A New Interpretation of the Selberg Trace Formula*

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N.B. This is a presentation by one of the authors [A.V.] of work done in collaboration [2, 3].

1 Notations

Take Σ as a compact surface of genus g with constant curvature K. The Gauss–Bonnet formula implies: $-K \times \text{Area}(\Sigma) = 4\pi(g-1)$. Below Σ will be either the sphere S^2 (with $K = +1$, $g = 0$), or a hyperbolic surface X (with $K = -1$, $g \geq 2$). If $-\Delta_\Sigma$ is the positive Laplacian on Σ, we associate with it the 'almost positive' operator:

$$P_\Sigma = \left(-\Delta_\Sigma + \frac{K}{4}\right)^{\frac{1}{2}} \tag{1}$$

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where the square root is taken positive on each eigenspace where P_Σ^2 is positive. In the hyperbolic case, denoting by $\{\lambda_n\}$ the spectrum of $-\Delta_X$ and by $\left\{\rho_n = (\lambda_n + K/4)^{1/2}\right\}$ the spectrum of P_X, we recall that the first few values $\lambda_n + K/4$ are negative (e.g., $\lambda_0 = 0$) and for these values we agree for instance that $\arg \rho_n = -\pi/2$.

We denote by $\mathcal{P}$ the set of periodic, oriented, primitive geodesics of the hyperbolic surface X and by $\tau(p)$ the length of $p \in \mathcal{P}$. The length spectrum of all the periodic geodesics, primitive or repeated, is the set

$$\{m\tau(p) |\, p \in \mathcal{P},\, m \in \mathbb{N} \setminus \{0\}\} \,.$$

This spectrum is placed in duality with the spectrum of eigenvalues of P_X by Selberg's trace formula [12, 8]

$$\begin{aligned} \mathrm{Tr}\; h(P_X) &= \sum_{n=0}^{\infty} h(\rho_n) \\ &= (g-1)\int_{-\infty}^{+\infty} h(\rho)\rho \tanh(\pi\rho)\, d\rho + \sum_{p\in\mathcal{P}}\sum_{m=1}^{\infty} R_{p,m}\hat{h}\left(m\tau(p)\right) \end{aligned} \tag{2}$$

where

$$\hat{h}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} h(\rho)e^{-i\tau\rho}\, d\rho$$

and $R_{p,m}$ is the 'hyperbolic weight'

$$R_{p,m} = \frac{\tau(p)}{2\sinh(m\tau(p)/2)} \,. \tag{3}$$

In (2) the test function $h(\rho)$ must be defined and analytic in some strip $|\mathrm{Im}\,\rho| < \frac{1}{2} + \epsilon$, in order for the series over the geodesics to converge, and decreasing of order $\mathcal{O}\left(|\rho|^{-2-\delta}\right)$ as $\rho \to \infty$, in order for the two other terms to converge. Moreover, $h(\rho)$ must be even.

The classical restrictions on the function $h(\rho)$ can be weakened, as we will show below, by using the analogy with Poisson's Summation Formula,

$$\sum_{n\in\mathbb{Z}} h(n) = 2\pi \sum_{m\in\mathbb{Z}} \hat{h}(2\pi m) \,. \tag{4}$$

2 The determinant formula [15, 7, 11, 14]

Poisson's Summation Formula can be understood from the two classical formulae for $\sinh(\pi\kappa)$

$$\sinh(\pi\kappa) = \pi\kappa \prod_{n=1}^{\infty}\left(1 + \frac{\kappa^2}{n^2}\right) \tag{5}$$

$$\sinh(\pi\kappa) = \frac{1}{2}e^{\pi\kappa}\left(1 - e^{-2\pi\kappa}\right) \,. \tag{6}$$

Supposing a little analyticity for $h(\rho)$, we evaluate the integral

$$\frac{1}{2\pi i}\int_{\mathrm{C}} h(-i\kappa)\, d\log\sinh(\pi\kappa)$$

taken on the contour of figure 1 by two different methods.

If we use (5) and the residue formula we immediately obtain the left-hand

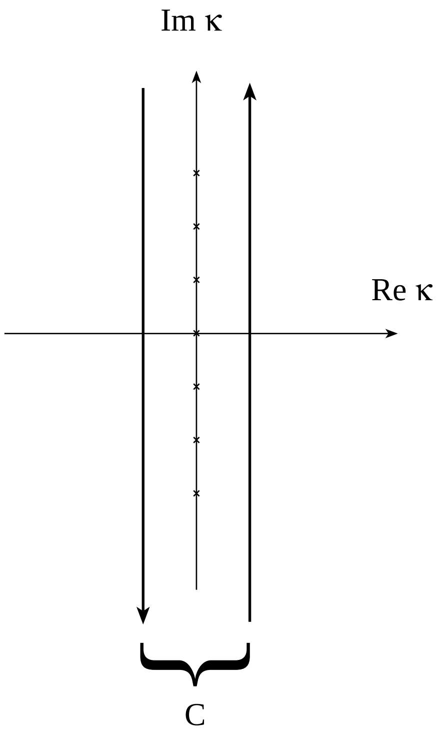

Figure 1: Contour of integration.

side of (4). On the other hand, if we use (6) in the half-plane $\mathrm{Re}\,\kappa > 0$ and the symmetric formula in $\mathrm{Re}\,\kappa < 0$ we get the right-hand side of (4). This proof is 'microanalytic' in the sense that it is based on the decomposition of the real distribution $\sum_{n\in\mathbb{Z}}\delta(\rho-n)$ as the difference of the boundary values of the analytic function $-1/(2\tanh i\pi\rho)$ as $\mathrm{Im}\,\rho\to\pm 0$.

Analogs of the formulae (5,6) for the Selberg case would arise by putting $h(\rho) = -\log(\rho^2+\kappa^2)$ in (2) (κ plays the role of a parameter)—were it not for the fact that the formula written this way diverges since $h(\rho)$ is not $\mathcal{O}\left(|\rho|^{-2-\delta}\right)$. However, it becomes convergent if we apply the operator $d/d(\kappa^2)$ twice. It is then transformed into the classic trace formula for $h(\rho) = \left(\rho^2+\kappa^2\right)^{-2}$ which expresses the function $\mathrm{Tr}\left(P_X^2+\kappa^2\right)^{-2}$.

If we introduce the generalised zeta function

$$\zeta_\Sigma(s,a) = \mathrm{Tr}\left(-\Delta_\Sigma + a\right)^{-s} = \mathrm{Tr}\left(P_\Sigma^2 - \frac{K}{4} + a\right)^{-s} \qquad (7)$$

where $\operatorname{Re} s > 1$ and $a \notin \mathbb{R}_-$, we see that we can remove the two differentiations in κ^2 thanks to the functional equation:

$$\zeta_\Sigma(s,a) = s\int_a^{+\infty} \zeta_\Sigma(s+1,a')\,da'\,. \tag{8}$$

In particular, by applying (8) twice to

$$\zeta_X\big(2,\kappa^2-\tfrac14\big) = \sum_{n=0}^{\infty}\big(\rho_n^2+\kappa^2\big)^{-2}$$

we get $\frac{\partial}{\partial s}\zeta_X\big(s,\kappa^2-\frac14\big)\big|_{s=0}$. Now, this by definition is $-\log\det\left(P_X^2+\kappa^2\right)$, where det designates the functional determinant of an operator (or the zeta-regularised determinant) [10].

After all calculations are completed, the trace formula regularised then exponentiated gives us:

$$\det\big(P_X^2+\kappa^2\big) = \Big(e^{\kappa^2}\det\left(P_{S^2}+\kappa\right)\Big)^{-(2g-2)} z_X\big(\tfrac12+\kappa\big) \tag{9}$$

where

$$z_X\big(\tfrac12+\kappa\big) = \prod_{p\in\mathcal{P}}\prod_{k=0}^{\infty}\Big(1-e^{\tau(p)\left(k+\frac12+\kappa\right)}\Big)\,,\quad (\operatorname{Re}\kappa > 1/2) \tag{10}$$

is the Selberg zeta function [12].

The formulae (9,10) are similar to (5,6). The appearance of the operator P_{S^2} on the sphere S^2 in (9) will be explained below.

3 An extension of the trace formula [2, 3]

If we suppose that the test function $h(\rho)$ is indeed even and analytic in $|\operatorname{Im}\rho| < \frac12+\epsilon$, we can recover the classic trace formula (2) from (9,10) by evaluating the integral

$$\frac{1}{4\pi i}\int_{\mathrm{C}} h(-i\kappa)\,d\log\det\big(P_X^2+\kappa^2\big)$$

taken on the contour C of figure 2 in two different ways, by analogy with the above proof of Poisson's Summation Formula.

But the same reasoning can extend to functions $h(\rho)$ having different analyticity properties, simply by changing the integration contour! The most interesting case is where $h(\rho)$ is not even, but is analytic in a sector $|\arg\rho| < \frac{\pi}{2}+\theta_0$, and the integration is along the contour C$'$ of figure 2. Under very mild decrease assumptions on $h(\rho)$ (just to allow some contour deformations), we thus obtain a new trace formula:

$$\sum_{n=0}^{\infty} h(\rho_n) = (2g-2)\int_0^{+\infty} h(\rho)\rho\tanh(\pi\rho)\,d\rho + \int_0^{+\infty} h_-(\kappa)\,d\log z_X\big(\tfrac12+\kappa-i0\big) \tag{11}$$

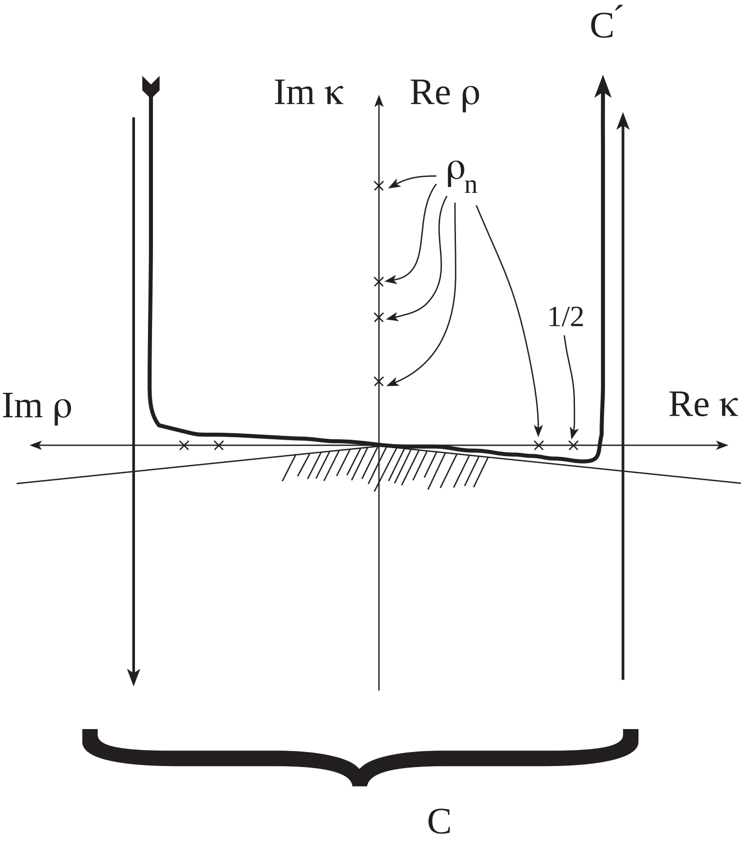

Figure 2: Contours of integration ($\kappa = i\rho$).

where we have defined

$$h_-(\kappa) = \frac{1}{2\pi i}\left(h(-i\kappa) - h(+i\kappa)\right) \ , \ |\arg \kappa| < \theta_0 \tag{12}$$

(the shift $\kappa \to \kappa - i0$ in (11) corresponds to the convention: $\arg \rho_n = -\pi/2$ if $\rho_n^2 < 0$).

4 A new interpretation of the trace formula

The extended trace formula (11) applies to the function $h(\rho) = e^{-t\rho}$. Thus we obtain a formula for $\Theta_X(t) := \mathrm{Tr}\ e^{-tP_X}$ when $\mathrm{Re}\, t > 0$

$$\begin{aligned} \Theta_X(t) &= (2g-2)\left(t^{-2} + 2\sum_{m=1}^{\infty}(-1)^m (t+2\pi m)^{-2}\right) \\ &\quad + \frac{1}{\pi}\int_0^{+\infty} \sin(t\kappa)\, d\log z_X\left(\tfrac{1}{2} + \kappa - i0\right) \ . \end{aligned} \tag{13}$$

The integral yields an odd meromorphic function of t with simple poles at the points $t = \pm i m \tau(p)$. The corresponding residues are $(1/2\pi) R_{p,m}$ (all of this follows from (10)). Poisson's formula on a manifold actually predicts these singularities which it situates at the lengths of the periodic geodesics of the said manifold [4, 5].

The formula (13) thus establishes a meromorphic extension for $\Theta_X(t)$ but now also displaying double poles on the negative real axis at the points $-2\pi m$, $m \in \mathbb{N}$. We interpret these poles as the contributions of *complex periodic geodesics* with lengths $2\pi i m$, $m \in \mathbb{Z}$ which appear through the change $t \to it$ implying $K \to -K$. Indeed, in the direction of imaginary time, the geometry of X becomes the geometry of the sphere S^2 for which the periodic geodesics are well known. The sphere moreover admits an exact Poisson summation formula

$$\Theta_{S^2}(t) = \mathrm{Tr}\, e^{-tP_{S^2}} = 2 \sum_{m\in\mathbb{Z}} (-1)^m (t - 2\pi i m)^{-2} \tag{14}$$

which strongly resembles the summation term in (13), as is made evident by the functional equation

$$\Theta_X(t) + \Theta_X(-t) = (2-2g)\Theta_{S^2}(it)\,. \tag{15}$$

These results confirm the hypothesis of Balian and Bloch [1] that the quantum evolution can be formulated in an exact way by a resummation of terms associated with real *and complex* classical trajectories. Here the quantum problem is to find the spectrum of the Laplacian on X and the classical trajectories are the periodic geodesics on the same surface. On the test functions $h(\rho) = e^{-t\rho}$ (parametrised by $t > 0$) Selberg's trace formula revealed itself to be a Poisson summation formula indexed by the real and complex periods of the geodesic flow on $X^{\mathbb{C}}$. We can also describe these contributions of the complexified flow as the manifestation of a 2-dimensional tunnel effect within the problem. Otherwise, Balian–Bloch decompositions have only been effectively achieved in one-dimensional cases, using ordinary-differential equation techniques [13, 6]. Our interpretation of Selberg's trace formula thus constitutes the first hint in favour of a generalisation of [13, 6] to partial differential equations.

5 The generalised zeta function

We briefly mention another application of formula (11), to generalised zeta functions $\zeta_\Sigma(s,a)$ defined by (7). We will see that from the point of view of their dependence on the variable s these functions become particularly simple, in the case of constant curvature surfaces, if we evaluate them at $a = K/4$, whereas the more traditional choice is $a = 0$ (Minakshisundaram–Pleijel).

Indeed, starting with $K = +1$, $\zeta_{S^2}(s,a)$ is expressed in terms of the Riemann zeta function for (and only for) $a = +1/4$:

$$\zeta_{S^2}\big(s, +\tfrac{1}{4}\big) = \big(2^{2s} - 2\big)\, \zeta(2s-1)\,. \tag{16}$$

But it is equally true in the hyperbolic case ($K = -1$) that $\zeta_X(s, a)$ takes its simplest form for $a = -1/4$; now, since $a < 0$, this result follows from the extended trace formula (11) which gives:

$$\zeta_X\big(s, -\tfrac{1}{4}\big) = \frac{1-g}{\cos(\pi s)} \zeta_{S^2}\big(s, +\tfrac{1}{4}\big) + \frac{\sin(\pi s)}{\pi} \int_0^{+\infty} \kappa^{-2s} \, d \log z_X\big(\tfrac{1}{2} + \kappa - i0\big) \tag{17}$$

(compare with [9] for the case $a \geq 0$). We must understand (17) as an identity linking two meromorphic extensions, that of $\zeta_X(s, -1/4)$ from $\mathrm{Re}\, s > 1$, and that of the Mellin integral from $\mathrm{Re}\, s < 1/2$! The analog of (17) for the sphere S^2 consists of writing Poisson's summation formula for $\zeta_{S^2}(s, +1/4)$, which restores Riemann's functional equation for $\zeta(s)$.

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