \frac{1}{2} 2z(s) \int_{\hat{s}}^{\hat{\hat{s}}} \sigma^{n-1} ds + \cdots \\
&\leq \epsilon + \frac{1}{2}\epsilon + \cdots + \frac{1}{2^n}\epsilon + \cdots = 2\epsilon
\end{aligned}$$

where we have used $2z(s) = z(\hat{s})$, $4z(s) = z(\hat{\hat{s}})$, ... But this contradicts our assumption that (4) is not zero giving the existence of $\{s_l\}$.
Let us take a subsequence $\{s_{l_k}\}$ such that

$$z\left(s_{l_{k+1}}\right) > 3\, z\left(s_{l_k}\right) \, . \tag{6}$$

We drop the extra subscript and denote this sequence by $\{s_k\}$ and also put $z_k = z(s_k)$. Let us choose a smooth real function η with support $(\frac{3}{4}, \frac{9}{4})$ which is equal to one on $[1, 2]$. Then we construct the sequence of functions

$$u_k(s, \xi) = \sqrt{z_k}\, \eta \left(\frac{z(s)}{z_k} \right) \, .$$

The Dirichlet integral is

$$\int |u_k'(s)|^2 d\mu = c_k \int \left| \eta' \left(\frac{z}{z_k} \right) \right|^2 \frac{1}{z_k} \frac{ds}{\sigma^{n-1}} = c_n \int |\eta'(y)|^2 dy = c_n c_1$$

for some constant c_1. Likewise the L^2 norms of the u_k

$$\begin{aligned}\int |u_k|^2 d\mu &= z_k \int \left|\eta\left(\frac{z}{z_k}\right)\right|^2 d\mu \geq z_k c_n \int_{s_k}^{\hat{s}_k} d\mu \\ &= c_n \int_0^{s_k} \frac{ds}{\sigma^{n-1}} \int_{s_k}^{\hat{s}_k} \sigma^{n-1} ds \geq c_n c_0\end{aligned}$$

are bounded below. Therefore we may normalise this sequence in H, $v_k = u_k / |u_k|_{\mathcal{M}}$, and in doing so the Dirichlet integral remains bounded by

$$\langle \mathcal{L} v_k, v_k \rangle_{\mathcal{M}} \leq \frac{c_1}{c_0},$$

ie. $\{v_k\} \subset N$. Furthermore we have from (6) that the support of v_l and v_m are disjoint so that the sequence is orthonormal, ie. we have an infinite orthonormal sequence in N which consequently cannot be compact. □

Corollary 2.1 *The Laplace-Beltrami operator $\mathcal{L}$ on H has only discrete spectrum iff the measure satisfies (4).*

Proof: From the previous theorem we have that the set of elements bounded with respect to the quadratic form

$$F[f] = \langle (\mathcal{L} + 1) f, f \rangle_{\mathcal{M}}$$

is compact in H. This means that the preimage under $\sqrt{\mathcal{L}+1}$ of a bounded set in H is a compact set in H, ie. the inverse operator is compact. This gives us the desired discreteness of the spectrum. □

Remark 2.1 *The weighted Hardy inequality by Muckenhoupt [4] generalises the inequality stated here to the case of L^p spaces. This raises the question of whether it is possible to use [4] to generalise the result in this paper to L^p spaces over a manifold.*

References

[1] M. Š. Birman. The spectrum of singular boundary problems. In *American Mathematical Society Translations Series 2*, volume 53, pages 23–80. American Mathematical Society, 1966.

[2] V. A. Kondratiev, V. Maz'ja, and M. A. Shubin. Discreteness of spectrum and strict positivity criteria for magnetic schrödinger operators. *Comm. Partial Differential Equations*, 29:489–521, 2004.

[3] V. G. Maz'ja. *Sobolev Spaces.* Springer, Berlin, New York, 1985.

[4] B. Muckenhoupt. Hardy's inequality with weights. *Studia Math.*, 44:31–38, 1972.

[5] M. Shubin. Spectral theory of the schrödinger operator on non-compact manifolds. In Brian Davies and Yuri Safarov, editors, *Spectral theory and geometry*, volume 273 of *London Mathematical Society Lecture Note Series*, pages 226–283, 1999.