QUASICONTINUOUS SELECTIONS OF UPPER CONTINUOUS SET-VALUED MAPPINGS

JILING CAO[1] AND WARREN B. MOORS

Abstract. In this paper, we extend a theorem of Matejdes on quasicontinuous selections of upper Baire continuous set-valued mappings from compact (or separable) metric range spaces to regular T_1 range spaces. In addition, we also prove a quasicontinuous selection theorem for a special class of upper semicontinuous set-valued mappings.

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1. Introduction

Let $T : X \to 2^Y$ be a set-valued mapping with non-empty values. By a *selection f of T*, we mean a single-valued mapping $f : X \to Y$ such that $f(x) \in T(x)$ for all $x \in X$. A well-known theorem of Micheal on selections in [7] claims that any lower semicontinuous set-valued mapping $T : X \to 2^Y$ with non-empty closed convex values acting from a paracompact space X into a Banach space Y has a continuous selection. However, the conclusion of this theorem fails when lower semicontinuity is replaced by upper semicontinuity. For example, the set-valued mapping $T : \mathbb{R} \to 2^{\mathbb{R}}$, defined by

$$T(x) := \begin{cases} \{1/x\}, & \text{if } x \neq 0, \\ \mathbb{R}, & \text{if } x = 0, \end{cases}$$

is upper semicontinuous with non-empty closed convex values. However, this mapping does not even possess a quasicontinuous selection. Recall that a (single-valued) mapping $f : X \to Y$ is *quasicontinuous* if for every pair of open sets $U \subseteq X$ and $W \subseteq Y$ with $f(U) \cap W \neq \emptyset$, there exists a non-empty open set $V \subseteq U$ such that $f(V) \subseteq W$. In a series of papers [3, 4, 5, 6], Matejdes studied the problem of when a set-valued mapping admits a quasicontinuous selection. To achieve his goal, Matejdes introduced the following definition, [3].

Definition 1.1 ([3])**.** A set-valued mapping $T : X \to 2^Y$ is called *upper Baire continuous* at a point $x \in X$ if for each pair of open sets U and W with $x \in U$ and $T(x) \subseteq W$, there is a subset $B \subseteq U$ of the second category, having the Baire property, such that $T(z) \subseteq W$ for all $z \in B$.

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We shall say that a set-valued mapping $T : X \to 2^Y$ is *upper Baire continuous* if it is upper Baire continuous at every point of X, and a Baire continuous single-valued mapping is just a special case of an upper Baire continuous set-valued mapping. Analogously, one can define lower Baire continuity for a set-valued mapping. However, we shall not do so here, since we are not going to use such a notion in this paper.

The following two facts on (upper) Baire continuity of mappings can be readily proved:

• *If $f : X \to 2^Y$ is upper Baire continuous with non-empty values, then X is Baire.*

• *If a (single-valued) mapping $f : X \to Y$ is Baire continuous, X is Baire and Y is regular, then f must be quasicontinuous,* [3].

By applying the previous two facts Matejdes proved the following theorem.

Theorem 1.2 ([3])**.** *Let X be a T_1-space and Y be a compact metric space. If $T : X \to 2^Y$ is upper Baire continuous with non-empty compact values, then T admits a quasicontinuous selection.*

In [4], it was further shown that the compactness of Y in the previous theorem can be relaxed to the separability of Y. The main purpose of this paper is to extend Theorem 1.2 using a different approach. Specifically, in Section 2, we show that the conclusion of Theorem 1.2 still holds when the condition "Y a compact (or separable) metric space" is weakened to "Y a regular T_1-space". The last section is dedicated to the study of quasicontinuous selections of a special class of upper semicontinuous set-valued mappings. Throughout the paper, $T : X \to 2^Y$ always denotes a set-valued mapping acting from a space X to a space Y and $f : X \to Y$ stands for a single-valued mapping from X into Y. The *graph* $Gr(T)$ of $T : X \to 2^Y$ is defined by $Gr(T) := \{(x, y) \in X \times Y : y \in T(x)\}$. All of our notation is standard and any undefined concepts may be found in the references.

2. An extension of Theorem 1.2

The following characterisation for upper Baire continuity of a set-valued mapping is easier to work with than the original definition in Definition 1.1.

Lemma 2.1. *A set-valued mapping $T : X \to 2^Y$ with non-empty values is upper Baire continuous if, and only if, X is Baire and for each pair of open subsets U and W with $x \in U$ and $T(x) \subseteq W$, there exist a non-empty open set $V \subseteq U$ and a residual set $R \subseteq V$ such that $T(z) \subseteq W$ for all $z \in R$.*

Proof. ($\Rightarrow$). Suppose that $T : X \to 2^Y$ is upper Baire continuous. First, by remarks in Section 1, X must be Baire. Furthermore, by the definition, for each pair of open sets U and W with $x \in U$ and $T(x) \subseteq W$, there exists some subset $B \subseteq U$ of the second category having the Baire property such that $T(z) \subseteq W$ for all $z \in B$. Let $B = G\Delta C$, where G is an open set and C is a set of the first category. Next, put $V = G \cap U$ and $R = G \setminus C$. Then $V \subseteq U$ is a non-empty open set and R is a residual set in V such that $T(z) \subseteq W$ for each $z \in R$.

($\Leftarrow$). Conversely, suppose that X is Baire and for each pair of open sets U and W with $x \in U$ and $T(x) \subseteq W$, there exists a non-empty open subset $V \subseteq U$ and a residual subset $R \subseteq V$ such that $T(z) \subseteq W$ for all $z \in R$. Since V is of the second category, then R must be of the second category. In addition, $R = V\Delta(V \setminus R)$. Thus, R has the Baire property as well. □

Our next theorem extends Theorem 1.2 from a compact (or separable) metric range space to an arbitrary regular T_1 range space.

Theorem 2.2. *Let X be a topological space and Y a regular T_1-space. If $T : X \to 2^Y$ is an upper Baire continuous set-valued mapping with non-empty compact values, then T admits a quasicontinuous selection.*

Proof. First, by Lemma 2.1, X must be a Baire space. Let $\mathscr{M}$ be the family of all upper Baire continuous set-valued mappings from X to Y with non-empty compact values such that for every $H \in \mathscr{M}$, $Gr(H) \subseteq Gr(T)$. Since $T \in \mathscr{M}$, $\mathscr{M} \neq \varnothing$. We define a partial order $\preceq$ on $\mathscr{M}$ by writing

$$H_1 \preceq H_2 \text{ if, and only if, } Gr(H_1) \subseteq Gr(H_2).$$

Next, we show that $\mathscr{M}$ has a minimal element. To this end, let $\mathscr{M}_0$ be any linearly ordered non-empty subfamily of $\mathscr{M}$. Then, define a set-valued mapping $H_{\mathscr{M}_0} : X \to 2^Y$ by letting

$$H_{\mathscr{M}_0}(x) := \bigcap\{H(x) : H \in \mathscr{M}_0\}$$

for all $x \in X$. Fix an arbitrary point $x_0 \in X$. Since $\{H(x_0) : H \in \mathscr{M}_0\}$ is a linearly ordered family of non-empty compact subsets of Y, $H_{\mathscr{M}_0}(x_0)$ is also a non-empty compact subset of Y. Now, suppose that $U \subseteq X$ and $W \subseteq Y$ are a pair of non-empty open subsets with $x_0 \in U$ and $H_{\mathscr{M}_0}(x_0) \subseteq W$. Then, there must be some element $H \in \mathscr{M}_0$ such that $H(x_0) \subseteq W$. By upper Baire continuity of H at x_0, there is a non-empty open set $V \subseteq U$ and a residual subset $R \subseteq V$ such that $H(x) \subseteq W$ for all $x \in R$. This implies that $H_{\mathscr{M}_0}(x) \subseteq W$ for all $x \in R$. Thus, $H_{\mathscr{M}_0} \in \mathscr{M}$. By Zorn's lemma, $\mathscr{M}$ has a minimal member, which we will denote by $\Phi_{\mathscr{M}}$.

Claim 1. *For each pair of open subsets $U \subseteq X$ and $W \subseteq Y$ such that $\Phi_{\mathscr{M}}(U) \cap W \neq \varnothing$, there exist a non-empty open subset $V \subseteq U$ and a residual set $R \subseteq V$ such that $\Phi_{\mathscr{M}}(x) \subseteq W$ for all $x \in R$.*

Proof. Suppose the contrary. Then, there is a pair of open subsets $U \subseteq X$ and $W \subseteq Y$ with $\Phi_{\mathscr{M}}(U) \cap W \neq \varnothing$ such that for every non-empty open subset $V \subseteq U$ and every residual subset $R \subseteq V$ there exists an $x \in R$ such that $\Phi_{\mathscr{M}}(x) \not\subseteq W$. Since $\Phi_{\mathscr{M}}$ is upper Baire continuous, this implies that $\Phi_{\mathscr{M}}(x) \not\subseteq W$ for any $x \in U$. Next, we define a set-valued mapping $\Gamma : X \to 2^Y$ by

$$\Gamma(x) := \begin{cases} \Phi_{\mathscr{M}}(x) \cap (Y \setminus W), & \text{if } x \in U, \\ \Phi_{\mathscr{M}}(x), & \text{otherwise.} \end{cases}$$

Then Γ has non-empty compact values. We will show that Γ is upper Baire continuous. Pick any point $x_0 \in X$. If $x_0 \notin U$, then the result is clear, since $\Phi_{\mathscr{M}}$ is upper Baire continuous and $\Gamma \preceq \Phi_{\mathscr{M}}$. Assume $x_0 \in U$. Let U' and W' be a pair of open sets with $x_0 \in U'$ and $\Gamma(x_0) \subseteq W'$. Then $\Phi_{\mathscr{M}}(x_0) \subseteq W \cup W'$, thus there exist a non-empty open set $V' \subseteq U'$ and a residual set $R' \subseteq V'$ such that $\Phi_{\mathscr{M}}(x) \subseteq W \cup W'$ for all $x \in R'$. Clearly, $\Gamma(x) \subseteq W'$ for every point $x \in R'$. This implies that Γ is upper Baire continuous at every point of U. Thus, we have shown that $\Gamma \in \mathscr{M}$. But this is impossible since $\Gamma \preceq \Phi_{\mathscr{M}}$ and $\Phi \neq \Phi_{\mathscr{M}}$. Hence we have obtained our desired contradiction. □

Claim 2. *$\Phi_{\mathscr{M}}$ is single-valued at every point $x \in X$.*

Proof. If not, there must exist a point $x_1 \in X$ such that $\Phi_{\mathscr{M}}(x_1)$ contains at least two points. Now, pick any point $y_1 \in \Phi_{\mathscr{M}}(x_1)$, and then define another set-valued mapping $\Psi : X \to 2^Y$ by

$$\Psi(x) := \begin{cases} \{y_1\}, & \text{if } x = x_1, \\ \Phi_{\mathscr{M}}(x), & \text{otherwise.} \end{cases}$$

It is clear that Ψ has non-empty compact images. Let $x \in X$ and consider open sets $U \subseteq X$ and $W \subseteq Y$ such that $x \in U$ and $\Psi(x) \subseteq W$. By Claim 1, there exist a non-empty open subset $V \subseteq U$ and a residual subset $R \subseteq V$ such that $\Phi_{\mathscr{M}}(x) \subseteq W$ for all $x \in R$. It follows that $\Psi(x) \subseteq W$ for all $x \in R$. Thus Ψ is upper Baire continuous. But, $\Psi \preceq \Phi_{\mathscr{M}}$ and $\Psi \neq \Phi_{\mathscr{M}}$; which contradicts the minimality of $\Phi_{\mathscr{M}}$. □

Finally, by Claim 2, $\Phi_{\mathscr{M}}$ is a Baire continuous selection of T. Therefore, since X is Baire and Y is regular, $\Phi_{\mathscr{M}}$ is quasicontinuous. □

3. Strongly injective set-valued mappings

In this section, we shall examine when an upper semicontinuous set-valued mapping acting between topological spaces admits a quasicontinuous selection. Our considerations are based upon the following notion.

Definition 3.1. A set-valued mapping $T : X \to 2^Y$ is *strongly injective* if $T(x_1) \cap T(x_2) = \varnothing$ for any two distinct points $x_1, x_2 \in X$.

Furthermore, we shall also require the definition of property $(**)$ introduced in [2]. Let X be a space, $\mathscr{F}$ a proper filter (or filterbase) in X. We shall consider the following *$G(\mathscr{F})$-game* played in X between players A and B: Player A goes first (always!) and chooses a point $x_1 \in X$. Player B responds by choosing a member $F_1 \in \mathscr{F}$. Following this, player A must select another (possibly the same) point $x_2 \in F_1$ and in turn player B must again respond to this by choosing a member $F_2 \in \mathscr{F}$. Repeating this procedure infinitely, the players A and B produce a sequence $p := ((x_n, F_n) : n \in \mathbb{N})$ with $x_{n+1} \in F_n$ for all $n \in \mathbb{N}$, called a *play* of the $G(\mathscr{F})$-game. We shall say that B *wins* a play of the $G(\mathscr{F})$-game if the sequence $(x_n : n \in \mathbb{N})$ has a cluster point in X. Otherwise, the player A is said to have *won* this play.

We shall call a pair $(\mathscr{F}, \sigma)$ a *σ-filter* (*σ-filterbase*) if $\mathscr{F}$ is a proper filter (filterbase) in X and σ is a winning strategy for player B in the $G(\mathscr{F})$-game. Finally, we say that a space X has *property* $(**)$ if $\bigcap\{\overline{F} : F \in \mathscr{F}\} \neq \varnothing$ for each σ-filterbase $(\mathscr{F}, \sigma)$ in X. The class of spaces having property $(**)$ includes all metric spaces [1], all Dieudoné-complete spaces, all function spaces $C_p(X)$ for compact Hausdorff spaces X, and all Banach spaces in their weak topologies [2]. Recall that a space X is a q-space if for every point $x \in X$, there is a sequence $(U_n : n \in \mathbb{N})$ of neighbourhoods of x such that if $x_n \in U_n$ for all $n \in \mathbb{N}$, the sequence $(x_n : n \in \mathbb{N})$ has a cluster point in X (which is not necessarily x itself). All first countable spaces and all Čech complete spaces are q-spaces.

The following theorem may be deduced from [2, Theorem 3.3].

Theorem 3.2 ([2]). *Let $T : X \to 2^Y$ be a strongly injective upper semicontinuous set-valued mapping with non-empty closed values. If X is a regular q-space and Y is a regular space with property $(**)$, then for any point $x_0 \in X$,*

$$K := \bigcap_{U \in \mathscr{U}(x_0)} \overline{T(U \setminus \{x_0\})}$$

is a compact subset of $T(x_0)$, where $\mathscr{U}(x_0)$ is the family of all neighbourhoods of x_0 in X and $\overline{T(U \setminus \{x_0\})}$ is the closure of $T(U \setminus \{x_0\})$ in Y. In addition, the mapping $T_K : X \to 2^Y$, defined by

$$T_K(x) := \begin{cases} K, & \text{if } x = x_0, \\ T(x), & \text{otherwise,} \end{cases}$$

is upper semicontinuous on X.

In the previous theorem it follows that if $x_0 \in X$ is not an isolated point then K is non-empty.

Theorem 3.3. *Let $T : X \to 2^Y$ be an upper semicontinuous set-valued mapping acting from a regular q-space X into a regular T_1-space Y with property $(**)$. If T is strongly injective, then it admits a quasicontinuous selection.*

Proof. For any isolated point $x \in X$, pick an arbitrary point $y_x \in T(x)$. Next, define the set-valued mapping $\Phi : X \to 2^Y$ by,

$$\Phi(x) := \begin{cases} \bigcap_{U \in \mathscr{U}(x)} \overline{T(U \setminus \{x\})}, & \text{if } x \text{ is not isolated,} \\ \{y_x\}, & \text{if } x \text{ is isolated.} \end{cases}$$

By Theorem 3.2 and the subsequent remark, Φ has non-empty compact values.

Now, fix an arbitrary point $x_0 \in X$. To show that Φ is upper semicontinuous at x_0, we consider two possible cases. If x_0 is an isolated point of X, then the upper semicontinuity of Φ at x_0 is trivial. In the case that x_0 is non-isolated, it follows from the second part of Theorem 3.2. Thus, Φ is an upper semicontinuous non-empty compact-valued set-valued mapping whose graph is contained in the graph of T. By Theorem 2.2, Φ has a quasicontinuous selection f, which is also a selection of T. □

If $f : X \to Y$ is a mapping with $f(X) = Y$, then $f^{-1} : Y \to 2^X$ is strongly injective. Conversely, for any strongly injective set-valued mapping $T : Y \to 2^X$ with non-empty values and $T(Y) = X$, there exists a mapping $f : X \to Y$ such that $T = f^{-1}$.

Corollary 3.4. *Let $f : X \to Y$ be a closed mapping from a regular T_1-space X with property $(**)$ onto a regular q-space Y. If $f^{-1}(y)$ is closed for every $y \in Y$, then there exists a quasicontinuous mapping $\varphi : Y \to X$ such that $(f \circ \varphi)(y) = y$ for all $y \in Y$.*

Proof. Note that $f^{-1} : Y \to 2^X$ is an upper semicontinuous strongly injective set-valued mapping with non-empty closed values. By applying Theorem 3.3, f^{-1} admits a quasicontinuous selection $\varphi : Y \to X$. Evidently, $(f \circ \varphi)(y) = y$ for all $y \in Y$. □

Remark 3.5. By [2, Theorem 1.2] and an argument similar to that in Theorem 3.3, one can show the following: Let $T : X \to 2^Y$ be an upper semicontinuous set-valued mapping from a first countable space X into a Hausdorff angelic space Y. If T is strongly injective, then it admits a quasicontinuous selection. As a consequence of this result, the condition "$f^{-1}(y)$ is closed for every $y \in Y$" in Corollary 3.4 can be dropped when X is Hausdorff angelic and Y is first countable, i.e., for any closed mapping $f : X \to Y$ from a Hausdorff angelic space X onto a first countable space Y, there exists a quasicontinuous mapping $\varphi : Y \to X$ such that $(f \circ \varphi)(y) = y$ for all $y \in Y$.

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Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland, New Zealand
Email: cao@math.auckland.ac.nz & moors@math.auckland.ac.nz