

# Separate Continuity, Joint Continuity and the Lindelöf Property

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**Abstract.** In this paper we prove a theorem more general than the following. Suppose that  $X$  is Lindelöf and  $\alpha$ -favourable and  $Y$  is Lindelöf and Čech-complete. Then for each separately continuous function  $f : X \times Y \rightarrow \mathbb{R}$  there exists a residual set  $R$  in  $X$  such that  $f$  is jointly continuous at each point of  $R \times Y$ .

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## 1 Introduction

If  $X$ ,  $Y$  and  $Z$  are topological spaces and  $f : X \times Y \rightarrow Z$  is a function then we say that  $f$  is *jointly continuous at*  $(x_0, y_0) \in X \times Y$  if for each neighbourhood  $W$  of  $f(x_0, y_0)$  there exists a product of open sets  $U \times V \subseteq X \times Y$  containing  $(x_0, y_0)$  such that  $f(U \times V) \subseteq W$  and we say that  $f$  is *separately continuous on*  $X \times Y$  if for each  $x_0 \in X$  and  $y_0 \in Y$  the functions  $y \mapsto f(x_0, y)$  and  $x \mapsto f(x, y_0)$  are both continuous on  $Y$  and  $X$  respectively. If the range space  $Z$  is a metric space, with metric  $d$ , and  $\varepsilon$  is a positive number then we say that  $f$  is  $\varepsilon$ -*jointly continuous at*  $(x_0, y_0) \in X \times Y$  if there exists a product of open sets  $U \times V \subseteq X \times Y$  containing  $(x_0, y_0) \in X \times Y$  such that  $d\text{-diam}f(U \times V) \leq \varepsilon$ .

Some form of our first lemma may be found in many of the papers written on separate and joint continuity.

**Lemma 1** *Let  $X$  and  $Y$  be topological spaces,  $\varepsilon$  be a positive number and  $(Z, d)$  be a metric space. If  $f : X \times Y \rightarrow Z$  is separately continuous on  $X \times Y$  but not  $\varepsilon$ -jointly continuous at  $(x_0, y_0) \in X \times Y$  then for each pair of open neighbourhoods  $U$  of  $x_0$  and  $V$  of  $y_0$  there exist points  $x$  and  $x'$  in  $U$  and  $y$  in  $V$  such that  $\varepsilon/3 < d(f(x, y), f(x', y))$ .*

**Proof:** Let  $U \times V \subseteq X \times Y$  be a product of open sets containing the point  $(x_0, y_0) \in X \times Y$ . Since  $f$  is separately continuous on  $X \times Y$  we can assume, by possibly making  $V$  smaller, that  $d(f(x_0, y), f(x_0, y_0)) < \varepsilon/6$  for all  $y \in V$ . Since  $f$  is not  $\varepsilon$ -jointly continuous at  $(x_0, y_0)$  there exist points  $(x, y)$  and  $(x', y')$  in  $U \times V$  such that  $\varepsilon < d(f(x, y), f(x', y'))$ . On the other hand,

$$\begin{aligned} d(f(x, y), f(x', y')) &\leq d(f(x, y), f(x_0, y)) + d(f(x_0, y), f(x_0, y')) + d(f(x_0, y'), f(x', y')) \\ &< d(f(x, y), f(x_0, y)) + d(f(x_0, y'), f(x', y')) + \varepsilon/3 \end{aligned}$$

Therefore, either  $\varepsilon/3 < d(f(x, y), f(x_0, y))$  or  $\varepsilon/3 < d(f(x_0, y'), f(x', y'))$ . ☺

For a topological space  $Y$  we shall denote by  $C(Y)$  the set of all real-valued continuous functions defined on  $Y$  and by  $C_p(Y)$  the set  $C(Y)$  endowed with the topology of pointwise convergence on  $Y$ . Further, if  $X$  is a topological space and  $f : X \rightarrow C(Y)$  then the mapping  $\tilde{f} : X \times Y \rightarrow \mathbb{R}$  defined by,  $\tilde{f}(x, y) := f(x)(y)$  is separately continuous on  $X \times Y$  if, and only if,  $f : X \rightarrow C_p(Y)$  is continuous. Hence there is a natural correspondence between the study of real-valued separately

continuous functions on  $X \times Y$  and the study of continuous mappings from  $X$  into  $C_p(Y)$ . With this in mind, we introduce the following definitions. We say that a mapping  $f : X \rightarrow C(Y)$  is *jointly continuous* at  $(x_0, y_0) \in X \times Y$  if the function  $\tilde{f}$  is jointly continuous at  $(x_0, y_0)$  and for each  $\varepsilon > 0$ , we will say that  $f$  is  $\varepsilon$ -*jointly continuous* at  $(x_0, y_0)$  if the function  $\tilde{f}$  is  $\varepsilon$ -jointly continuous at  $(x_0, y_0)$ .

With these definitions under our belt we can rephrase Lemma 1 as follows.

**Lemma 2** *Let  $X$  and  $Y$  be topological spaces and let  $f : X \rightarrow C_p(Y)$  be continuous. If for some  $\varepsilon > 0$ ,  $f$  is not  $\varepsilon$ -jointly continuous at  $(x_0, y_0) \in X \times Y$  then for each pair of open neighbourhoods  $U$  of  $x_0$  and  $V$  of  $y_0$  there exist points  $x$  and  $x'$  in  $U$  and  $y$  in  $V$  such that  $\varepsilon/3 < f(x)(y) - f(x')(y)$ .*

In addition to the previously mention notions of continuity we shall also require a weaker form of continuity. If  $f : X \rightarrow Y$  is a function acting from a topological space  $X$  into a topological  $Y$  then we say that  $f$  is *quasi-continuous* on  $X$  if for each pair of open subsets  $U$  of  $X$  and  $W$  of  $Y$  such that  $f(U) \cap W \neq \emptyset$  there exists a non-empty open set  $V \subseteq U$  such that  $f(V) \subseteq W$ . Although in general quasi-continuous functions are not obliged to have any points of continuity, the next lemma shows that when the domain space is Baire and the range space is metric such mappings must have many points of continuity.

**Lemma 3** [1] *Let  $f : X \rightarrow Y$  be a quasi-continuous mapping acting from a Baire space  $X$  into a metric space  $Y$ . Then  $\{x \in X : f \text{ is continuous at } x\}$  is residual in  $X$  (i.e., contains a countable intersection of dense open subsets of  $X$ ).*

For more information on the continuity of quasi-continuous mappings see, [4].

## 2 Main Result

The main result of this paper (Theorem 1) is based upon Lemma 6 which in turn is based upon the following lemma.

**Lemma 4** [2, Corollary 3] *Let  $Y$  be a compact and  $D$  be a dense and countable subset of  $Y$ . Then every subset of  $C(Y)$  which is compact with respect to the topology of pointwise convergence on  $D$  and Lindelöf with respect to the topology of pointwise convergence on  $Y$  is separable in  $(C(K), \|\cdot\|_\infty)$ .*

The proof of Lemma 6 relies upon the careful handling of second category sets (i.e., sets that are **not** of the first category). So here we shall introduce some notation that that will facilitate this. Let  $X$  be a topological space and let  $U$  be an open subset of  $X$ . We say that a subset  $A$  of  $X$  is *everywhere second category in  $U$*  if  $A \cap W$  is second category in  $X$  for each non-empty open subset  $W$  of  $U$ . For a subset  $A$  of  $X$  we shall denote by  $D(A)$  the union of all open subsets  $W$  for which  $A$  is everywhere second category in  $W$ . It is readily seen that  $A$  is everywhere second category in  $D(A)$  (i.e.,  $D(A)$  is the largest open subset of  $X$  in which  $A$  is everywhere second category). Although in general  $D(A)$  may be empty, it follows from [3, Proposition 3.2.5] that if  $A$  is second category in  $X$  then  $D(A) \neq \emptyset$ .

Lemma 6 also requires a version of the pigeonhole principle.

**Lemma 5** (Pigeonhole principle for second category sets) *Let  $f : X \rightarrow Y$  be a mapping from a second category set  $X$  into a non-empty set  $Y$ . If  $(V_n)_{n \in \mathbb{N}}$  is a cover of  $Y$  then for at least one  $n \in \mathbb{N}$ ,  $f^{-1}(V_n)$  is second category in  $X$ .*

**Lemma 6** *Let  $Y$  be a Lindelöf Čech-complete space,  $(X, d)$  a complete metric space and  $f : X \rightarrow C_p(Y)$  a quasi-continuous mapping. If there exists a Lindelöf subspace  $L$  of  $C_p(Y)$  such that  $f(X) \subseteq L$ , then there exists a residual subset  $R$  of  $X$  such that  $f$  is jointly continuous at each point of  $R \times Y$ .*

**Proof:** Let  $\beta Y$  denote the Stone-Čech-compactification of  $Y$  and let  $(G_n)_{n \geq 0}$  be a decreasing sequence of open subsets of  $\beta Y$  such that: (i)  $G_0 := \beta Y$  and (ii)  $Y = \bigcap_{n \geq 0} G_n$ . For each  $\varepsilon > 0$  consider the set

$$R_\varepsilon := \{x \in X : f \text{ is } \varepsilon\text{-jointly continuous at each point of } \{x\} \times Y\}.$$

Clearly,  $f$  is jointly continuous at each point of  $(\bigcap_{n \in \mathbb{N}} R_{1/n}) \times Y$ . Therefore, it will be sufficient to show that for each  $\varepsilon > 0$ ,  $R_\varepsilon$  is residual in  $X$ . To this end, let us fix  $\varepsilon > 0$ . In order to obtain a contradiction let us assume that  $X \setminus R_\varepsilon$  is second category in  $X$  [Note that for each  $x \in X \setminus R_\varepsilon$  there exists an element  $y \in Y$  that that  $f$  is not  $\varepsilon$ -jointly continuous at  $(x, y)$ ]. Let  $T$  be the set of all finite sequences of 0's and 1's. We shall inductively (on the length  $|t|$  of  $t \in T$ ) define the following: second category subsets  $X_t$  of  $X \setminus R_\varepsilon$ ; points  $x_t$  and  $x'_t$  in  $D(X_t)$ ; non-empty open subsets  $Y_t$  of  $\beta Y$ ; elements  $y_t \in Y_t \cap Y$  and sequences  $(O_n^t)_{n \in \mathbb{N}}$  of dense open subsets of  $X$  that fulfil the following properties:

- (i)  $X_t \subseteq X_{t'}$  and  $Y_t \subseteq Y_{t'}$  whenever  $t' < t$  (i.e., whenever  $t$  is an extension of  $t'$ );
- (ii)  $d\text{-diam}(X_t) < 1/2^{|t|}$  and  $\overline{Y_t}^{\beta Y} \subseteq G_{|t|}$ , where  $\overline{Y_t}^{\beta Y}$  denotes the closure of  $Y_t$  in  $\beta Y$ ;
- (iii)  $\overline{X_{t_0}} \cap \overline{X_{t_1}} = \emptyset$ ;
- (iv) for each  $x \in X_t$  there exists a  $y_x \in Y_t \cap Y$  such that  $f$  is not  $\varepsilon$ -jointly continuous at  $(x, y_x)$ ;
- (v)  $\varepsilon/3 < f(x_t)(y_t) - f(x'_t)(y_t)$ ;
- (vi)  $\varepsilon/3 < f(x)(y_t) - f(x')(y_t)$  for all  $x \in \overline{X_{t_1}}$  and all  $x' \in \overline{X_{t_0}}$ ;
- (vii) the mapping  $x \mapsto f(x)(y_t)$  is continuous at the points of  $\bigcap_{n \in \mathbb{N}} O_n^t$ ;
- (viii) if  $|t'| < |t|$  then  $\overline{X_t} \subseteq \bigcap \{O_k^{t'} : 1 \leq k \leq |t|\}$ .

*Base Step.* Let  $X_\emptyset$  be any second category subset of  $X \setminus R_\varepsilon$  with  $d$ -diameter less than 1 (Note: such a subset exists. For example, if  $W$  is any non-empty open subset of  $D(X \setminus R_\varepsilon)$  of  $d$ -diameter less than 1 then one could set  $X_\emptyset := W \cap (X \setminus R_\varepsilon)$ ) and let  $Y_\emptyset := \beta Y$ . Since  $X_\emptyset$  is a subset of  $X \setminus R_\varepsilon$ , for every  $x \in X_\emptyset$  there exists a  $y_x \in Y_\emptyset \cap Y$  such that  $f$  is not  $\varepsilon$ -jointly continuous at  $(x, y_x)$ . Therefore, by Lemma 2 there exist points  $x_\emptyset$  and  $x'_\emptyset$  in  $D(X_\emptyset)$  and an element  $y_\emptyset \in Y_\emptyset \cap Y$  such that

$$\varepsilon/3 < f(x_\emptyset)(y_\emptyset) - f(x'_\emptyset)(y_\emptyset).$$

Also since the mapping  $x \mapsto f(x)(y_\emptyset)$  is quasi-continuous it follows from Lemma 3 that there exists a sequence  $(O_n^\emptyset)_{n \in \mathbb{N}}$  of dense open subsets of  $X$  such that the mapping  $x \mapsto f(x)(y_\emptyset)$  is continuous at the points of  $\bigcap_{n \in \mathbb{N}} O_n^\emptyset$ .

Assuming that we have defined the second category subsets  $X_t$  of  $X$ , the points  $x_t$  and  $x'_t \in D(X_t)$ , the non-empty open subsets  $Y_t$  of  $\beta Y$ , the elements  $y_t \in Y_t \cap Y$  and the sequences  $(O_n^t)_{n \in \mathbb{N}}$  of dense open subsets of  $X$ , that satisfy the properties (i)-(viii) for each  $t \in T$  with  $|t| \leq n$ , we shall proceed to the next step.

*Inductive Step.* Consider  $t \in T$  of length  $n$ . Since  $\varepsilon/3 < f(x_t)(y_t) - f(x'_t)(y_t)$  and  $x \mapsto f(x)(y_t)$  is quasi-continuous there exist non-empty open sets  $W_0$  and  $W_1$  with  $d$ -diameter less than  $1/2^{n+1}$  such that

$$\emptyset = \overline{W_0} \cap \overline{W_1} \subseteq \overline{W_0} \cup \overline{W_1} \subseteq D(X_t) \cap \bigcap \{O_k^t : 1 \leq k \leq n+1 \text{ and } |t'| < n+1\}$$

and  $\varepsilon/3 < f(x)(y_t) - f(x')(y_t)$  for all  $x \in \overline{W_1}$  and all  $x' \in \overline{W_0}$ . Now, since  $\overline{Y_t \cap Y}$  (i.e., the closure of  $Y_t \cap Y$  with respect to  $Y$ ) is Lindelöf there exist open subsets  $(V_n)_{n \in \mathbb{N}}$  of  $\beta Y$  such that

$$Y_t \cap Y \subseteq \overline{Y_t \cap Y}^Y \subseteq \bigcup_{n \in \mathbb{N}} V_n \subseteq \bigcup_{n \in \mathbb{N}} \overline{V_n}^{\beta Y} \subseteq G_{n+1}$$

Hence by Lemma 5 for each  $i \in \{0, 1\}$  there exists a second category subset  $X_{ti} \subseteq X_t \cap W_i$  and a  $n_i \in \mathbb{N}$  such that for each  $x \in X_{ti}$  there exists a  $y_x \in V_{n_i} \cap Y_t \cap Y$  for which  $f$  is not  $\varepsilon$ -jointly continuous at  $(x, y_x)$ . Therefore by Lemma 2, for each  $i \in \{0, 1\}$  there exist points  $x_{ti}$  and  $x'_{ti}$  in  $D(X_{ti})$  and an element  $y_{ti} \in V_{n_i} \cap Y_t \cap Y$  such that  $\varepsilon/3 < f(x_{ti})(y_{ti}) - f(x'_{ti})(y_{ti})$ . Also since the mapping  $x \mapsto f(x)(y_{ti})$  is quasi-continuous it follows from Lemma 3 that there exists a sequence  $(O_n^{ti})_{n \in \mathbb{N}}$  of dense open subsets of  $X$  such that the mapping  $x \mapsto f(x)(y_{ti})$  is continuous at the points of  $\bigcap_{n \in \mathbb{N}} O_n^{ti}$ . The inductive step is completed by defining  $Y_{ti} := V_{n_i} \cap Y_t$ .

Let  $D := \{y_t : t \in T\}$ . We claim that  $\overline{D}^Y$  (i.e., the closure of  $D$  with respect to  $Y$ ) is a compact subset of  $Y$ . To see this note that for each  $n \in \mathbb{N}$

$$\overline{D}^{\beta Y} \subseteq \bigcup_{|t|=n} \overline{Y_t}^{\beta Y} \cup \{y_t : 0 \leq |t| < n\} \subseteq \bigcup_{|t|=n} \overline{Y_t}^{\beta Y} \cup D \subseteq \bigcup_{|t|=n} \overline{Y_t}^{\beta Y} \cup Y \subseteq G_n$$

and so  $\overline{D}^{\beta Y} \subseteq \bigcap_{n \geq 0} G_n = Y$ . Therefore,  $\overline{D} = \overline{D}^{\beta Y}$ ; which is compact in  $\beta Y$ .

By the construction, the set  $K := \bigcap_{n \in \mathbb{N}} K_n$ , where  $K_n := \bigcup_{|t|=n} \overline{X_t}$ , is a closed and totally bounded subset of  $X$  (and hence compact, since  $X$  is complete). Furthermore, the construction also yields that for each  $t \in T$ ,  $K \subseteq \bigcap_{n \in \mathbb{N}} O_n^t$ . Thus, for each  $t \in T$  the mapping  $x \mapsto f(x)(y_t)$  is continuous on  $K$ . Note also that for each pair of distinct points  $x$  and  $x'$  in  $K$  there exists a  $t \in T$  such that  $\varepsilon/3 < |f(x)(y_t) - f(x')(y_t)|$ . Next we consider the continuous mapping  $\mathcal{R} : C_p(Y) \rightarrow C_p(\overline{D})$  defined by,  $\mathcal{R}(f) := f|_{\overline{D}}$ . Then  $(\mathcal{R} \circ f)(K)$  is a non-separable subset of  $(C(\overline{D}), \|\cdot\|_\infty)$  that is compact with respect to the topology of pointwise convergence on  $D$ . However, by Lemma 4 this is impossible since  $(\mathcal{R} \circ f)(K) \subseteq \mathcal{R}(L)$ ; which is Lindelöf. Hence it must be the case that for each  $\varepsilon > 0$ ,  $R_\varepsilon$  is residual in  $X$ . ☺ .

To formulate the statement of our main theorem we will need to consider the following topological game.

Let  $X$  be a topological space. On  $X$  we shall consider the *Choquet game* played between two players  $\alpha$  and  $\beta$ . A *play* of this game is a decreasing sequence of, alternately chosen, non-empty open subsets  $A_n \subseteq B_n \subseteq \dots B_2 \subseteq A_1 \subseteq B_1$ , where the sets  $A_n$  are chosen by player  $\alpha$  and the sets  $B_n$  by player  $\beta$ . The player  $\alpha$  is said to have *won* a play of the Choquet game if  $\bigcap_{n \in \mathbb{N}} B_n \neq \emptyset$ . Otherwise player  $\beta$  is said to have won the play. A *strategy*  $s$  for the player  $\alpha$  is a rule that tells him or her how to play (possibly depending on all the previous moves of player  $\beta$ ). Since the moves of player  $\alpha$  may depend on the previous moves of player  $\beta$ , we denote the  $n^{\text{th}}$  move of player  $\alpha$  by,  $s(B_1, B_2, \dots B_n)$ . Any sequence of non-empty open subsets  $(B_n)_{n \in \mathbb{N}}$  of  $X$  that satisfy  $B_{n+1} \subseteq s(B_1, B_2, \dots B_n)$  for all  $n \in \mathbb{N}$  is called an *s-play*. We say that  $s$  is a *winning strategy*, if using it, player  $\alpha$  wins every play, independently of the moves of player  $\beta$ , (i.e.,  $\bigcap_{n \in \mathbb{N}} B_n \neq \emptyset$  for

each  $s$ -play  $(B_n)_{n \in \mathbb{N}}$ . A topological space  $X$  is called an  $\alpha$ -favourable space if  $\alpha$  has a winning strategy in the Choquet game played on  $X$ . More information on the Choquet game can be found in [7].

Let  $X$  be an  $\alpha$ -favourable space and let  $s$  be a winning strategy for the player  $\alpha$  in the Choquet game played on  $X$ . We shall denote by  $P$  the space of all  $s$ -plays endowed with the Baire metric  $d$ , that is, if  $p := (B_n)_{n \in \mathbb{N}}$  and  $p' := (B'_n)_{n \in \mathbb{N}}$  then  $d(p, p') := 0$  if  $p = p'$  and  $d(p, p') := 1/n$  otherwise, where  $n := \min\{i \in \mathbb{N} : B_i \neq B'_i\}$ . It is straight forward to verify that  $(P, d)$  is a complete metric space, [4].

**Lemma 7** [5, Proposition 2.3] *Let  $X$  be an  $\alpha$ -favourable space,  $s$  be a winning strategy for the player  $\alpha$  in the Choquet game played on  $X$  and let  $P$  denote the space of all  $s$ -plays, endowed with the Baire metric. If  $f : X \rightarrow Z$  is a quasi-continuous mapping into a topological space  $Z$  then the set-valued mapping  $F : P \rightarrow 2^Z$  defined by,*

$$F(p) := \bigcap_{n \in \mathbb{N}} f(B_n), \quad \text{where } p := (B_n)_{n \in \mathbb{N}} \text{ is an } s\text{-play.}$$

*has non-empty values and is a “minimal mapping” (i.e., for every pair of open subsets  $U$  of  $P$  and  $W$  of  $Z$  such that  $F(U) \cap W \neq \emptyset$  there exists a non-empty open subset  $V \subseteq U$  such that  $F(V) \subseteq W$ ).*

To exploit the previous lemma we need to establish the connection between minimal mappings and the continuity of their selections.

**Lemma 8** [6, Lemma 1.1] *Let  $F : X \rightarrow 2^Z$  be a minimal mapping acting from a topological space  $X$  into non-empty subsets of a Hausdorff space  $Z$  and let  $\sigma : X \rightarrow Z$  be any selection of  $F$  (i.e.,  $\sigma(x) \in F(x)$  for all  $x \in X$ ). If  $\sigma$  is continuous at  $x_0 \in X$  then  $F(x_0) = \{\sigma(x_0)\}$ .*

**Theorem 1** *Let  $Y$  be a Lindelöf Čech-complete space,  $X$  an  $\alpha$ -favourable space and  $f : X \rightarrow C_p(Y)$  a quasi-continuous mapping. If there exists a Lindelöf subspace  $L$  of  $C_p(Y)$  such that  $f(X) \subseteq L$ , then there exists a residual subset  $R$  of  $X$  such that  $f$  is jointly continuous at each point of  $R \times Y$ .*

**Proof:** Let  $s$  be a winning strategy for the player  $\alpha$  in the Choquet game played on  $X$  and let  $P$  denote the space of all  $s$ -plays, endowed with the Baire metric. On the complete metric space  $P$  we define the set-valued mapping  $F : P \rightarrow 2^{C(Y)}$  by,

$$F(p) := \bigcap_{n \in \mathbb{N}} f(B_n), \quad \text{where } p := (B_n)_{n \in \mathbb{N}} \text{ is an } s\text{-play}$$

By Lemma 7 the mapping  $F$  has non-empty values and is a minimal mapping. Let  $\sigma : P \rightarrow C(Y)$  be any selection of  $F$ . Then since  $F$  is minimal,  $\sigma$  is quasi-continuous. Hence by Lemma 6 there exists a residual subset  $R'$  of  $P$  such that  $\sigma$  is jointly continuous at each point of  $R' \times Y$ . Note that in particular this means that  $\sigma$  is continuous on  $R'$  when  $C(Y)$  is considered with the topology of pointwise convergence on  $Y$ . Therefore, by Lemma 8,  $F(p) = \{\sigma(p)\}$  for each  $p \in R'$ . We are now in a position to apply the “Generic Factorisation Theorem” (i.e., Theorem 1.2 in [5]) to obtain a continuous function  $g : R \rightarrow R'$  defined on a residual subset  $R$  of  $X$  such that  $f(x) = \sigma(g(x))$  for each  $x \in R$ . Thus it follows that  $f|_R$  is jointly continuous at each point of  $R$ . Then with a small amount of extra effort we can deduce that  $f$  is in fact jointly continuous at each point of  $R$ . ☺

**Remark** If one really wanted to “squeeze the pips” out of the previous theorem one could prove the slightly more general statement given below.

“Let  $Y$  be a Lindelöf Čech-complete space,  $D \subseteq Y$  be a dense subset,  $X$  an  $\alpha$ -favourable space and  $f : X \rightarrow C(Y)$  a quasi-continuous mapping with respect to the topology of pointwise convergence on  $D$ . If  $\{f(x)|_{\overline{E}} : x \in X\}$  is contained in a Lindelöf subset of  $C_p(\overline{E})$  for each countable subset  $E$  of  $D$  that is relatively compact in  $Y$  then there exists a residual subset  $R$  of  $X$  such that  $f$  is jointly continuous at each point of  $R \times Y$ .”

**Corollary 1** *Suppose that  $X$  is Lindelöf and  $\alpha$ -favourable and  $Y$  is Lindelöf and Čech-complete. Then for each separately continuous function  $f : X \times Y \rightarrow \mathbb{R}$  there exists a residual set  $R$  in  $X$  such that  $f$  is jointly continuous at each point of  $R \times Y$ .*

**Proof:** Consider the mapping  $\hat{f} : X \rightarrow C_p(Y)$  defined by,  $\hat{f}(x)(y) := f(x, y)$ . Since  $f$  is separately continuous  $\hat{f} : X \rightarrow C_p(Y)$  is continuous and hence  $\hat{f}(X)$  is Lindelöf. The result now follows from Theorem 1. ☺

**Corollary 2** *Suppose that  $X$  is separable and  $\alpha$ -favourable and  $Y$  is Lindelöf and Čech-complete. Then for each separately continuous function  $f : X \times Y \rightarrow \mathbb{R}$  there exists a residual set  $R$  in  $X$  such that  $f$  is jointly continuous at each point of  $R \times Y$ .*

**Proof:** Consider the mapping  $\hat{f} : X \rightarrow C_p(Y)$  defined by,  $\hat{f}(x)(y) := f(x, y)$ . Since  $f$  is separately continuous  $\hat{f} : X \rightarrow C_p(Y)$  is continuous and hence  $\hat{f}(X)$  is separable in  $C_p(Y)$ . Thus for each compact subset  $K$  of  $Y$ ,  $\{f(x)|_K : x \in X\}$  is separable in  $(C(K), \|\cdot\|_\infty)$  and so Lindelöf in  $C_p(K)$ . The result now follows from the Remark. ☺

## References

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