

Asymptotics of the triangle groups and the associated spectra

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Abstract

We consider the asymptotics of the (p, q, r) -triangle groups as one of the vertices goes to a cusp $p \rightarrow \infty$. We introduce new coordinates on the fundamental domain of the (p, q, r) -triangle group which in the limit approach the natural coordinates on the (∞, q, r) -triangle group and show that in these coordinates the fundamental solutions on the (p, q, r) -triangle group approach the fundamental solutions on the (∞, q, r) -triangle group. We use these coordinates to study how eigensolutions in the co-compact case accumulate onto the continuous spectrum in the non co-compact case.

1 Introduction

It is well known that a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ acts on the hyperbolic plane \mathbb{H} via Möbius transformations [1, 8]. The quotient \mathbb{H}/Γ of the hyperbolic plane by this group action can be identified with the so called fundamental domain, \mathcal{F} , of Γ , a subset of \mathbb{H} which covers the hyperbolic plane without any intersections under the action of Γ , $\mathbb{H} = \cup\{\gamma\bar{\mathcal{F}}; \gamma \in \Gamma\}$. Here we consider triangle groups $\Gamma_{p,q,r}$ which are generated by *pairs* of reflections across the sides of the hyperbolic triangle $\Delta_{p,q,r}$ with angles π/p , π/q and π/r at the vertices, ([1], pg. 276). The p , q and r are positive integers with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. The fundamental domain of $\Gamma_{p,q,r}$ consists of the union of a copy of $\Delta_{p,q,r}$ with its reflection through any one of its sides.

The $\Gamma_{p,q,r}$ are, for finite p , q and r , co-compact, ie. the fundamental domain \mathcal{F} is compact in the topology of \mathbb{H} and consequently the spectrum of the Laplace-Beltrami operator on \mathcal{F} is discrete. If we allow one of the vertices to become a cusp, for instance $p \rightarrow \infty$, then the fundamental domain becomes

non-compact and the spectrum picks up a continuous branch on the interval $[\frac{1}{4}, \infty)$ [6]. Here we plan to study how the discrete spectrum of the $\Gamma_{p,q,r}$ approaches the various branches of the spectrum of $\Gamma_{\infty,q,r}$ as $p \rightarrow \infty$ (here of course we mean the spectra of the associated Laplace-Beltrami operators). One approach to this question is to consider the Selberg zeta function $Z_{\Gamma}(s)$ [3, 5, 6]. This function has so called non-trivial zeroes on the critical line corresponding to the discrete eigenvalues, with multiplicity, of the associated group Γ . Unfortunately, due to the complicated nature of the Selberg zeta function there seems little hope of studying its convergence properties as $p \rightarrow \infty$. Moreover, the zeta function for the non co-compact triangle group ([6], pg. 153) has a pole at the centre of the critical strip $s = \frac{1}{2}$ which is absent for all the zeta functions of the co-compact triangle groups. Another approach is to consider Weyl's law which states that for co-compact triangle groups ([6], pg. 159)

$$N_{\Gamma}(T) = \frac{|\mathcal{F}|}{4\pi} T^2 + O(T \log T)$$

where $|\mathcal{F}|$ is the hyperbolic area of the associated fundamental domain and $N_{\Gamma}(T)$ is the number of eigenvalues $\lambda_i = \frac{1}{4} + t_i^2$ with $t_i < T$. In the non co-compact case we have, for the modular group $(p, q, r) = (\infty, 2, 3)$ (and its congruence subgroups), exactly the same asymptotics of the counting function. Consequently, for a given energy, we expect the non co-compact group and the co-compact groups which approach it to have approximately the same number of *discrete* eigenvalues of lower energy (at least in the case of the modular group). What then happen to the continuous spectrum, in particular are there any discrete eigenvalues of the co-compact groups which accumulate onto it? Weyl's law does not preclude this, it just puts a *finite* limit on the number of discrete eigenvalues of the co-compact group which tend to points of the continuous spectrum as $p \rightarrow \infty$ —in particular the discrete spectrum of the co-compact group may not become dense in the continuous spectrum! That this is unlike the case of the Laplacean on an interval tending to the line has probably something to do with the fact that the fundamental domain of the non co-compact group has finite volume. For non co-compact triangle groups other than the modular group the exact form of Weyl's law is not known, in particular it is not known whether the term due to the continuous spectrum is dominated by the term due to the discrete spectrum [6]. Nevertheless the asymptotic form is the same (it is just the constant in front of the asymptotic which is unknown) so that as before there is no possibility that the discrete spectrum becomes dense in the continuous spectrum as $p \rightarrow \infty$.

Here we make a heuristic study of the asymptotics of the spectra of the co-compact (p, q, r) -triangle. By taking hyperbolic polar coordinates about the order p vertex we can expand the eigenfunctions of $\Gamma_{p,q,r}$ in terms of associated Legendre functions. Using these polar coordinates we can easily find a coordinate system depending on p , (x_p, y_p) , which approaches the

usual coordinates in the upper half-plane model of the hyperbolic plane as $p \rightarrow \infty$. There are well known expansions in the upper half-plane model for the eigenfunctions, both continuous and discrete, of the non co-compact $\Gamma_{\infty,q,r}$. We compare these with our expansions for $\Gamma_{p,q,r}$ in terms of (x_p, y_p) and see that it is possible to get eigenfunctions which in the limit approach the *discrete* eigenfunctions of $\Gamma_{\infty,q,r}$. Surprisingly it also appears that there are eigenfunctions of $\Gamma_{p,q,r}$ which approach the continuous spectrum of $\Gamma_{\infty,q,r}$ and moreover become dense in the continuous spectrum! This contradiction with Weyl's law can only be resolved if we assume that these functions are not actually eigenfunctions, for instance they may be Green's functions with a logarithmic singularity at some point in the fundamental domain (we say eigensolutions to distinguish them).

2 Expansion of eigenfunctions for the triangle groups

2.1 Co-compact case

Given a co-compact (p, q, r) -triangle group we can write the Laplace-Beltrami operator in terms of hyperbolic polar coordinates about the order p vertex [6]

$$\mathcal{L} \equiv - \left((u^2 - 1) \frac{\partial^2}{\partial u^2} + 2u \frac{\partial}{\partial u} + \frac{1}{4(u^2 - 1)} \frac{\partial^2}{\partial \varphi^2} \right)$$

where $u = \cosh(\rho)$, ρ is the hyperbolic distance from the order p vertex and $\varphi \in \left[-\frac{\pi}{2p}, \frac{\pi}{2p}\right)$ is the angular coordinate—we assume $\varphi = 0$ corresponds to the arc joining the order p and order q vertices. The following ansatz $\psi(u, \varphi) = P(u)e^{2npi\varphi}$ ensures that in neighbourhoods of the order p vertex automorphy is satisfied. We see then that the solutions of

$$\mathcal{L}\psi = s(1 - s)\psi \tag{1}$$

are given by $P(u) = P_{-s}^{np}(u)$, $Q_{-s}^{np}(u)$, the associated Legendre functions [7]. Since the eigenfunctions are smooth ([6], pg. 58) we can immediately disregard the $Q_{-s}^{np}(u)$ which have order np poles or logarithmic singularities at $r = 0$ ([2], pg. 163). This allows us to write the discrete eigenfunctions for $\Gamma_{p,q,r}$ in the even

$$\psi_{p,+}(z; s) = \sum_{n=0}^{\infty} \alpha_{p,n}(s) P_{-s}^{np}(u) \cos(2np\varphi) \tag{2}$$

and odd

$$\psi_{p,-}(z; s) = \sum_{n=1}^{\infty} \beta_{p,n}(s) P_{-s}^{np}(u) \sin(2np\varphi) \tag{3}$$

cases. Here the $\alpha_{p,n}$ ($\beta_{p,n}$) are found by the solution of an infinite dimensional linear problem which is equivalent to the vanishing of the derivative of the eigenfunction (vanishing of the eigenfunction) on the boundary arc from the order q vertex to the order r vertex. This problem only has solutions for distinguished $s = s_i$, the eigenvalues, which lie in the set $s_i \in \{[0, 1] \cup \{\frac{1}{2} + i\mathbb{R}\}\}$ or $s_i(1 - s_i) \in [0, \infty)$.

2.2 Non co-compact case

We place the fundamental domain of the non co-compact (∞, q, r) -triangle group on the upper half z -plane so that the cusp is at $i\infty$, the order q vertex is at i and the order r vertex lies on the vertical lines $x = \pm\frac{\lambda}{2}$. Simple geometry [1] shows that

$$\lambda = 2 \frac{\cos\left(\frac{\pi}{q}\right) + \cos\left(\frac{\pi}{r}\right)}{\sin\left(\frac{\pi}{q}\right)}. \quad (4)$$

The Laplace-Beltrami operator in this context is

$$\mathcal{L} \equiv -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

where $z = x + iy$. We adopt the ansatz $\psi(x, y) = K(y)e^{\frac{2\pi i n x}{\lambda}}$ so that we have automorphy in neighbourhoods of the cusp. Then, for $n \neq 0$, the solutions of (1) are given by $K(y) = \left(\frac{ny}{\lambda}\right)^{\frac{1}{2}} I_{s-\frac{1}{2}}\left(\frac{2\pi n y}{\lambda}\right)$, $\left(\frac{ny}{\lambda}\right)^{\frac{1}{2}} K_{s-\frac{1}{2}}\left(\frac{2\pi n y}{\lambda}\right)$, the Bessel functions of imaginary argument [6, 7]. In this case we may disregard $I_{s-\frac{1}{2}}$ as it has exponential growth for large y . For $n = 0$ we get the two linearly independent solutions $K(y) = y^s, y^{1-s}$ both of which play a role (these degenerate at $s = \frac{1}{2}$ but that is not important here).

It is well known that in the case of a non co-compact group the spectrum splits into three parts: the continuous, residual and cuspidal spectra [6]. The cuspidal spectrum is discrete and is defined by the condition that integrals of the eigenfunctions across horocycles are zero, ie. integrating on $x \in [-\frac{\lambda}{2}, \frac{\lambda}{2}]$ gives zero. Expanding in Bessel functions this is just equivalent to the absence of the $n = 0$ term giving

$$\psi_+(z; s) = \sum_{n=1}^{\infty} \alpha_n(s) \left(\frac{ny}{\lambda}\right)^{\frac{1}{2}} K_{s-\frac{1}{2}}\left(\frac{2\pi n y}{\lambda}\right) \cos\left(\frac{2\pi n x}{\lambda}\right) \quad (5)$$

for the even and

$$\psi_-(z; s) = \sum_{n=1}^{\infty} \beta_n(s) \left(\frac{ny}{\lambda}\right)^{\frac{1}{2}} K_{s-\frac{1}{2}}\left(\frac{2\pi n y}{\lambda}\right) \sin\left(\frac{2\pi n x}{\lambda}\right) \quad (6)$$

for the odd cusp forms. Again the α_n (β_n) are solutions of an infinite dimensional problem which is only defined for distinguished values of s , the

discrete eigenvalues, with $s_i(1 - s_i) \in [0, \infty)$.

The eigenfunctions of the continuous spectrum are known as Eisenstein series and in our case can be expanded in terms of Bessel functions as

$$E(z; s) = y^s + \phi(s)y^{1-s} + \sum_{n=1}^{\infty} \gamma_n(s) \left(\frac{ny}{\lambda}\right)^{\frac{1}{2}} K_{s-\frac{1}{2}}\left(\frac{2\pi ny}{\lambda}\right) \cos\left(\frac{2\pi nx}{\lambda}\right) \quad (7)$$

(see [6] (3.20)). Here the spectral parameter lies in the interval $s \in \{\frac{1}{2} + i\mathbb{R}\}$ or $s(1 - s) \in [\frac{1}{4}, \infty)$. The function $\phi(s)$ is known as the scattering matrix.

We note that in our case the Eisenstein series is an *even* function.

The Eisenstein series has poles in the interval $(\frac{1}{2}, 1]$ which form the residual spectrum. The corresponding eigenfunctions can be written

$$\psi_r(z; s) = \eta_0(s)y^{1-s} + \sum_{n=1}^{\infty} \eta_n(s) \left(\frac{ny}{\lambda}\right)^{\frac{1}{2}} K_{s-\frac{1}{2}}\left(\frac{2\pi ny}{\lambda}\right) \cos\left(\frac{2\pi nx}{\lambda}\right) \quad (8)$$

([6], pg. 90). We note that this is an even function which, since $s_i \in (\frac{1}{2}, 1]$, is square integrable with respect to the hyperbolic measure on the fundamental domain.

Evenness of the continuous and residual spectra follows from the symmetry of the fundamental domain whereby every eigenspace can be decomposed into odd and even parts. Only the even part will have a zero order, $n = 0$, term. Consequently, the Eisenstein series and residual spectrum, which by definition have non-zero integral across horocycles, must be even.

3 Coordinates in the co-compact case and the limit $p \rightarrow \infty$

We introduce new coordinates on the fundamental domain of the (p, q, r) -triangle group by

$$x_p = \lambda \frac{p\varphi}{\pi}, \quad y_p = \exp(\rho_p - \rho).$$

Here ρ_p is the hyperbolic distance from the order p vertex to the order q vertex and is given by the hyperbolic cosine law [1]

$$\cosh(\rho_p) = \frac{\cos\left(\frac{\pi}{p}\right) \cos\left(\frac{\pi}{q}\right) + \cos\left(\frac{\pi}{r}\right)}{\sin\left(\frac{\pi}{p}\right) \sin\left(\frac{\pi}{q}\right)}. \quad (9)$$

We note that $x_p \in [-\frac{\lambda}{2}, \frac{\lambda}{2}]$ and that at the order q vertex $(x_p, y_p) = (0, 1)$. Furthermore, in the upper half plane model we use to describe the fundamental domain of $\Gamma_{\infty, q, r}$ the logarithm of the y -coordinate measures the hyperbolic distance above i , the order q vertex, while y_p measures the hyperbolic distance above the order q vertex for $\Gamma_{p, q, r}$.

From these facts we see that, ‘in some sense’, $(x_p, y_p) \rightarrow (x, y)$ as $p \rightarrow \infty$. The precise sense of this convergence is not important, rather we are interested in whether the fundamental solutions for the (p, q, r) -triangle group in terms of these coordinates approach the fundamental solutions for the (∞, q, r) -triangle group. This is obvious for the angular or x -coordinate, the case for the radial or y -coordinate is the subject of the following lemma:

Lemma 3.1 *For $n \neq 0$ we have the asymptotic formula*

$$\frac{(-1)^{np}}{(np-1)!} P_{-s}^{np}(u) = \frac{2 \sin(\pi s)}{\pi} \left(\frac{ny}{\lambda}\right)^{\frac{1}{2}} K_{s-\frac{1}{2}}\left(\frac{2\pi ny}{\lambda}\right) + O(p^{-1}). \quad (10)$$

For $n = 0$ and $\Re s \geq \frac{1}{2}$ we have two cases; if $\Re s = \sigma > \frac{1}{2}$

$$p^{1-s} P_{-s}(u) = \frac{\Gamma\left(s - \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(s)} \left(\frac{\pi y}{\lambda}\right)^{1-s} + O(p^{1-2\sigma}). \quad (11)$$

For $s = \frac{1}{2} + it$, $t \in \mathbb{R}$ we have

$$p^{1-s} P_{-s}(u) = \frac{\Gamma(-it)}{\sqrt{\pi} \Gamma\left(\frac{1}{2} - it\right)} \left(\frac{\pi y}{\lambda}\right)^{\frac{1}{2}+it} p^{-2it} + \frac{\Gamma(it)}{\sqrt{\pi} \Gamma\left(\frac{1}{2} + it\right)} \left(\frac{\pi y}{\lambda}\right)^{\frac{1}{2}-it} + O(p^{-2}). \quad (12)$$

Here

$$u = \frac{1}{2} \left(\frac{e^{\rho p}}{y} + \frac{y}{e^{\rho p}} \right)$$

and y is fixed independent of p .

We consign the proof to an appendix.

We see immediately from the first limit (10) that it is possible for the even and odd eigenfunctions of the co-compact group (2,3) to go to the cusp forms (5,6) of the non co-compact group. Likewise, from (11) we see that it is possible for the even eigenfunctions of the co-compact group to go to the residual eigenfunctions (8) of the non co-compact group.

The asymptotic (12) is more difficult to interpret as the limit $p \rightarrow \infty$ clearly does not exist. We would like to know whether there are *eigensolutions*¹—we know by Weyl’s law that there are no, or very few, eigenfunctions—of the co-compact groups (2) with the same zero order asymptotics

$$E(z; s) = y^s + \phi(s)y^{1-s} + O(e^{-2\pi y})$$

as the Eisenstein series (7) in a given interval \mathcal{I} of the continuous spectrum. It is clear that there are eigensolutions filling the whole interval \mathcal{I} , we may just take the Green’s function, but it is not clear that there are eigensolutions with

¹We recall that by eigensolutions we mean solutions of the eigenvalue equation which are not smooth, eg. Green’s functions, so that they are not eigenfunctions.

the same asymptotics as the Eisenstein series. From (12) these eigensolutions occur when we have solutions $\frac{1}{4} + t^2 \in \mathcal{I}$ of

$$\frac{\Gamma(it)\Gamma\left(\frac{1}{2} - it\right)}{\Gamma(-it)\Gamma\left(\frac{1}{2} + it\right)} \exp(2it \log p - \log(\pi/\lambda)) = \phi\left(\frac{1}{2} + it\right).$$

It is clear from that as $p \rightarrow \infty$ the solutions of this equation become dense in the interval \mathcal{I} .

In summary we conjecture that in the limit $p \rightarrow \infty$ we have a dense set of points on the continuous spectrum where the Greens function of the co-compact group has the same asymptotics for large y as the Eisenstein series.

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Appendix: Proof of lemma

We begin by noting from (4,9) that

$$\exp(\rho_p) = \frac{\lambda p}{\pi} + O(p^{-1}).$$

As in the lemma we drop the subscript from y_p so that

$$\exp(\rho) = \frac{\lambda p}{\pi y} + O(p^{-1})$$

or

$$u = \cosh(\rho) = \frac{\lambda p}{2\pi y} + O(p^{-1}).$$

The asymptotic (11) is an easy consequence of equation (7.6.3) in [7]

$$\begin{aligned} P_{-s}(u) &= \frac{\Gamma\left(\frac{1}{2} - s\right)}{\sqrt{\pi}\Gamma(1-s)} \frac{1}{(2u)^s} F\left(\frac{1}{2} + \frac{s}{2}, \frac{s}{2}; \frac{1}{2} + s; \frac{1}{u^2}\right) + \\ &\quad \frac{\Gamma\left(s - \frac{1}{2}\right)}{\sqrt{\pi}\Gamma(s)} \frac{1}{(2u)^{1-s}} F\left(\frac{1}{2} - \frac{s}{2}, \frac{1}{2} - \frac{s}{2}; \frac{3}{2} - s; \frac{1}{u^2}\right) \\ &= \frac{\Gamma\left(\frac{1}{2} - s\right)}{\sqrt{\pi}\Gamma(1-s)} \left(\frac{\pi y}{\lambda p}\right)^s \frac{1}{(1 + O(p^{-2}))^s} (1 + O(p^{-2})) + \\ &\quad \frac{\Gamma\left(s - \frac{1}{2}\right)}{\sqrt{\pi}\Gamma(s)} \left(\frac{\pi y}{\lambda p}\right)^{1-s} \frac{1}{(1 + O(p^{-2}))^{1-s}} (1 + O(p^{-2})). \end{aligned}$$

Here F is the hypergeometric function and in the second equality we use the hypergeometric series which is clearly valid in the region under consideration. Multiplying by p^{1-s}

$$p^{1-s}P_{-s}(u) = \frac{\Gamma\left(\frac{1}{2} - s\right)}{\sqrt{\pi}\Gamma(1-s)} \left(\frac{\pi y}{\lambda}\right)^s p^{1-2s} + \frac{\Gamma\left(s - \frac{1}{2}\right)}{\sqrt{\pi}\Gamma(s)} \left(\frac{\pi y}{\lambda}\right)^{1-s} + O(p^{-2})$$

we get (11,12).

Equation (10) can be derived from equation 8.777.1 in [4]

$$P_{-s}^m(\cosh(r)) = \frac{2^m \Gamma\left(\frac{1}{2} - s\right)}{\sqrt{\pi} \Gamma(1 - s - m)} \frac{\sinh^m(r)}{e^{(m+s)r}} F\left(\frac{1}{2} + m, m + s; \frac{1}{2} + s; e^{-2r}\right) + \frac{2^m \Gamma\left(s - \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(s - m)} \frac{\sinh^m(r)}{e^{(m+1-s)r}} F\left(\frac{1}{2} + m, m + 1 - s; \frac{3}{2} - s; e^{-2r}\right).$$

We denote $m = np$ in the following and use the above asymptotics for ρ to give

$$P_{-s}^m(u) = \frac{1}{\sqrt{\pi}} \left[\frac{\Gamma\left(\frac{1}{2} - s\right)}{\Gamma(1 - s - m)} \left(\frac{\pi ny}{\lambda m}\right)^s F\left(\frac{1}{2} + m, m + s; \frac{1}{2} + s; \left(\frac{\pi ny}{\lambda m}\right)^2\right) + \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s - m)} \left(\frac{\pi ny}{\lambda m}\right)^{1-s} F\left(\frac{1}{2} + m, m + 1 - s; \frac{3}{2} - s; \left(\frac{\pi ny}{\lambda m}\right)^2\right) \right] + O(p^{-1}).$$

Here we have used the fact that $\left(1 - \left(\frac{\pi y}{\lambda p}\right)^2\right)^{np} = 1 + O(p^{-1})$. The terms $\Gamma(1 - s - m)$, $\Gamma(s - m)$ are manipulated using [7], equation (1.2.2), to get

$$\frac{(-1)^m}{\Gamma(m)} P_{-s}^m(u) = \frac{\sin(\pi s)}{\sqrt{\pi} \pi} \left[\Gamma\left(\frac{1}{2} - s\right) \frac{\Gamma(s + m)}{\Gamma(m) m^s} \left(\frac{\pi ny}{\lambda}\right)^s F\left(\frac{1}{2} + m, m + s; \frac{1}{2} + s; \left(\frac{\pi ny}{\lambda m}\right)^2\right) + \Gamma\left(s - \frac{1}{2}\right) \frac{\Gamma(1 - s + m)}{\Gamma(m) m^{1-s}} \left(\frac{\pi ny}{\lambda}\right)^{1-s} F\left(\frac{1}{2} + m, m + 1 - s; \frac{3}{2} - s; \left(\frac{\pi ny}{\lambda m}\right)^2\right) \right] + O(p^{-1}).$$

We consider just the first term in the brackets, the second is treated similarly. Using the hypergeometric series [7], equation (9.1.1), this term can be written

$$\begin{aligned} \Gamma\left(\frac{1}{2} - s\right) \frac{\Gamma(s + m)}{\Gamma(m) m^s} \left(\frac{\pi ny}{\lambda}\right)^s F\left(\frac{1}{2} + m, m + s; \frac{1}{2} + s; \left(\frac{\pi ny}{\lambda m}\right)^2\right) \\ = \Gamma\left(\frac{1}{2} - s\right) \frac{(m)_s}{m^s} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} + m\right)_k (m + s)_k}{m^k m^k} \frac{1}{\left(\frac{1}{2} + s\right)_k k!} \left(\frac{\pi ny}{\lambda}\right)^{s+2k}. \end{aligned}$$

Here $(m + a)_k = \frac{\Gamma(m+a+k)}{\Gamma(m+a)}$ and we have the well known asymptotic [7], pg. 15,

$$\frac{(m + a)_k}{m^k} = 1 + O(m^{-1}).$$

This gives us

$$\begin{aligned} \Gamma\left(\frac{1}{2} - s\right) \frac{\Gamma(s + m)}{\Gamma(m) m^s} \left(\frac{\pi ny}{\lambda}\right)^s F\left(\frac{1}{2} + m, m + s; \frac{1}{2} + s; \left(\frac{\pi ny}{\lambda m}\right)^2\right) \\ = \Gamma\left(\frac{1}{2} - s\right) \Gamma\left(\frac{1}{2} + s\right) \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(\frac{1}{2} + s + k\right) \Gamma(k + 1)} \left(\frac{\pi ny}{\lambda}\right)^{s+2k} + O(p^{-1}) \\ = \frac{\pi e^{\frac{i\pi}{2}\left(\frac{1}{2}-s\right)}}{\cos(\pi s)} \left(\frac{\pi ny}{\lambda}\right)^{\frac{1}{2}} J_{s-\frac{1}{2}}\left(\frac{2i\pi ny}{\lambda}\right) + O(p^{-1}) \end{aligned}$$

where J_ν is the Bessel function. Similarly the second term gives us

$$\begin{aligned} \Gamma\left(s - \frac{1}{2}\right) \frac{\Gamma(1-s+m)}{\Gamma(m)} \frac{\left(\frac{\pi ny}{\lambda}\right)^{1-s}}{m^{1-s}} F\left(\frac{1}{2} + m, m + 1 - s; \frac{3}{2} - s; \left(\frac{\pi ny}{\lambda m}\right)^2\right) \\ = -\frac{\pi e^{\frac{i\pi}{2}(s-\frac{1}{2})}}{\cos(\pi s)} \left(\frac{\pi ny}{\lambda}\right)^{\frac{1}{2}} J_{\frac{1}{2}-s}\left(\frac{2i\pi ny}{\lambda}\right) + O(p^{-1}) \end{aligned}$$

so that the Legendre function becomes

$$\begin{aligned} \frac{(-1)^m}{\Gamma(m)} P_{-s}^m(u) &= \frac{\tan(\pi s)}{\sqrt{\pi}} \left(\frac{\pi ny}{\lambda}\right)^{\frac{1}{2}} \left[e^{\frac{i\pi}{2}(\frac{1}{2}-s)} J_{s-\frac{1}{2}}\left(\frac{2i\pi ny}{\lambda}\right) - \right. \\ &\quad \left. e^{\frac{i\pi}{2}(s-\frac{1}{2})} J_{\frac{1}{2}-s}\left(\frac{2i\pi ny}{\lambda}\right) \right] + O(p^{-1}). \end{aligned}$$

On the other hand [7] we can write the K-Bessel function in terms of the J-Bessel function as

$$\begin{aligned} \left(\frac{ny}{\lambda}\right)^{\frac{1}{2}} K_{s-\frac{1}{2}}\left(\frac{2\pi ny}{\lambda}\right) &= \frac{\sqrt{\pi}}{2 \cos(\pi s)} \left(\frac{\pi ny}{\lambda}\right)^{\frac{1}{2}} \left[e^{\frac{i\pi}{2}(\frac{1}{2}-s)} J_{s-\frac{1}{2}}\left(\frac{2i\pi ny}{\lambda}\right) - \right. \\ &\quad \left. e^{\frac{i\pi}{2}(s-\frac{1}{2})} J_{\frac{1}{2}-s}\left(\frac{2i\pi ny}{\lambda}\right) \right] \end{aligned}$$

which immediately gives us (10).

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