Rankings of Multisets and Discrete Cones

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Abstract: We study additive representability of orders on multisets (of size k drawn from a set of size n) which satisfy the condition of Independence of Equal Submultisets (IES) introduced by Sertel and Slinko (2002). Here we take a geometric view of those orders, and relate them to certain combinatorial objects which we call discrete cones. Following Fishburn (1996) and Conder and Slinko (2003), we define functions $f(n,k)$ and $g(n,k)$ which measure the maximal possible deviation of an arbitrary order satisfying the IES and an arbitrary almost representable order satisfying the IES, respectively, from a representable order. We prove that $g(n,k) = n - 1$ whenever $n \geq 3$ and $(n,k) \neq (5,2)$. In the exceptional case, $g(5,2) = 3$. We also prove that $g(n,k) \leq f(n,k) \leq n$ and establish that for small n and k the functions $g(n,k)$ and $f(n,k)$ coincide.

Key words: multiset, linear orders, Independence of Equal Submultisets, additive representability

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1   Introduction

The most common preference ordering in economics is that of bundles of k commodities, and the most common problem is to rationalise such an ordering, representing it by a numerical order-preserving function of utilities of those commodities. The space of commodity bundles can be naturally represented as a subset of a Cartesian product $X = \prod_{i=1}^{n} X_i$, where X_i can vary depending on the model. It is often assumed that each X_i has a very rich structure and is isomorphic to $\mathbb{R}$, so that X becomes a finite-dimensional Euclidean space $\mathbb{R}^n$. This case was extensively studied by J. von Neumann and O. Morgenstern [16], J. Marshak [11], I.N. Herstein and J. Milnor [6] and others. Another particular case, when the X_i have no structure at all and are arbitrary finite sets, was studied by P. Fishburn [5].

In this paper we take the middle ground and assume that each X_i has the structure of the monoid of nonnegative integers $\mathbb{N}$. This model has several useful interpretations.

Example 1. *We assume that there are n different types of goods, which are indivisible. Each $\mathbf{x} \in X$ is a commodity bundle $\mathbf{x} = (k_1, \dots, k_n)$, where i is the type of good and k_i is its quantity — which, due to the indivisibility assumption, is a nonnegative integer, that is, $k_i \in \mathbb{N}$. The total number of goods is $k = \sum_{i=1}^{n} k_i$. The relation $\mathbf{x} \succeq \mathbf{y}$ means that a consumer thinks $\mathbf{x}$ is at least as good as $\mathbf{y}$. (See W.D. Katzner [7].)*

Example 2. *We assume a dynamic process with discrete time, say an income stream. Here i is a point of time and $k_i \in \mathbb{N}$ is the dollar amount of income at this point of time. The total amount to be received is $k = \sum_{i=1}^{n} k_i$. The relation $\mathbf{x} \succeq \mathbf{y}$ means that a beneficiary thinks income stream $\mathbf{x}$ is at least as good as income stream $\mathbf{y}$. (See T.C. Koopmans [8].)*

Example 3. *In this example we consider the composition of a k-member parliament or committee. There are n political parties to which the members are affiliated. Here i is the type of political party and k_i is the number of members of this political party elected to the parliament. The total number of elected parliamentarians is $k = \sum_{i=1}^{n} k_i$. The relation $\mathbf{x} \succeq \mathbf{y}$ means that a voter thinks parliament $\mathbf{x}$ is at least as good as parliament $\mathbf{y}$. (See M.R. Sertel and E. Kalaycıoğlu [13].)*

Mathematically speaking, in all these examples we are talking about multisets on the set $[n] = \{1, 2, \dots, n\}$. Unlike sets, multisets allow multiple entry of elements, so in each example the object under consideration — the bundle, income stream or committee — can be represented as a multiset $M = \{1^{k_1}, 2^{k_2}, \dots, n^{k_n}\}$, where i^{k_i} means that element i enters the multiset

k_i times. The number k_i is called the multiplicity of i in M. The multiset M can also be described as $([n], \mu)$ where $\mu \colon [n] \to \mathbb{N}$ is the multiplicity function given by $\mu(i) = k_i$ for all $i \in [n]$. The sum of multiplicities $k = k_1 + k_2 + \ldots + k_n$ is called the cardinality of M. In all three examples we deal with orderings of the set of all multisets of fixed cardinality k on the set $[n]$, which will be denoted by $\mathcal{P}_k[n]$.

By an *order* we understand any reflexive, complete and transitive binary relation, which will be denoted as $\succeq$. If it is also antisymmetric, it will be called a *linear order*. The notation $\mathbf{x} \succ \mathbf{y}$ will mean $\mathbf{x} \succeq \mathbf{y}$ but not $\mathbf{y} \succeq \mathbf{x}$. Orders describe preferences on $\mathcal{P}_k[n]$.

In this paper we will be interested in additive representation of preferences (See the monograph [17] for a systematic discussion of the idea and the history of the subject.)

Definition 1. *An order $\succeq$ on $\mathcal{P}_k[n]$ is said to be* representable *if there exist nonnegative real numbers $u_1, \ldots, u_m$ such that for all $M_1 = ([n], \mu_1)$ and $M_2 = ([n], \mu_2)$ belonging to $\mathcal{P}_k[n]$,*

$$M_1 \succeq M_2 \iff \sum_{i=1}^{n} \mu_1(i) u_i \geq \sum_{i=1}^{n} \mu_2(i) u_i. \tag{1}$$

The coefficients $u_1, \ldots, u_n$ have different meanings in our three basic examples (namely bundles of goods, income streams, or committees), as they represent utilities of goods, individual perceptions of the beneficiary about inflation, or coefficients of influence of political parties, respectively. Nevertheless, we will refer to them simply as *utilities*. The problem considered here and in many similar situations (as in [3, 5, 9, 10] for example) is to impose minimal conditions on the order $\succeq$ to guarantee its representability. This means that the consumer, beneficiary or voter must be (to a certain extent) rational in order for their preferences to be representable. The question is, just how rational they should be.

To formulate our main rationality condition we need the operation of multiset union. The union $M_1 \cup M_2$ of two multisets $M_1 = ([n], \mu_1)$ and $M_2 = ([n], \mu_2)$ is again a multiset on $[n]$ whose multiplicity function is defined as $\mu_1 + \mu_2$. Similarly, the intersection $M_1 \cap M_2$ is given by the multiplicity function $\min(\mu_1, \mu_2)$, where this minimum is defined pointwise on $[n]$. Next, we say that M_1 is a submultiset of M_2, if $\mu_1(i) \leq \mu_2(i)$ for all $i \in [n]$, and we denote this by $M \subseteq N$. For more information on multisets see, for example, [15].

The following rationality condition was suggested by Sertel and Slinko [14], who called it *consistency*. Here we give slightly different (but equivalent)

definition of this concept, which makes it a close relative to the concept of the Independence of Equal Coordinates [17, p.30].

Definition 2. *An order $\succeq$ on $\mathcal{P}_k([n])$ is said to satisfy the* Independence of Equal Submultisets *condition (IES) if for every two multisets U and V of the same cardinality j with $1 \leq j \leq k-1$, the relation $U \cup W \succeq V \cup W$ holds for one particular multiset W of cardinality $k-j$ if and only if this relation holds for every multiset W of cardinality $k-j$.*

This is also an analogue of de Finetti's axiom for comparative probability orders [3, 4].

If $\succeq$ is an order on $\mathcal{P}_k([n])$ which satisfies the IES, then, for all j in the range $1 \leq j \leq k-1$, it induces an order $\succeq_j$ on $\mathcal{P}_j([n])$, which also satisfies the IES. To define $\succeq_j$ we take any $W \in \mathcal{P}_{k-j}([n])$, and then for any $U, V \in \mathcal{P}_j([n])$, we set

$$U \succeq_j V \quad \text{if and only if} \quad U \cup W \succeq V \cup W. \tag{2}$$

By the IES, this will not depend on the choice of the particular multiset W.

Without loss of generality, we assume that $\succeq$ induces a linear order on $[n]$ (which is naturally identified with $\mathcal{P}_1([n])$), and that this linear order satisfies the condition

$$1 \succ_1 2 \succ_1 3 \succ_1 \ldots \succ_1 n. \tag{3}$$

It is important that we rule out the possibility of indifferences here. This does not restrict our framework, since if $j \succeq i$ for some $j > i$, then we may view the elements $i, i+1, \ldots, j$ as indistinguishable, and then we may simplify the framework by not making any distinction between these elements.

The next definition is technical but very important.

Definition 3. *We say that an order $\succeq$ on $\mathcal{P}_k[n]$ satisfies the mth cancellation condition C_m if for no m distinct comparisons $A_i \succeq B_i$, $i = 1, 2, \ldots, m$, among which $A_i \succ B_i$ for at least one i, there exist positive integers $a_1, \ldots, a_m$ such that the following two multiset unions coincide*

$$\bigcup_{i=1}^{m} \underbrace{(A_i \cup \ldots \cup A_i)}_{a_i} = \bigcup_{i=1}^{m} \underbrace{(B_i \cup \ldots \cup B_i)}_{a_i}. \tag{4}$$

This is a complete analogue of the mth cancellation condition formulated by Kraft, Pratt and Seidenberg [9] for orders on the power set $2^{[n]}$. As in [9, Theorem 2] it is easy to show that for an order $\succeq$ on $\mathcal{P}_k([n])$ to be

representable it is necessary and sufficient that all cancellation conditions $C_2, C_3, \ldots$ are satisfied. This will become obvious later, when we reformulate this statement in vector form, in which the cancellation conditions look much more natural.

Example 4. *The nonrepresentable linear order A_4 on $\mathcal{P}_2[4]$ constructed in [14] does not satisfy the condition C_3, since it contains the following comparisons:*

$$\{1,3\} \succ \{2^2\}, \quad \{2,3\} \succ \{1,4\}, \quad \{2,4\} \succ \{3^2\}. \tag{5}$$

Indeed, the union of the multisets on the right and the union of the multisets on the left are both equal to the multiset $\{1, 2^2, 3^2, 4\}$. Thus C_3 is violated with $a_1 = a_2 = a_3 = 1$.

In order to convince the reader that the above cancellation conditions are natural, we note that the IES for $\succeq$ follows from C_2, since it cannot be true that $A \cup W_1 \succeq B \cup W_1$ and $B \cup W_2 \succ A \cup W_2$. When $W_1 = W_2 = \emptyset$ we obtain also antisymmetry. It is also obvious that C_2 follows from the IES and antisymmetry, and hence for antisymmetric orders, condition C_2 and the IES are equivalent. Transitivity of $\succeq$ is implied by C_3 as it is impossible to have simultaneously $A \succeq B$, $B \succeq C$, and $C \succ A$. In fact, the above representability result can be formulated for any complete relation $\succeq$.

It should be noted that, although transitivity follows from C_3, it is weaker than C_3 (see Example 4). It is of special interest when the IES implies $C_3, C_4, \ldots$, and hence representability of $\succeq$. For $n = 3$, Sertel and Slinko [14] showed that the IES implies C_i for all $i \geq 3$, and hence all orders on $\mathcal{P}_k[3]$ which satisfy the IES are representable, for all $k \geq 1$. This is no longer true for $n > 3$. Nevertheless, sometimes something good can be said about $\succeq$ even when it is not representable.

Definition 4. *We will say that an order $\succeq$ on $\mathcal{P}_k[n]$ is* almost representable *if there exist nonnegative real numbers $u_1, \ldots, u_m$ such that for all $M_1 = ([n], \mu_1)$ and $M_2 = ([n], \mu_2)$ belonging to $\mathcal{P}_k[n]$,*

$$M_1 \succeq M_2 \Longrightarrow \sum_{i=1}^{n} \mu_1(i) u_i \geq \sum_{i=1}^{n} \mu_2(i) u_i. \tag{6}$$

This is a complete analogue of the almost representability condition for comparative probability orders [9, 12].

Almost representable orders satisfying the IES need not be representable (see [14, Sect. 3]). Such linear orders may exist only when the representable

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order associated with utilities $u_1, \ldots, u_m$ is non-linear and some multisets are tied. There are also linear orders which satisfy the IES but still fail to be almost representable (see [14, Sect. 4]). More precisely, examples are given in [14] of a non-representable but almost representable linear order in $\mathcal{P}_2[4]$, and a linear order in $\mathcal{P}_3[4]$ that fails to be almost representable; both satisfy the IES but fail to satisfy C_3.

Thus it is clear that to secure representability, sometimes we have to assume more than the IES (or C_2, which is the same), and we may need also some of the conditions $C_3, C_4, \ldots$. It is important to find out how far do we have to go, and how many cancellation conditions we need to assume before we can guarantee the representability of an order on $\mathcal{P}_k[n]$. The current paper is devoted to this particular question. All results will be formulated in terms of the following two functions.

For $n \geq 3$ and $k \geq 2$, let $f(n, k)$ be the smallest positive integer such that any linear order on $\mathcal{P}_k[n]$ which satisfies $C_2, C_3, \ldots, C_{f(n,k)}$ is representable. Similarly, let $g(n, k)$ be the smallest positive integer such that any *almost* representable linear order on $\mathcal{P}_k[n]$ which satisfies $C_2, C_3, \ldots, C_{g(n,k)}$ is representable. Clearly, $g(n, k) \leq f(n, k)$ for all n and k. Sertel and Slinko [14] showed in this notation that $f(3, k) = g(3, k) = 2$ for all $k \geq 1$.

In this paper we will completely describe the function $g(n, k)$. Specifically, we will show that $g(n, k) = n - 1$ for all $n \geq 3$ and $k \geq 2$ apart from the pair $(n, k) = (5, 2)$, for which $g(5, 2) = 3$. As for the function $f(n, k)$, the best we can prove in general is that $n - 1 \leq f(n, k) \leq n$ whenever $(n, k) \neq (5, 2)$. Computer-assisted calculations show that $g(n, k) = f(n, k)$ for small values of n and k (namely, for $(n, k) = (4, 2), (4, 3), (5, 2), (5, 3), (6, 2)$ and $(7, 2)$), and so we conjecture that this is true in general.

To obtain these results we take a geometric view of linear orders, and define *discrete cones* (similar to those defined by Fishburn [4] for comparative probability orders on a set). We will show that to every order $\succeq$ on $\mathcal{P}_k[n]$ satisfying the IES there corresponds a discrete cone $C(\succeq)$, and vice versa.

2   Discrete Cones

To define discrete cones we need the following notation. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$, let $M_k = \{-k, \ldots, -1, 0, 1, \ldots, k\}$ and let M_k^n be

the nth Cartesian product of M_k. Next, for $\mathbf{x} = (x_1, \dots, x_n) \in M_k^n$, define

$$
\begin{aligned}
|\mathbf{x}| &= \sum_{i=1}^{n} |x_i|, \\
\mathbf{x}^+ &= (v_1, \dots, v_n), \quad \text{where } v_i = \max(x_i, 0), \text{ and} \\
\mathbf{x}^- &= (u_1, \dots, u_n), \quad \text{where } u_i = \min(x_i, 0),
\end{aligned}
$$

so that $\mathbf{x} = \mathbf{x}^+ + \mathbf{x}^-$ and $|\mathbf{x}| = |\mathbf{x}^+| + |\mathbf{x}^-|$. Similarly, for any two vectors $\mathbf{x} = (x_1, \dots, x_n) \in M_k^n$ and $\mathbf{y} = (y_1, \dots, y_n) \in M_k^n$, let

$$
\max(\mathbf{x}, \mathbf{y}) = (w_1, \dots, w_n), \quad \text{where } w_i = \max(x_i, y_i).
$$

Finally, we define T_k^n, a set in which the cones associated with orders on $\mathcal{P}_k[n]$ will live, by

$$
T_k^n = \left\{ \mathbf{x} \in M_k^n \mid |\mathbf{x}^-| = |\mathbf{x}^+| \leq k \right\}.
$$

This consists of all vectors in M_k^n which are orthogonal to the vector $\mathbf{n} = (1, 1, \dots, 1)$ and whose sum of positive entries is no greater than k.

Definition 5. *A subset $C \subseteq T_k^n$ is said to be a* discrete cone *(or simply, a* cone*) in T_k^n if the following conditions hold:*

D1. $\mathbf{e}_i - \mathbf{e}_{i+1} \in C \setminus (-C)$ *for* $1 \leq i \leq n-1$,

D2. $\mathbf{x} \in C$ *or* $-\mathbf{x} \in C$ *for every* $\mathbf{x} \in T_k^n$,   *and*

D3. $\mathbf{x} + \mathbf{y} \in C$ *whenever* $\mathbf{x}, \mathbf{y} \in C$ *and* $|\max(-\mathbf{x}^-, \mathbf{y}^+)| \leq k$.

Given a multisetset $M = ([n], \mu) \in \mathcal{P}_k[n]$, let $\chi(M) = (\mu(1), \dots, \mu(n))$ be the characteristic function of M. Given an order $\succeq$ on $\mathcal{P}_k[n]$, for every comparison $M \succeq N$ we construct a vector $\chi(M, N) = \chi(M) - \chi(N) \in T_k^n$. We can now define $C(\succeq)$ to be the set of all vectors $\chi(M, N)$ for all valid comparisons $M \succeq N$, where $M, N \in \mathcal{P}_k[n]$. Note that if $\succeq$ satisfies the IES, then $C(\succeq)$ is well-defined. Moreover, we have the following:

Proposition 1. *If $\succeq$ satisfies the IES, then $C(\succeq)$ satisfies conditions D1 to D3 and so is a discrete cone. On the other hand, every discrete cone $C \subseteq T_k^n$ defines a linear order $\succeq$ on $\mathcal{P}_k[n]$ which satisfies the IES, and in which $A \succeq B$ if and only if $\chi(A, B) \in C$, for every two multisets $A, B \in \mathcal{P}_k[n]$.*

Proof. Let $\succeq$ be a linear order on $\mathcal{P}_k[n]$ which satisfies the IES. Then conditions D1 and D2 obviously hold for the set $C(\succeq)$, and only D3 is not immediately clear. So suppose that $\mathbf{x} = (x_1, \ldots, x_n) = \chi(A, B) \in C(\succeq)$ and that $\mathbf{y} = (y_1, \ldots, y_n) = \chi(D, E) \in C(\succeq)$, with $|\max(-\mathbf{x}^-, \mathbf{y}^+)| \leq k$. In fact, we may suppose that $A \cap B = \emptyset$ and $D \cap E = \emptyset$, so that $\mathbf{x}^+ = \chi(A)$ and $-\mathbf{x}^- = \chi(B)$ while $\mathbf{y}^+ = \chi(D)$ and $-\mathbf{y}^- = \chi(E)$, and that $|\mathbf{x}^-| = |\mathbf{x}^+| = |A| = |B| = s$ while $|\mathbf{y}^-| = |\mathbf{y}^+| = |D| = |E| = t$, say, where $s \leq k$ and $t \leq k$. This means that $A, B \in \mathcal{P}_s[n]$ and $D, E \in \mathcal{P}_t[n]$ with $A \succeq_s B$ and $D \succeq_t E$, where $\succeq_s$ and $\succeq_t$ are the orders induced by $\succeq$ on $\mathcal{P}_s[n]$ and $\mathcal{P}_t[n]$, respectively. The fact that $|\max(-\mathbf{x}^-, \mathbf{y}^+)| \leq k$ implies that there exist two multisets F_1 and F_2 such that $F_1 \cup B = F_2 \cup D$, and the cardinality ℓ of this common union is not greater than k. By Lemma 1 of [14],

$$F_1 \cup A \succeq_\ell F_1 \cup B = F_2 \cup D \succeq_\ell F_2 \cup E,$$

and hence by transitivity, $F_1 \cup A \succeq_\ell F_2 \cup E$. It now follows that

$$
\begin{aligned}
\mathbf{x} + \mathbf{y} &= \chi(A \cup D, B \cup E) = \chi(A \cup F_2 \cup D, B \cup F_2 \cup E) \\
&= \chi(A \cup F_1 \cup B, B \cup F_2 \cup E) = \chi(A \cup F_1, F_2 \cup E) \in C(\succeq).
\end{aligned}
$$

For the second part, let C be any cone in T_k^n. We define an order on each of the sets $\mathcal{P}_\ell[n]$ for $\ell \leq k$ by setting for any two multisets $A, B \in \mathcal{P}_\ell[n]$

$$A \succeq_\ell B \quad \text{whenever} \quad \chi(A, B) \in C.$$

By its definition, the order $\succeq_k (=\succeq)$ on $\mathcal{P}_k[n]$ will satisfy the IES, but we need to prove its transitivity. So suppose $A \succeq B$ and $B \succeq D$ for some $A, B, D \in \mathcal{P}_k[n]$. Let $\mathbf{x} = \chi(A, B)$ and $\mathbf{y} = \chi(B, D)$. Then $|\max(-\mathbf{x}^-, \mathbf{y}^+)| \leq k$ since each entry of both $-\mathbf{x}^-$ and $\mathbf{y}^+$ is bounded above by the corresponding entry of $\chi(B)$, and $|\chi(B)| = k$, so $\chi(A, D) = \chi(A) - \chi(B) + \chi(B) - \chi(D) = \mathbf{x} + \mathbf{y} \in C$, and therefore $A \succeq D$. $\square$

Example 5. *For the linear order $\succeq$ on $\mathcal{P}_2[4]$ from Example 4, the three vectors*

$$\mathbf{x} = (1, -2, 1, 0), \ \mathbf{y} = (0, 1, -2, 1), \ \mathbf{z} = (-1, 1, 1, -1) \tag{7}$$

belong to $C(\succeq)$, which is a cone in T_2^4. These vectors correspond to the three comparisons given in (5). Note that for each pair of distinct vectors $\mathbf{u}, \mathbf{v}$ in the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ we have $|\max(\mathbf{u}^+, \mathbf{v}^-)| = 3$, and that explains why no sum of two vectors from this set belongs to $C(\succeq)$.

Continuing, we will say that a cone $C \subseteq T_k^n$ is *generated* by a set of vectors $V = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ if C is the smallest cone that contains V. It follows that the cone $C(\succeq)$ associated with an order $\succeq$ satisfying the IES always contains the cone U in T_k^n generated by

$$\mathbf{e}_1 - \mathbf{e}_2, \ \dots, \ \mathbf{e}_{n-1} - \mathbf{e}_n. \tag{8}$$

It is easy to see that a vector $\mathbf{x} = (x_1, \dots, x_n) \in T_k^n$ lies in U if and only if

$$x_1 \geq 0, \ \ x_1 + x_2 \geq 0, \ \dots, \ \text{and} \ \ x_1 + x_2 + \dots + x_{n-1} \geq 0, \tag{9}$$

and at least one of these inequalities is strict. More generally, vectors in U represent consequences of the IES and condition (3).

We can reformulate now the cancellation conditions as follows:

Proposition 2. *An order $\succeq$ satisfies the mth cancellation condition if and only if for no m-subset $\{\mathbf{x}_1, \dots, \mathbf{x}_m\} \subseteq C(\succeq)$ do there exist positive integers $a_1, \dots, a_m$ such that*

$$a_1 \mathbf{x}_1 + a_2 \mathbf{x}_2 + \dots + a_m \mathbf{x}_m = 0. \tag{10}$$

Proof. If m distinct comparisons $A_i \succeq B_i$ exist as in Definition 3, then we take vectors $\mathbf{x}_i = \chi(A_i, B_i) \in C(\succeq)$ for $1 \leq i \leq m$, and condition (4) implies that $a_1 \mathbf{x}_1 + a_2 \mathbf{x}_2 + \dots + a_m \mathbf{x}_m = \mathbf{0}$. The converse is also clear. $\square$

Definition 6. *Suppose for an order $\succeq$ on $\mathcal{P}_k[n]$ there exists a set of m vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_m\} \subseteq C(\succeq)$ and positive integers $a_1, \dots, a_m$ such that (10) holds. Then we say that the order $\succeq$ violates the mth cancellation condition for vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ with multiplicities $(a_1, \dots, a_m)$.*

Example 6. *The three vectors from $C(\succeq)$ given in Example 5 add up to the zero vector, hence the order $\succeq$ violates C_3 with multiplicities $(1, 1, 1)$, and is not representable.*

Geometrically, what is happening is clear. An order $\succeq$ is representable provided there exists a positive-integer-valued vector $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$ such that $w_1 > w_2 > \dots > w_n$ and such that

$$\mathbf{x} \in C(\succeq) \iff (\mathbf{w}, \mathbf{x}) > 0 \ \text{ for every } \mathbf{x} \in T_k^n \setminus \{\mathbf{0}\}, \tag{11}$$

that is, all non-zero vectors in the cone $C(\succeq)$ lie in the open half-space $H_\mathbf{w} = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{w}, \mathbf{x}) > 0\}$. The cone $C(\succeq)$ in this case is a pointed cone.

Similarly, for any order $\succeq$ that is almost representable, there exists a nonnegative-integer-valued vector $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$ such that both $w_1 \geq w_2 \geq \ldots \geq w_n \geq 0$ and

$$\mathbf{x} \in C(\succeq) \implies (\mathbf{w}, \mathbf{x}) \geq 0 \quad \text{for every } \mathbf{x} \in T_k^n, \tag{12}$$

so that in this case the whole of the cone $C(\succeq)$ lies in the closed half-space $\overline{H_{\mathbf{w}}} = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{w}, \mathbf{x}) \geq 0\}$.

Note that in both cases, the vector $\mathbf{w}$ represents the vector of utilities.

3 Construction Theorem

Proposition 3. *Let $\succeq$ be an almost representable order with a vector of utilities $\mathbf{w}$. Suppose that the mth cancellation condition is violated for vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ with multiplicities $(a_1, \ldots, a_m)$. Then all the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ lie in the hyperplane $H_{\mathbf{w}}$.*

Proof. First we note that $(\mathbf{w}, \mathbf{x}) > 0$ for all those $\mathbf{x} \in C(\succeq)$ which do not belong to $H_{\mathbf{w}}$. Next, condition (10) implies that

$$\sum_{i=1}^{n} a_i(\mathbf{w}, \mathbf{x}_i) = 0,$$

and since $a_i > 0$ for $1 \leq i \leq n$, this can hold only when all $(\mathbf{w}, \mathbf{x}_i) = 0$ for all i, and hence we find that $\mathbf{x}_i \in H_{\mathbf{w}}$ for $1 \leq i \leq n$. $\square$

Corollary 1. *Any almost representable order on $\mathcal{P}_k[n]$ which satisfies C_3, $C_4, \ldots, C_{n-1}$ is representable.*

Proof. Suppose that for some m, the mth cancellation condition is violated by vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m \in C(\succeq)$ with multiplicities $(a_1, \ldots, a_m)$, that is, with $a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m = \mathbf{0}$. Now since all the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ belong to the $(n-2)$-dimensional subspace $H_{\mathbf{w}} \cap H_{\mathbf{n}}$, where $H_{\mathbf{n}}$ is the hyperplane with normal vector $\mathbf{n} = (1, 1, \ldots, 1)$, we may use standard linear algebra to reduce the number of vectors in this linear combination to at most $n-1$ vectors, while keeping all coefficients positive. We claim that if $m \geq n$, then one of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ can be excluded from the linear combination. Indeed in this case the set of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_{m-1}\}$ is linearly dependent, and so there exist real numbers $b_1, \ldots, b_{m-1}$ with at least one b_i being positive, such that $b_1\mathbf{x}_1 + \cdots + b_{m-1}\mathbf{x}_{m-1} = \mathbf{0}$. Hence for all $\epsilon \in \mathbb{R}$ we have

$$(a_1 - \epsilon b_1)\mathbf{x}_1 + \ldots + (a_m - \epsilon b_m)\mathbf{x}_m + a_m\mathbf{x}_m = \mathbf{0}.$$

When $\epsilon = 0$, all coefficients in this equation are positive. If we choose the smallest positive ϵ for which $a_i - \epsilon b_i = 0$ for some i, then the resulting linear combination will not contain $\mathbf{x}_i$, and will still be nontrivial since $a_m > 0$. Hence we may suppose that $m \leq n - 1$. Since $\succeq$ satisfies $C_4, \ldots, C_{n-1}$, it follows that $\succeq$ is representable. $\square$

Proposition 4. $g(n, k) \leq n - 1$ *and* $f(n, k) \leq n$ *for all n and k.*

Proof. The first inequality $g(n, k) \leq n - 1$ follows from Corollary 1, and the second inequality $f(n, k) \leq n$ follows from a similar argument in the $(n - 1)$-dimensional subspace $H_{\mathbf{n}}$. $\square$

The following theorem enables us to construct orderings that are almost representable but fail C_m for particular values of m.

Theorem 1 (Construction method). *For $m \geq 3$, let $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ be a system of m non-zero vectors from T_k^n satisfying the following conditions:*

(a) $\displaystyle\sum_{i=1}^{m} a_i \mathbf{x}_i = \mathbf{0}$ *for some positive integers $a_1, \ldots, a_m$,*

(b) $|\max(-\mathbf{x}_i^-, \mathbf{x}_j^+)| > k$ *for every pair $\{i, j\} \subset \{1, 2, \ldots, m\}$ with $i \neq j$, and*

(c) *no proper subsystem $X' \subset X$ is linearly dependent with positive coefficients.*

Suppose further that the $m \times n$ matrix A whose rows are the given vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ has the property that $A\mathbf{w} = \mathbf{0}$ for some positive-integer-valued vector $\mathbf{w} = (w_1, \ldots, w_n)$ with $w_1 > w_2 > \ldots > w_n > 0$, and that

$$H_{\mathbf{w}} \cap (T_k^n \setminus \{\mathbf{0}\}) = \{\pm\mathbf{x}_1, \ldots, \pm\mathbf{x}_m\}.$$

Let $\succeq$ be the (nonlinear) order on $\mathcal{P}_k[n]$ with the vector of utilities $\mathbf{w}$, and let $C(\succeq) = \{\mathbf{x} \in T_k^n \mid (\mathbf{x}, \mathbf{w}) \geq 0\}$ be the discrete cone associated with $\succeq$. Then the set

$$C' = C(\succeq) \setminus \{-\mathbf{x}_1, \ldots, -\mathbf{x}_m\}$$

is a discrete cone that corresponds to an almost representable order on $\mathcal{P}_k[n]$ which satisfies C_j for all $j < m$, but does not satisfy C_m.

Proof. First we note that $\{\pm\mathbf{x}_1, \ldots, \pm\mathbf{x}_m\} \subseteq H_\mathbf{w} = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{w}, \mathbf{x}) = 0\}$. Next $(\mathbf{e}_i - \mathbf{e}_{i+1}, \mathbf{w}) = w_i - w_{i+1} > 0$ for $1 \le i < n$, so the property D1 holds for the cone C'. Property D2 holds for C' because it holds for $C(\succeq)$ and, when we remove vectors $-\mathbf{x}_1, \ldots, -\mathbf{x}_m$ from C', the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ remain in C'. Now let us prove D3. Suppose that $\mathbf{y}, \mathbf{z} \in C'$ and $|\max(-\mathbf{y}^-, \mathbf{z}^+)| \le k$, but $\mathbf{y} + \mathbf{z} \notin C'$. Since the cone $C(\succeq)$ satisfies D3 we know that $\mathbf{y} + \mathbf{z} \in C(\succeq)$, so $\mathbf{y} + \mathbf{z} \in C(\succeq) \setminus C'$, and hence $\mathbf{y} + \mathbf{z} = -\mathbf{x}_i$ for some i. Moreover, $\mathbf{y}, \mathbf{z} \in H_\mathbf{w}$, since otherwise $-(\mathbf{x}_i, \mathbf{w}) = (\mathbf{y} + \mathbf{z}, \mathbf{w}) = (\mathbf{y}, \mathbf{w}) + (\mathbf{z}, \mathbf{w}) > 0$. It follows from (c) that $\mathbf{y} = \mathbf{x}_j$ for some j and $\mathbf{z} = \mathbf{x}_\ell$ for some ℓ. This contradicts hypothesis (b), however, since $|\max(-\mathbf{y}^-, \mathbf{z}^+)| \le k$. Thus D3 holds for C'.

Suppose now that some violation of C_j occurs, say

$$c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 + \cdots + c_j \mathbf{y}_j = \mathbf{0},$$

with positive integers $c_1, \ldots, c_j$ and $\mathbf{y}_i \in C'$ for $1 \le i \le j$, where $j < m$. If $(\mathbf{y}_i, \mathbf{w}) > 0$ for some i, then $0 = (\mathbf{0}, \mathbf{w}) = (c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 + \cdots + c_j \mathbf{y}_j, \mathbf{w}) = c_1(\mathbf{y}_1, \mathbf{w}) + c_2(\mathbf{y}_2, \mathbf{w}) + \ldots + c_j(\mathbf{y}_j, \mathbf{w}) > 0$, a contradiction, hence $\mathbf{y}_i \in H_\mathbf{w}$ for all i, and therefore $\{\mathbf{y}_1, \ldots, \mathbf{y}_j\} \subseteq \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$, which contradicts (c). Finally, C_m fails by hypothesis (a). $\square$

4   Characterisation of $g(n, k)$

In this section we will use the construction method from the previous section and Proposition 4 to characterise the function $g(n, k)$ as follows:

Theorem 2. *For all $n \ge 3$ and $k \ge 2$,*

$$g(n, k) = \begin{cases} n - 2 & \text{if } (n, k) = (5, 2), \\ n - 1 & \text{otherwise.} \end{cases}$$

We prove this theorem partly by theoretical means and partly by direct calculations for small n and k (with the help of the computer algebra system MAGMA [1], using the same techniques of enumeration as described in [2]).

As was mentioned above, the case $n = 3$ of this theorem was proved in [14], so we need only consider cases where $n \ge 4$.

First we present the theoretical results. We will use the following notation extensively: for any vector $\mathbf{v}$, let the expression $x_i(\mathbf{v})$ denote the ith coordinate entry of $\mathbf{v}$.

4.1   Proof of Theorem 2 for $n = 4$.

Consider the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ that make up the rows of the 3×4 matrix

$$A = \begin{bmatrix} 0 & -(k-1) & k & -1 \\ 1 & -1 & -(k-1) & k-1 \\ -1 & k & -1 & -k+2 \end{bmatrix}. \tag{13}$$

It is clear that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. It can also be seen (by inspection of the first and fourth co-ordinates) that no linear combination $y_1\mathbf{v}_1 + y_2\mathbf{v}_2$ of $\mathbf{v}_1$ and $\mathbf{v}_2$ can be a vector of integers unless both y_1 and y_2 are integers. The row-space of A has a normal vector $\mathbf{w} = (k^2 - k + 2,\, k + 1,\, k,\, 1)$, satisfying the requirement $w_1 > w_2 > \ldots > w_n > 0$ of Theorem 1 since $k^2 - k + 1 > k$ for all $k > 1$. Similarly the given vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ satisfy the requirement that $|\max(-\mathbf{v}_i^-, \mathbf{v}_j^+)| > k$ for every two i and j, since $2k - 1 > k$ for all $k > 1$.

Next, because $|\mathbf{v}_1^+| = |\mathbf{v}_1^-| = |\mathbf{v}_2^+| = |\mathbf{v}_2^-| = k$, no proper multiple of $\mathbf{v}_1$ or $\mathbf{v}_2$ can lie in T_k^4, and hence any vector $\mathbf{v}$ in $H_{\mathbf{w}} \cap T_k^4$ other than $\mathbf{0}$, $\pm\mathbf{v}_1$ or $\pm\mathbf{v}_2$ must be of the form $\mathbf{v} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$ with non-zero coefficients $m_1, m_2 \in \mathbb{Z}$. Now if $m_1 > 0$ and $m_2 > 0$, then since $-k \leq x_2(\mathbf{v}) = -m_1(k-1) - m_2 < 0$ we find that $m_1 = m_2 = 1$, giving $\mathbf{v} = (1, -k, 1, k-2) = -\mathbf{v}_3$. Similarly, if $m_1 < 0$ and $m_2 < 0$, then inspection of $x_2(\mathbf{v})$ gives $m_1 = m_2 = -1$ and $\mathbf{v} = \mathbf{v}_3$. On the other hand, if $m_1 > 0 > m_2$ or $m_1 < 0 < m_2$ then $x_3(\mathbf{v}) = m_1 k - m_2(k-1)$ lies outside the acceptable range M_k for co-ordinates of vectors in T_k^4. Hence the only non-zero vectors in $H_{\mathbf{w}} \cap T_k^4$ are $\pm\mathbf{v}_1$, $\pm\mathbf{v}_2$ and $\pm\mathbf{v}_3$.

By Theorem 1, these vectors give rise to an almost representable order on $\mathcal{P}_k[4]$ which fails to satisfy C_3, and by Proposition 4 it follows that $g(4, k) = 3$ for all $k \geq 2$.

4.2   Proof of Theorem 2 for $n = 5$ and $k \geq 4$.

Consider the four vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ forming the rows of the 4×5 matrix

$$A = \begin{bmatrix} 1 & -(k-1) & k-3 & 2 & -1 \\ 0 & 1 & -k & k-2 & 1 \\ 0 & 0 & 1 & -k & k-1 \\ -1 & k-2 & 2 & 0 & -(k-1) \end{bmatrix}.$$

We note that $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$, any three of the given vectors are linearly independent, and the vector $\mathbf{w} = (k^3 - 3k^2 + 6k - 3,\, k^2 - k + 3,\, k + 1,\, 2,\, 1)$ is a normal vector to the row-space of A. Also the given vectors can easily be seen to satisfy the requirement that $|\max(-\mathbf{v}_i^-, \mathbf{v}_j^+)| > k$ whenever $i \neq j$. Next, we need the following:

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Lemma 1. $H_{\mathbf{w}} \cap T_k^5 = \{\mathbf{0}, \pm\mathbf{v}_1, \pm\mathbf{v}_2, \pm\mathbf{v}_3, \pm\mathbf{v}_4\}$.

Proof. Clearly $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for the 3-dimensional subspace $H_{\mathbf{w}} \cap H_{\mathbf{n}}$, so every vector $\mathbf{v}$ in $H_{\mathbf{w}} \cap T_k^5$ must be of the form $\mathbf{v} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + y_3\mathbf{v}_3$, and moreover, from inspection of $x_1(\mathbf{v})$, $x_2(\mathbf{v})$ and $x_3(\mathbf{v})$ it is obvious that y_1, y_2 and y_3 have to be integers. Without loss of generality (since $-T_k^5 = T_k^5$), we may assume that $y_1 \geq 0$. We proceed case-by-case:

Case 1: Suppose $y_1 = 0$. If $y_2 \neq 0$ then y_2 and y_3 must have the same sign (for otherwise $x_3(\mathbf{v}) = -y_2k + y_3$ lies outside M_k), and then inspection of $x_2(\mathbf{v}) + x_5(\mathbf{v}) = 2y_2 + (k-1)y_3$ forces $y_3 = 0$ (because membership of T_k^5 requires $|\mathbf{v}^-| = |\mathbf{v}^+| \leq k$). Hence either $y_2 = 0$ or $y_3 = 0$, giving $\mathbf{v} = \mathbf{0}, \pm\mathbf{v}_2$ or $\pm\mathbf{v}_3$.

Case 2: Suppose $y_1 > 0$ and $y_2 = 0$. Then inspection of $x_2(\mathbf{v}) = -y_1(k-1)$ gives $y_1 = 1$. Also if $y_3 \neq 0$ then $y_3 > 0$ (for otherwise $x_4(\mathbf{v}) = 2y_1 - y_3k > k$), but then $\mathbf{v}^- = (0, -(k-1), 0, 2 - y_3k, 0)$ so $|\mathbf{v}^-| = k - 3 + y_3k > k$, a contradiction. Hence $\mathbf{v} = \mathbf{v}_1$ in this case.

Case 3: Suppose $y_1 > 0$, $y_2 \neq 0$ and $y_3 = 0$. Then inspection of $x_3(\mathbf{v}) = y_1(k-3) - y_2k$ gives $y_2 > 0$, following which inspection of $x_4(\mathbf{v}) = 2y_1 + y_2(k-2)$ gives $y_1 = y_2 = 1$, but then $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 = (1, -(k-2), -3, k, 0)$ so $|\mathbf{v}^+| = 1 + k > k$, a contradiction. Hence no such $\mathbf{v}$ exists in this case.

Case 4: Suppose $y_1 > 0$, $y_2 < 0$ and $y_3 \neq 0$. Then since $-k \leq x_2(\mathbf{v}) = -y_1(k-1) + y_2 < 0$ we find that $y_1 = 1$ and $y_2 = -1$, and therefore $\mathbf{v} = (1, -k, 2k-3+y_3, 4 - k(1+y_3), -2 + y_3(k-1))$. Now as $x_2(\mathbf{v}) = -k$, no other co-ordinates of $\mathbf{v}$ can be negative, so $4 - k(1+y_3) = x_4(\mathbf{v}) \geq 0$ and $-2 + y_3(k-1) = x_5(\mathbf{v}) \geq 0$. The first of these two inequalities gives $y_3 \leq 0$ but the second gives $y_3 > 0$, a contradiction, so no such $\mathbf{v}$ exists in this case.

Case 5: Suppose $y_1 > 0$, $y_2 > 0$ and $y_3 \neq 0$. Then since $k \geq x_4(\mathbf{v}) = 2y_1 + y_2(k-1) - y_3k$ we find $y_3 > 0$, so all three coefficients are positive. If $y_1 = y_2 = y_3 = 1$, then we have $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = -\mathbf{v}_4$. Otherwise $\max(y_1, y_2, y_3) > 1$. If $y_2 > y_1$ then $x_5(\mathbf{v}) = -y_1 + y_2 + y_3(k-1) \geq 1 + k - 1 = k$ and then $|\mathbf{v}^+| \geq x_1(\mathbf{v}) + x_5(\mathbf{v}) \geq 1 + k > k$, a contradiction, hence $y_2 \leq y_1$. Similarly if $y_3 = \max(y_1, y_2, y_3) > 1$, then $x_5(\mathbf{v}) = -y_1 + y_2 + y_3(k-1) \geq -y_3 + 1 + y_3(k-1) = y_3(k-2)+1 \geq 2(k-2)+1 = 2k-3 > k$, a contradiction, and hence we may suppose that $y_1 = \max(y_1, y_2, y_3) > 1$. Now this implies $-k \leq x_2(\mathbf{v}) = -y_1(k-1)+y_2 \leq -y_1(k-1)+y_1 = (2-k)y_1 \leq (2-k)2 = 4-2k$, and therefore $k \geq 4$. In fact equality must occur, so $y_1 = y_2 = 2$, but then $\mathbf{v} = (2, -4, -6+y_3, 8-4y_3, 3y_3)$, and requiring $x_3(\mathbf{v}) \geq -k$ and $x_5(\mathbf{v}) \leq k$ forces the contradictory inequalities $y_3 \geq 2$ and $y_3 \leq 1$.

As these five cases cover all possibilities, the proof is complete. $\square$

By Theorem 1, there exists an almost representable order on $\mathcal{P}_k[5]$ which fails C_4, and by Proposition 4 it follows that $g(5, k) = 4$ for all $k \geq 4$.

4.3   Proof of Theorem 2 for $n \geq 6$ and $k \geq 3$.

The proof for this case is split into a number of observations and lemmata. Let us consider the following $(n - 2) \times n$ matrix:

$$
A = \begin{bmatrix}
1 & -(k-1) & k-3 & 1 & 0 & \ldots & 0 & 0 & 0 & 1 & -1 \\
0 & 1 & -k & k-2 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -k & k-2 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -k & k-2 & 1 & 0 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -k & k-2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -k & k-1
\end{bmatrix}.
$$

Let $\mathbf{v}_i$ be the ith row of A. Clearly the vectors $\mathbf{v}_1, \mathbf{v}_1, \ldots, \mathbf{v}_{n-2}$ are linearly independent, and therefore span a subspace of $\mathbb{R}^n$ of dimension $n - 2$. Also it is easy to check that $|\mathbf{v}_i^+| = |\mathbf{v}_i^-| = k$ for all i, and that $|\max(-\mathbf{v}_i^-, \mathbf{v}_j^+)| > k$ whenever $i \neq j$. For later use, we make note the following:

$$
\begin{aligned}
x_1(\mathbf{v}) &= y_1, \\
x_2(\mathbf{v}) &= -y_1(k - 1) + y_2, \\
x_3(\mathbf{v}) &= y_1(k - 3) - y_2 k + y_3, \\
x_i(\mathbf{v}) &= y_{i-3} + y_{i-2}(k - 2) - y_{i-1}k + y_i \quad \text{for } 4 \leq i \leq n - 2, \\
x_{n-1}(\mathbf{v}) &= y_1 + y_{n-4} + y_{n-3}(k - 2) - y_{n-2}k, \quad \text{and} \\
x_n(\mathbf{v}) &= -y_1 + y_{n-3} + y_{n-2}(k - 1).
\end{aligned}
$$

In addition, we will use particular sums of these co-ordinates, including

$$
\begin{aligned}
x_1(\mathbf{v}) + x_3(\mathbf{v}) &= y_1(k - 2) - y_2 k + y_3, \\
x_2(\mathbf{v}) + x_3(\mathbf{v}) &= -2y_1 - y_2(k - 1) + y_3, \\
x_1(\mathbf{v}) + x_3(\mathbf{v}) + x_4(\mathbf{v}) &= y_1(k - 1) - 2y_2 - y_3(k - 1) + y_4,
\end{aligned}
$$

and (for $4 \leq i \leq n-2$) the sums

$$
x_i(\mathbf{v}) + x_{i+1}(\mathbf{v}) = y_{i-3} + y_{i-2}(k-1) - 2y_{i-1} - y_i(k-1) + y_{i+1},
$$

$$
x_2(\mathbf{v}) + x_3(\mathbf{v}) + x_i(\mathbf{v}) = -2y_1 - y_2(k-1) + y_3 + y_{i-3} + y_{i-2}(k-2) - y_{i-1}k + y_i,
$$

$$
\text{and also } \quad x_2(\mathbf{v}) + x_3(\mathbf{v}) + \ldots + x_i(\mathbf{v}) = -y_1 - y_{i-2} - y_{i-1}(k-1) + y_i.
$$

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All of these sums must lie in the interval $M_k = \{-k, \ldots, -1, 0, 1, \ldots, k\}$, irrespective of whether their terms $x_j(\mathbf{v})$ are positive or negative (since, for example, if the sum exceeds k then the sub-sum of positive terms exceeds k).

Lemma 2. *Suppose that some linear combination* $\mathbf{v} = \sum_{i=1}^{n-2} y_i \mathbf{v}_i$ *of the* $\mathbf{v}_i$ *belongs to* T_k^n. *Then* $y_i \in \mathbb{Z}$ *for all* i.

Proof. We have $y_1 = x_1(\mathbf{v}) \in \mathbb{Z}$, and as $-y_1(k-1) + y_2 = x_2(\mathbf{v}) \in \mathbb{Z}$ we find $y_2 \in \mathbb{Z}$. Clearly this argument can be repeated, to show $y_i \in \mathbb{Z}$ for all i. $\square$

Lemma 3. *There is a vector* $\mathbf{w} = (w_1, \ldots, w_n)$ *belonging to the null space of* A *such that* $w_1 > w_2 > \ldots > w_n > 0$.

Proof. The null space of A has dimension 2 (since $\mathbf{v}_1, \ldots, \mathbf{v}_{n-2}$ are linearly independent), and contains the vector $\mathbf{n} = (1, 1, \ldots, 1)$. To construct another vector $\mathbf{u} = (u_1, \ldots, u_n)$ in this null space, let us set $u_n = 0$ and $u_{n-1} = 1$. Then from orthogonality with the last row of A we see that $u_{n-2} = k$. We proceed by reverse induction to prove that $u_i > 2u_{i+1}$ for $2 \leq i \leq n-1$. This is clearly true for $i = n-1$, and for $i = n-2$ since $k > 2$. Now let us assume that $u_{i+1} \geq 2u_{i+2}$ and $u_{i+2} \geq 2u_{i+3}$ for some $i \leq n-3$. Then by orthogonality with the ith row of A we have the inductive step

$$
\begin{aligned}
u_i &= ku_{i+1} - (k-2)u_{i+2} - u_{i+3} \\
&= 2u_{i+1} + (k-2)(u_{i+1} - u_{i+2}) - u_{i+3} \\
&\geq 2u_{i+1} + (k-2)u_{i+2} - u_{i+3} \\
&\geq 2u_{i+1} + u_{i+2} - u_{i+3} \\
&\geq 2u_{i+1},
\end{aligned}
$$

as required. Similarly, orthogonality with the first row of A gives

$$
\begin{aligned}
u_1 &= (k-1)u_2 - (k-3)u_3 - u_4 = 2u_2 + (k-3)(u_2 - u_3) - u_4 \\
&\geq 2u_2 + (k-3)u_3 - u_4 \geq 2u_2 - u_4 > u_2 + u_3 - u_4 \geq u_2.
\end{aligned}
$$

Thus we have $u_1 > u_2 > \ldots > u_n = 0$, and we can simply take $\mathbf{w} = \mathbf{u} + \mathbf{n}$ to satisfy the requirement that all co-ordinates of $\mathbf{w}$ are positive. $\square$

Lemma 4. $H_{\mathbf{w}} \cap T_k^n = \{\mathbf{0}, \pm\mathbf{v}_1, \pm\mathbf{v}_2, \ldots, \pm\mathbf{v}_{n-2}, \pm\mathbf{v}_{n-1}\}$, *where* $\mathbf{v}_{n-1}$ *is the vector* $-(\sum_{i=1}^{n-2} \mathbf{v}_i) = (-1, k-2, 2, 0, \ldots, 0, -k+1)$.

Proof. Clearly $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-2}\}$ is a basis for the $(n-2)$-dimensional subspace $H_{\mathbf{w}} \cap H_{\mathbf{n}}$, so every vector $\mathbf{v}$ in $H_{\mathbf{w}} \cap T_k^n$ is of the form $\mathbf{v} = \sum_{i=1}^{n-2} y_i \mathbf{v}_i$, and by Lemma 2, all the y_i are integers. To find all possibilities for $\mathbf{v}$, we may assume without loss of generality that $y_1 \geq 0$ (since $-T_k^n = T_k^n$). As before, we consider several cases:

Case 1: Suppose $y_1 = 0$. If $y_i = 0$ for all i then $\mathbf{v} = \mathbf{0}$. Otherwise let j be the smallest positive integer for which $y_j \neq 0$. Then without loss of generality we may suppose that $y_j \geq 1$. With these hypotheses, we will prove that $\mathbf{v}$ is one of $\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_{n-3}$ or $\mathbf{v}_{n-2}$. To do so, assume the contrary, and further, suppose that $\sum_{i=2}^{n-2} |y_i|$ is as small as possible.

Now assume also for the time being that $y_j \geq 2$. If $j = n-2$, then $x_n(\mathbf{v}) = -y_1 + y_{n-3} + y_{n-2}(k-1) = 0 + 0 + y_{n-2}(k-1) \geq 2(k-1) > k$, a contradiction, and so $j \leq n-3$. The requirement $x_{j+1}(\mathbf{v}) \geq -k$ then gives $y_{j+1} \geq y_j k - k \geq 2k - k = k \geq 3$. But now if $y_\ell \geq 3$ and $y_i \geq 2$ whenever $j \leq i < \ell$, where $3 \leq \ell \leq n-3$, then $x_2(\mathbf{v}) + x_3(\mathbf{v}) + \ldots + x_{\ell+1}(\mathbf{v}) \geq -k$ gives

$$y_{\ell+1} \geq -k + y_1 + y_{\ell-1} + y_\ell(k-1) \geq -k + 0 + 2 + 3(k-1) = 2k - 1 \geq 3,$$

and hence by induction, $y_\ell \geq 3$ whenever $j < \ell \leq n-2$. This, however, gives

$$x_n(\mathbf{v}) = -y_1 + y_{n-3} + y_{n-2}(k-1) \geq 0 + 2 + 3(k-1) = 3k - 1 > k,$$

another contradiction. Hence $y_j = 1$. Also $j \neq n-2$, since we have assumed $\mathbf{v} \neq \mathbf{v}_{n-2}$, and therefore $2 \leq j \leq n-3$.

Next, because $x_{j+1}(\mathbf{v}) \geq -k$ we have $y_{j+1} \geq y_j k - k \geq k - k = 0$. Assume for the time being that $y_{j+1} \geq 2$. If $j = n-3$, then we find $x_n(\mathbf{v}) = -y_1 + y_{n-3} + y_{n-2}(k-1) = 0 + 1 + y_{n-2}(k-1) \geq 1 + 2(k-1) > k$, a contradiction, and so $j \leq n-4$. Moreover, from $x_{j+1}(\mathbf{v}) + x_{j+2}(\mathbf{v}) \geq -k$ we find that $y_{j+2} \geq 2y_j + y_{j+1}(k-1) - k \geq 2 + 2(k-1) - k = k \geq 3$, and by induction (as above), it follows that $y_\ell \geq 3$ whenever $j + 1 < \ell \leq n-2$, and again $x_n(\mathbf{v}) = -y_1 + y_{n-3} + y_{n-2}(k-1) \geq 0 + 2 + 3(k-1) = 3k - 1 > k$, a contradiction. Hence $y_{j+1} = 0$ or 1.

If $y_{j+1} = 0$, then $x_j(\mathbf{v}) = x_j(\mathbf{v}_j) = 1$ and $x_{j+1}(\mathbf{v}) = x_{j+1}(\mathbf{v}_j) = -k$, and these co-ordinates cancel each other when we take the difference $\mathbf{u} = \mathbf{v} - \mathbf{v}_j$. Moreover, as $|\mathbf{v}| \leq 2k$ and $|\mathbf{v}_j| = 2k$ we see that $|\mathbf{u}| = |\mathbf{v} - \mathbf{v}_j| \leq (2k - (k+1)) + (2k - (k+1)) < 2k$, and since $\mathbf{u}$ lies in $H_\mathbf{w}$ (so that $|\mathbf{u}^+| = |\mathbf{u}^-|$), it follows that $\mathbf{u}$ lies in T_k^n, and hence also in $H_\mathbf{w} \cap T_k^n$. But $\mathbf{u} = \mathbf{v} - \mathbf{v}_j = \sum_{i=j+2}^{n-2} y_i \mathbf{v}_i$, and as $\sum_{i=j+2}^{n-2} |y_i|$ is smaller than $\sum_{i=2}^{n-2} |y_i|$ we deduce that $\mathbf{u} = \mathbf{0}$ or $\mathbf{u} = \pm\mathbf{v}_i$ for some $i \geq j+2$, and hence $\mathbf{v} = \mathbf{v}_j$ or $\mathbf{v}_j \pm \mathbf{v}_i$ for some $i \geq j+2$. The former possibility has been ruled out already, and the latter possibility is easily ruled out since $x_{j+1}(\mathbf{v}_j + \mathbf{v}_i) + x_{i+1}(\mathbf{v}_j + \mathbf{v}_i) \leq -k + 1 - k = 1 - 2k < -k$ and $x_{j+1}(\mathbf{v}_j - \mathbf{v}_i) + x_{i+2}(\mathbf{v}_j - \mathbf{v}_i) \leq -k - (k-2) = 2 - 2k < -k$.

Similarly if $y_{j+1} = 1$, then $x_j(\mathbf{v}) = x_j(\mathbf{v}_j) = 1$ while $x_{j+1}(\mathbf{v}) = -(k-1)$ and $x_{j+1}(\mathbf{v}_j) = -k$, so that when we take the difference $\mathbf{u} = \mathbf{v} - \mathbf{v}_j$, we find $|\mathbf{u}| = |\mathbf{v} - \mathbf{v}_j| \leq (2k - k) + (2k - k) = 2k$, and again $\mathbf{u}$ lies in $H_\mathbf{w} \cap T_k^n$, but $\mathbf{u} = \mathbf{v} - \mathbf{v}_j = \sum_{i=j+1}^{n-2} y_i \mathbf{v}_i$, with $\sum_{i=j+1}^{n-2} |y_i|$ smaller than $\sum_{i=2}^{n-2} |y_i|$, so $\mathbf{u} = \mathbf{0}$ or $\mathbf{u} = \pm\mathbf{v}_i$ for some $i \geq j+1$, and hence $\mathbf{v} = \mathbf{v}_j$ or $\mathbf{v}_j \pm \mathbf{v}_i$ for

some $i \geq j + 1$. In fact because $y_{j+1} = 1$ we must have $\mathbf{v} = \mathbf{v}_j + \mathbf{v}_{j+1}$, but then $x_j(\mathbf{v}) + x_{j+3}(\mathbf{v}) + x_{j+4}(\mathbf{v})) = 1 + (k-1) + 1 = k + 1 > k$, another contradiction.

Hence there is no other such $\mathbf{v}$; the only possibilities are $\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_{n-3}$ and $\mathbf{v}_{n-2}$.

Case 2: Suppose $y_1 = 1$ and $y_2 < 0$. Then the requirement $x_2(\mathbf{v}) \geq -k$ gives $y_2 = -1$ and $x_2(\mathbf{v}) = -k$, so that no other co-ordinates of $\mathbf{v}$ can be negative. Similarly, from $x_1(\mathbf{v}) + x_3(\mathbf{v}) \leq k$ we find that $y_3 \leq k - y_1(k-2) + y_2 k \leq k - (k-2) - k = 2 - k \leq -1$, and from $x_1(\mathbf{v}) + x_3(\mathbf{v}) + x_4(\mathbf{v}) \leq k$ we find that $y_4 \leq k - y_1(k-1) + 2y_2 + (k-1)y_3 \leq k - (k-1) - 2 - (k-1) = -k \leq -3$. It now follows by induction that $y_j \leq -3$ for $4 \leq j \leq n-2$, for if $y_j \leq -3$ and $y_i \leq -1$ for $2 \leq i < j$, then from the requirement $x_2(\mathbf{v}) + x_3(\mathbf{v}) + \ldots + x_{j+1}(\mathbf{v}) \geq -k$ we find that

$$y_{j+1} \leq k + y_1 + y_{j-1} + y_j(k-1) \leq k + 1 - 1 - 3(k-1) \leq 3 - 2k \leq -3.$$

This, however, gives

$$x_n(\mathbf{v}) = -y_1 + y_{n-3} + y_{n-2}(k-1) \leq -1 - 1 - 3(k-1) = 1 - 3k < -k,$$

a contradiction. Hence this case is impossible.

Case 3: Suppose $y_1 = 1$ and $y_2 = 0$. As $\mathbf{v}$ and $\mathbf{v}_1$ then have the same first two co-ordinates, namely 1 and $-(k-1)$, these cancel each other when taking their difference $\mathbf{u} = \mathbf{v} - \mathbf{v}_1$. Moreover, as $|\mathbf{v}| \leq 2k$ and $|\mathbf{v}_1| = 2k$ we see that $|\mathbf{u}| = |\mathbf{v} - \mathbf{v}_1| \leq (2k - k) + (2k - k) = 2k$, and since $\mathbf{u}$ lies in $H_{\mathbf{w}}$ (so that $|\mathbf{u}^+| = |\mathbf{u}^-|$), it follows that $\mathbf{u}$ lies in T_k^n. By the argument in Case 1, we find that $\mathbf{u} = \mathbf{0}$ or $\pm\mathbf{v}_i$ for some $i \geq 3$, and hence $\mathbf{v} = \mathbf{v}_1$ or $\mathbf{v}_1 \pm \mathbf{v}_i$ for some $i \geq 3$. The latter possibility, however, is easily ruled out since $x_2(\mathbf{v}_1 + \mathbf{v}_i) + x_{i+1}(\mathbf{v}_1 + \mathbf{v}_i) \leq -(k-1) + 1 - k = 2 - 2k < -k$ and $x_1(\mathbf{v}_1 - \mathbf{v}_i) + x_{i+1}(\mathbf{v}_1 - \mathbf{v}_i) \geq 1 + k > k$. Hence we have only $\mathbf{v} = \mathbf{v}_1$.

Case 4: Suppose $y_1 = 1$ and $y_2 > 0$. Assuming $y_2 \geq 2$, then the requirement $x_2(\mathbf{v}) + x_3(\mathbf{v}) \geq -k$ gives $y_3 \geq 2y_1 + y_2(k-1) - k \geq 2 + 2(k-1) - k = k \geq 3$. It then follows by induction that $y_j \geq 3$ for $4 \leq j \leq n-2$, for if $y_i \geq 3$ for $3 \leq i \leq j-1$, then from
$$x_2(\mathbf{v}) + x_3(\mathbf{v}) + \ldots + x_j(\mathbf{v}) \geq -k \text{ we find that}$$

$$y_j \geq -k + y_1 + y_{j-2} + y_{j-1}(k-1) \geq -k + 1 + 2 + 3(k-1) \geq 2k \geq 3.$$

This, however, gives

$$x_n(\mathbf{v}) = -y_1 + y_{n-3} + y_{n-2}(k-1) \geq -1 + 3 + 3(k-1) = 3k - 1 > k,$$

a contradiction. Thus $y_2 = 1$.

If all $y_i = 1$ for all i then $\mathbf{v} = \sum_{i=1}^{n-2} \mathbf{v}_i = -\mathbf{v}_{n-1}$. Otherwise suppose that j is the smallest positive integer for which $y_j \neq 1$. If $j = 3$, then from $x_2(\mathbf{v}) + x_3(\mathbf{v}) \geq -k$ we find that $y_3 \geq 2y_1 + y_2(k-1) - k = 1$, and so $y_j \geq 2$ in that case, while if $3 \leq j \leq n-2$ then $x_2(\mathbf{v}) + x_3(\mathbf{v}) + x_j(\mathbf{v}) \geq -k$ gives

$$
\begin{aligned}
y_j &\geq -k + 2y_1 + y_2(k-1) - y_3 - y_{j-3} - y_{j-2}(k-2) + y_{j-1}k \\
&= -k + 2 + (k-1) - 1 - 1 - (k-2) + k = 1,
\end{aligned}
$$

and so $y_j \geq 2$ in that case also. But now if $y_\ell \geq 2$ and $y_i \geq 1$ for all $i < \ell$, where $3 \leq \ell \leq n-3$, then $x_2(\mathbf{v}) + x_3(\mathbf{v}) + \ldots + x_{\ell+1}(\mathbf{v}) \geq -k$ gives

$$
y_{\ell+1} \geq -k + y_1 + y_{\ell-1} + y_\ell(k-1) \geq -k + 1 + 1 + 2(k-1) \geq k \geq 3,
$$

and hence by induction, $y_\ell \geq 2$ whenever $j \leq \ell \leq n-2$. This, however, gives

$$
x_n(\mathbf{v}) = -y_1 + y_{n-3} + y_{n-2}(k-1) \geq -1 + 1 + 2(k-1) > k,
$$

a contradiction. Hence no such j exists, and we have only $\mathbf{v} = -\mathbf{v}_{n-1}$.

Case 5: Suppose $y_1 \geq 2$. Then since $x_2(\mathbf{v}) \geq -k$ we have $y_2 \geq y_1(k-1) - k = (y_1-1)(k-1) - 1 \geq 2(y_1-1) - 1 = 2y_1 - 3 \geq 1$, and since $x_2(\mathbf{v}) + x_3(\mathbf{v}) \geq -k$ we have $y_3 \geq 2y_1 + y_2(k-1) - k = 2y_1 + (y_2-1)(k-1) - 1 \geq 2y_1 - 1 \geq y_1 + 1 \geq 3$. It follows by induction that $y_j \geq y_1 + 1 \geq 3$ for $4 \leq j \leq n-2$, for if $y_i \geq y_1 + 1 \geq 3$ for $3 \leq i \leq j-1$, then from $x_2(\mathbf{v}) + x_3(\mathbf{v}) + \ldots + x_j(\mathbf{v}) \geq -k$ we find that

$$
\begin{aligned}
y_j &\geq -k + y_1 + y_{j-2} + y_{j-1}(k-1) \geq -k + y_1 + 1 + 3(k-1) \\
&\geq y_1 + 2(k-1) \geq y_1 + 1 \geq 3.
\end{aligned}
$$

This, however, gives

$$
x_n(\mathbf{v}) = -y_1 + y_{n-3} + y_{n-2}(k-1) > 1 + 3(k-1) = 3k - 2 > k,
$$

a contradiction.

As these five cases cover all possibilities, the proof is complete. $\qquad\square$

4.4 Proof of Theorem 2 for $n \geq 8$ and $k = 2$.

Consider the $n-2$ vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{n-2}$ that make up the rows of the following $(n-2) \times n$ matrix:

$$
A = \begin{bmatrix}
-1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & -2 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & -2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & -2 & 1
\end{bmatrix}.
$$

Clearly these vectors are linearly independent, and so the row-space of A a subspace of $\mathbb{R}^n$ of dimension $n - 2$. Also clearly $|\mathbf{v}_i^+| = |\mathbf{v}_i^-| = 2$ for all i, and it is easy to check that $|\max(-\mathbf{v}_i^-, \mathbf{v}_j^+)| > 2$ whenever $i \neq j$.

In what follows, we use the Fibonacci sequence, that is, the sequence (f_m) of integers defined by $f_1 = f_2 = 1$ and $f_{m+2} = f_{m+1} + f_m$ for all $m \geq 1$.

Lemma 5. *There is a vector* $\mathbf{w} = (w_1, \ldots, w_n)$ *belonging to the null space of A such that $w_1 > w_2 > \ldots > w_n > 0$. In fact, such a vector is given by*

$$w_1 = 2f_{n-1} - 2, \quad w_2 = 2f_{n-1} - f_{n-2} - 1, \quad and \quad w_i = f_{n-i+2} \qquad (14)$$

for $3 \leq i \leq n$, where f_m is the mth term of the Fibonacci sequence.

Proof. Let us take $w_n = f_2 = 1$ and $w_{n-1} = f_3 = 2$. Then by orthogonality with $\mathbf{v}_{n-2}$ we have $w_{n-2} = 2w_{n-2} - w_n = 3 = f_4$. It follows by induction that $w_i = f_{n-i+2}$ whenever $3 \leq i \leq n$, for if this is true for all j such that $i < j \leq n$, then by orthogonality with $\mathbf{v}_i$ we have

$$
\begin{aligned}
w_i & = 2w_{i+1} - w_{i+3} = 2f_{n-(i+1)+2} - f_{n-(i+3)+2} \\
& = 2f_{n-i+1} - (f_{n-i+1} - f_{n-i}) = f_{n-i+1} + f_{n-i} \\
& = f_{n-i+2}.
\end{aligned}
$$

Next, by orthogonality with $\mathbf{v}_2$ we have $w_1 = 2w_3 - w_{n-1} = 2f_{n-1} - 2$, and by orthogonality with $\mathbf{v}_1$ we have $w_2 = w_1 - w_4 + w_n = 2f_{n-1} - 2 - f_{n-2} + 1 = 2f_{n-1} - f_{n-2} - 1 = f_{n-1} + f_{n-3} - 1$. Finally, it is an easy exercise to verify that $w_1 > w_2 > \ldots > w_n > 0$, because (f_m) is an increasing sequence of positive integers. $\square$

Lemma 6. $H_{\mathbf{w}} \cap T_2^n = \{\mathbf{0}, \pm\mathbf{v}_1, \pm\mathbf{v}_2, \ldots, \pm\mathbf{v}_{n-2}, \pm\mathbf{v}_{n-1}\}$, *where $\mathbf{v}_{n-1}$ is the vector* $-(\sum_{i=1}^{n-2} \mathbf{v}_i) = (0, 1, -1, 0, -1, 0, \ldots, 0, 1)$.

Proof. Let $\mathbf{v}$ be any non-zero vector in $H_{\mathbf{w}} \cap T_2^n$, and let i be the smallest positive integer such that $x_i(\mathbf{v}) \neq 0$. Without loss of generality we may assume that $x_i(\mathbf{v}) > 0$, and therefore $x_i(\mathbf{v}) = 1$ or 2. If $x_i(\mathbf{v}) = 2$, however, then orthogonality of $\mathbf{v}$ with $\mathbf{w}$ must correspond to an equation of the form $2w_i - w_r - w_s = 0$ with $i < r < s$ or $2w_i - 2w_r$ with $i < r$, both of which are impossible since the co-ordinates of $\mathbf{w}$ are strictly decreasing. Thus $x_i(\mathbf{v}) = 1$, and orthogonality of $\mathbf{v}$ with $\mathbf{w}$ must correspond to an equation of the form $w_i - w_r - w_s + w_t$ with $i < r < s < t$, or $w_i - 2w_r + w_t = 0$ with $i < r < t$. These two possibilities can be amalgamated into one by considering the equation as $w_i - w_r - w_s + w_t$ with $i < r \leq s < t$. Again we proceed case-by-case.

Case 1: Suppose $i = 1$. Then $w_1 = w_r + w_s - w_t < 2w_r$ so $2f_{n-1} - 2 < 2f_{n-r+2}$ and therefore $f_{n-1} < f_{n-r+2} + 1$, which implies $r \leq 3$. Next, if $r = 3$ then $w_1 = w_3 + w_s - w_t < w_3 + w_s$, so $2f_{n-1} - 2 < f_{n-1} + f_{n-s+2}$ and therefore $f_{n-s+2} > f_{n-1} - 2$, which implies $s \leq 3$ (because $n \geq 8$), giving $r = s = 3$ and then $w_t = 2w_3 - w_1 = 2f_{n-1} - 2f_{n-1} + 2 = 2$, so $t = n - 1$ and $\mathbf{v} = (1, 0, -2, 0, \ldots, 0, 1, 0) = \mathbf{v}_2$. Similarly, if $r = 2$ then we find that $w_1 = w_2 + w_s - w_t < w_2 + w_s$, so $2f_{n-1} - 2 < 2f_{n-1} - f_{n-2} - 1 + f_{n-s+2}$ and therefore $f_{n-s+2} > f_{n-2} - 1$, which implies $s \leq 4$. If $r = s = 2$, however, then $w_t = 2w_2 - w_1 = 2(2f_{n-1} - f_{n-2} - 1) - 2f_{n-1} + 2 = 2f_{n-1} - 2f_{n-2} = 2f_{n-3}$, which is impossible (since $2f_{n-3} > f_{n-3} + f_{n-4} = f_{n-2}$ while also $2f_{n-3} < f_{n-3} + f_{n-2} = f_{n-1}$). Similarly, if $r = 2$ and $s = 3$ then $w_t = w_2 + w_3 - w_1 = 2f_{n-1} - f_{n-2} - 1 + f_{n-1} - 2f_{n-1} + 2 = f_{n-1} - f_{n-2} + 1 = f_{n-3} + 1$, which is also impossible (since $n - 3 \geq 5$). Hence if $r = 2$ then $s = 4$, in which case $w_t = w_2 + w_4 - w_1 = 2f_{n-1} - f_{n-2} - 1 + f_{n-2} - 2f_{n-1} + 2 = 1$, giving $t = n$ and $\mathbf{v} = (1, -1, 0, -1, 0, \ldots, 0, 1) = -\mathbf{v}_1$.

Case 2: Suppose $i = 2$. Then $2w_r > w_r + w_s - w_t = w_2$ so

$$
\begin{aligned}
2f_{n-r+2} &> 2f_{n-1} - f_{n-2} - 1 = 2(f_{n-2} + f_{n-3}) - f_{n-2} - 1 \\
&= f_{n-2} + 2f_{n-3} - 1 > f_{n-2} + f_{n-3} + f_{n-4} = 2f_{n-2},
\end{aligned}
$$

and therefore $r \leq 3$, so $r = 3$, and

$$
w_s - w_t = w_2 - w_3 = f_{n-1} - f_{n-2} - 1 = f_{n-3} - 1.
$$

If $s \geq 6$, then $n - s + 2 < n - 3$ and so $w_s - w_t = f_{n-s+2} - f_{n-t+2} < f_{n-3} - 1$, a contradiction, hence $3 \leq s \leq 5$. If $s = 5$, then $f_{n-3} - 1 = w_s - w_t = f_{n-s+2} - f_{n-t+2} = f_{n-3} - f_{n-t+2}$ so $f_{n-t+2} = 1$, giving $t = n$, and then $\mathbf{v} = (0, 1, -1, 0, -1, 0, \ldots, 0, 1) = \mathbf{v}_{n-1}$. If $s = 4$, then $f_{n-t+2} = w_t = w_s - f_{n-3} + 1 = f_{n-2} - f_{n-3} + 1 = f_{n-4} + 1$, which is impossible since $n \geq 8$. Finally (in the case where $i = 2$) if $s = 3$, then $t \geq 4$ so $w_t \leq f_{n-2}$ and therefore $w_s - w_t \geq f_{n-1} - f_{n-2} = f_{n-3} > f_{n-3} - 1$, another contradiction.

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Case 3: Suppose $i \geq 3$. If $r \geq i+2$, then $f_{n-s+2} = w_s > w_s - w_t = w_i - w_r \geq w_i - w_{i+2} = f_{n-i+2} - f_{n-i} = f_{n-i+1}$, which implies $n - s + 2 > n - i + 1$, so $s < i + 1 < r$, a contradiction. Thus $r = i + 1$, and $w_s - w_t = w_i - w_r = f_{n-i+2} - f_{n-i+1} = f_{n-i}$. In particular, $f_{n-s+2} = w_s > w_s - w_t = f_{n-i}$, so $n - s + 2 > n - i$, giving $s < i + 2$, and therefore $r = s = i + 1$. It follows that $f_{n-t+2} = w_t = w_r + w_s - w_i = 2w_{i+1} - w_i = 2f_{n-i+1} - f_{n-i+2} = f_{n-i+1} + (f_{n-i+1} - f_{n-i+2}) = f_{n-i+1} - f_{n-i} = f_{n-i-1}$, and thus either $t = i + 3$ (if $i \leq n - 3$) or $t = i + 2 = n$ (if $i = n - 2$), giving $\mathbf{v} = \mathbf{v}_i$ in both cases.

Hence the only non-zero vectors in $H_{\mathbf{w}} \cap T_2^n$ are $\pm \mathbf{v}_1, \pm \mathbf{v}_2, \ldots, \pm \mathbf{v}_{n-1}$. $\quad\square$

By Theorem 1, we have $g(n, 2) = n - 1$ for all $n \geq 8$.

4.5 Remaining cases

The results for the remaining cases were obtained with the help of the computer algebra system MAGMA [1], using the same techniques of enumeration as described by Fishburn in [4, Section 4], but also checking the results for both representability and almost representability (using techniques of linear programming) as in [2].

In the case where $n = 5$ and $k = 2$, we have found that there are precisely 286 orders on $\mathcal{P}_2[5]$ which satisfy the IES, of which 114 are representable and 172 are not, and that all of the latter fail C_3, and all but 40 are all almost representable. One example of an almost representable order that fails C_3 is the following (in which ij denotes the obvious multiset of cardinality 2): $11 \succeq 12 \succeq 13 \succeq 14 \succeq 22 \succeq 15 \succeq 23 \succeq 24 \succeq 33 \succeq 34 \succeq 25 \succeq 35 \succeq 44 \succeq 45 \succeq 55$; this is almost representable via the vector $\mathbf{w} = (6, 4, 3, 2, 1)$. Thus $f(5, 2) = g(5, 2) = 3$.

For $n = 5$ and $k = 3$, there are precisely 68820 orders on $\mathcal{P}_3[5]$ satisfying the IES, of which 6588 are representable and 62232 are not. Of those which are not representable, 62208 fail C_3, while 24 satisfy C_3 but fail C_4. All of the latter 24 are almost representable, and one such example is given by $111 \succeq 112 \succeq 113 \succeq 114 \succeq 115 \succeq 122 \succeq 123 \succeq 124 \succeq 133 \succeq 125 \succeq 134 \succeq 222 \succeq 144 \succeq 135 \succeq 223 \succeq 145 \succeq 224 \succeq 233 \succeq 155 \succeq 225 \succeq 234 \succeq 244 \succeq 333 \succeq 235 \succeq 334 \succeq 245 \succeq 344 \succeq 335 \succeq 255 \succeq 444 \succeq 345 \succeq 445 \succeq 355 \succeq 455 \succeq 555$; this is almost representable via $\mathbf{w} = (20, 10, 6, 4, 1)$. Thus $f(5, 3) = g(5, 3) = 4$.

For $n = 6$ and $k = 2$, there are precisely 33592 orders on $\mathcal{P}_2[6]$ which satisfy the IES, of which 2608 are representable and 30984 are not. Of those which are not representable, 30980 fail C_3, while the other four satisfy C_3 and C_4 but fail C_5. All of the latter four are almost representable, and one

such example is $11 \succeq 12 \succeq 13 \succeq 14 \succeq 15 \succeq 22 \succeq 23 \succeq 24 \succeq 33 \succeq 34 \succeq 16 \succeq 25 \succeq 44 \succeq 35 \succeq 45 \succeq 26 \succeq 36 \succeq 55 \succeq 46 \succeq 56 \succeq 66$; this is almost representable via $\mathbf{w} = (12, 8, 7, 6, 4, 1)$. Thus $f(6, 2) = g(6, 2) = 5$.

For $n = 7$ and $k = 2$, there are 23178480 orders on $\mathcal{P}_2[7]$ which satisfy the IES, of which 107498 are representable and 23070982 are not. Of those which are not representable, 23069816 fail C_3, 1138 fail C_5 but satisfy C_3 and C_4, and 28 fail C_6 but satisfy C_3, C_4 and C_5. Also, of the 1138 that fail C_5 but satisfy C_3 and C_4, 1078 are almost representable, while the other 60 are not almost representable. Finally, all of the 28 that fail C_6 but satisfy C_3, C_4 and C_5 are almost representable; one such example is $11 \succeq 12 \succeq 13 \succeq 14 \succeq 15 \succeq 16 \succeq 22 \succeq 23 \succeq 24 \succeq 25 \succeq 33 \succeq 34 \succeq 17 \succeq 35 \succeq 44 \succeq 45 \succeq 26 \succeq 55 \succeq 36 \succeq 46 \succeq 56 \succeq 27 \succeq 37 \succeq 47 \succeq 66 \succeq 57 \succeq 67 \succeq 77$; this is almost representable via $\mathbf{w} = (22, 14, 12, 11, 10, 6, 1)$. Thus $f(7, 2) = g(7, 2) = 6$.

This completes the proof of Theorem 2. It is interesting to note that for each of the cases $(n, k) = (5, 2), (6, 2)$ and $(7, 2)$, there are no orderings that fail C_4 but satisfy C_3. We do not yet understand why that happens.

5   A Conjecture

The results of this paper give us reason to formulate the following:

Conjecture 1. $f(n, k) = g(n, k)$ for all $n \geq 3$ and $k \geq 1$.

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