SURFACE AREA AND CAPACITY OF ELLIPSOIDS IN n DIMENSIONS

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Abstract

The surface area of a general n-dimensional ellipsoid is represented as an Abelian integral, which can readily be evaluated numerically. If there are only 2 values for the semi-axes then the area is expressed as an elliptic integral, which reduces in most cases to elementary functions.

The capacity of a general n-dimensional ellipsoid is represented as a hyperelliptic integral, which can readily be evaluated numerically. If no more than 2 lengths of semi-axes occur with odd multiplicity, then the capacity is expressed in terms of elementary functions. If only 3 or 4 lengths of semi-axes occur with odd multiplicity, then the capacity is expressed as an elliptic integral.

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1 Spheroids and ellipses

Consider an ellipsoid centred at the coordinate origin, with rectangular Cartesian coordinate axes along the semi-axes $a,\ b,\ c$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \ = \ 1. \tag{1}$$

1.1 Surface area of spheroid

In 1714, Roger Cotes found the surface area for ellipsoids of revolution [Cotes], called spheroids.

For the case in which two axes are equal $b = c$, the surface is generated by rotation around the x–axis of the half–ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \ = \ 1$ with $y \geq 0$. On that half-ellipse, $\mathrm{d}y/\mathrm{d}x = -b^2x/(a^2y)$, and hence the surface area of the spheroid is

$$A \ = \ 2\int_0^a 2\pi y\sqrt{1 + \frac{b^4x^2}{a^4y^2}}\ \mathrm{d}x \ = \ 4\pi\int_0^a \sqrt{y^2 + \frac{b^4}{a^4}x^2}\ \mathrm{d}x$$

$$= 4\pi b \int_0^a \sqrt{1 - \frac{x^2}{a^2} + \frac{b^2}{a^2}\frac{x^2}{a^2}}\,\mathrm{d}x$$

$$= 4\pi ab \int_0^1 \sqrt{1 - \left(1 - \frac{b^2}{a^2}\right) u^2}\,\mathrm{d}u \;=\; 4\pi ab \int_0^1 \sqrt{1 - \delta u^2}\,\mathrm{d}u, \qquad (2)$$

where $u = x/a$ and $\delta = 1 - b^2/a^2$. Therefore, the surface areas for prolate spheroids ($a > b$), spheres ($a = b$) and oblate spheroids ($a < b$) are:

$$A \;=\; \begin{cases} 2\pi b \left(a \times \dfrac{\arcsin\sqrt{\delta}}{\sqrt{\delta}} + b\right) & \text{(prolate)}, \\ 2\pi b(a+b) \;=\; 4\pi a^2 & \text{(sphere)}, \\ 2\pi b \left(a \times \dfrac{\mathrm{arcsinh}\sqrt{-\delta}}{\sqrt{-\delta}} + b\right) & \text{(oblate)}\ . \end{cases} \qquad (3)$$

Neither the hyperbolic functions nor their inverses had then been invented, and Cotes gave a logarithmic formula for the oblate spheroid [Cotes, pp. 169-171]. In modern notation [Cotes, p.50],

$$A \;=\; \pi \left[2a^2 + b^2 \frac{1}{\sqrt{-\delta}} \log\left(\frac{1+\sqrt{-\delta}}{1-\sqrt{-\delta}}\right)\right]. \qquad (4)$$

For $|\delta| \ll 1$, either use the power series for $(\arcsin x)/x$ to get

$$A \;=\; 2\pi b \left(a\left[1 + \tfrac{1}{6}\delta + \tfrac{3}{40}\delta^2 + \tfrac{5}{112}\delta^3 + \cdots\right] + b\right), \qquad (5)$$

or else expand the integrand in (2) as a power series in u^2 and integrate that term by term:

$$\begin{aligned} A \;&=\; 4\pi ab \int_0^1 \left(1 - \delta u^2\right)^{1/2} \mathrm{d}u \\ &= 4\pi ab \int_0^1 \left(1 - \tfrac{1}{2}\delta u^2 + \frac{\frac{1}{2}\frac{-1}{2}}{2!}\delta^2 u^4 - \frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}}{3!}\delta^3 u^6 \right. \\ &\qquad\qquad \left. + \frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}\frac{-5}{2}}{4!}\delta^4 u^8 - \frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}\frac{-5}{2}\frac{-7}{2}}{5!}\delta^5 u^{10} + \cdots\right) \mathrm{d}u \\ &= 4\pi ab \left(1 - \frac{1}{2}\frac{\delta}{3} - \frac{1}{8}\frac{\delta^2}{5} - \frac{1}{16}\frac{\delta^3}{7} - \frac{5}{128}\frac{\delta^4}{9} - \frac{7}{256}\frac{\delta^5}{11} - \cdots\right). \end{aligned} \qquad (6)$$

1.2 Circumference of ellipse

In 1742, Colin MacLaurin constructed a definite integral for the circumference of an ellipse [MacLaurin]. Consider an ellipse with semi-axes a and b, with Cartesian coordinates along the axes:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \;=\; 1\ , \qquad (7)$$

On that ellipse, $2x\,\mathrm{d}x/a^2 + 2y\,\mathrm{d}\,y/b^2 = 0$, and hence $\mathrm{d}y/\mathrm{d}x = -b^2x/(a^2y)$, and the circumference is 4 times the ellipse quadrant with $x \geq 0$ and $y \geq 0$. That quadrant has arclength

$$I = \int_0^a \sqrt{1 + \frac{b^4 x^2}{a^4 y^2}}\, \mathrm{d}x = \int_0^a \sqrt{1 + \frac{b^2 (x/a)^2}{a^2 (y/b)^2}}\, \mathrm{d}x$$
$$= \int_0^a \sqrt{1 + \frac{(b/a)^2 (x/a)^2}{1 - (x/a)^2}}\, \mathrm{d}x \ . \tag{8}$$

Substitute $z = x/a$, and the circumference becomes

$$4I = 4a \int_0^1 \sqrt{1 + \frac{(b/a)^2 z^2}{1 - z^2}}\, \mathrm{d}z = 4a \int_0^1 \sqrt{\frac{1 - m z^2}{1 - z^2}}\, \mathrm{d}z \ , \tag{9}$$

where

$$m = 1 - \frac{b^2}{a^2} \ . \tag{10}$$

With $a \geq b$ this gives $0 \leq m < 1$.

That integral could not be expressed finitely in terms of standard functions.

Many approximations for the circumference $L(a, b)$ of an ellipse have been published, and some of those give very close upper or lower bounds for $L(a, b)$ [Barnard, Pearce & Schovanec]. A close approximation was given by Thomas Muir in 1883:

$$L(a, b) \approx M(a, b) \stackrel{\text{def}}{=} 2\pi \left(\frac{a^{3/2} + b^{3/2}}{2} \right)^{2/3} \ . \tag{11}$$

That is a very close lower bound for all values of $m \in (0, 1)$. Indeed, [Barnard, Pearce & Schovanec, (2)]:

$$0{\cdot}00006 m^4 < \frac{L(a, b) - M(a, b)}{a} < 0{\cdot}00666 m^4 \ . \tag{12}$$

1.3 Legendre on elliptic integrals

Adrien-Marie Legendre (1742-1833) worked on elliptic integrals for over 40 years, and summarized his work in [Legendre 1825]. He investigated systematically the integrals of the form $\int R(t, y)\, \mathrm{d}t$, where R is a general rational function and $y^2 = P(t)$, where P is a general polynomial of degree 3 or 4. Legendre called them "fonctions élliptique", because the formula (9) is of that form — now they are called elliptic integrals. He shewed how to express any such integral in terms of elementary functions, supplemented by 3 standard types of elliptic integral.

Each of Legendre's standard integrals has 2 (or 3) parameters, including $x = \sin \phi$. Notation for those integrals varies considerably between various authors. Milne-Thomson's notation for Legendre's elliptic integrals [Milne-Thomson, §**17.2**] uses the parameter m, where Legendre (and many other authors) had used k^2.

Each of the three kinds is given as two integrals. In each case, the second form is obtained from the first by the substitutions $t = \sin \theta$ and $x = \sin \phi$.

The Incomplete Elliptic Integral of the First Kind is:

$$F(\phi | m) \stackrel{\text{def}}{=} \int_0^\phi \frac{\mathrm{d}\theta}{\sqrt{1 - m \sin^2 \theta}} = \int_0^x \frac{\mathrm{d}t}{\sqrt{(1 - t^2)(1 - m t^2)}} \ . \tag{13}$$

The Incomplete Elliptic Integral of the Second Kind is:

$$E(\phi|m) \stackrel{\text{def}}{=} \int_0^\phi \sqrt{1 - m\sin^2\theta}\ \mathrm{d}\theta \ = \ \int_0^x \sqrt{\frac{1-mt^2}{1-t^2}}\ \mathrm{d}t\ . \tag{14}$$

That can be rewritten as

$$\int_0^x \frac{1-mt^2}{\sqrt{(1-t^2)(1-mt^2)}}\ \mathrm{d}t\ , \tag{15}$$

which is of the form $\int R(t,y)\ \mathrm{d}t$, where $y^2 = (1-t^2)(1-mt^2)$.

The Incomplete Elliptic Integral of the Third Kind is:

$$\begin{aligned} \Pi(n;\phi|m) &\stackrel{\text{def}}{=} \int_0^\phi \frac{\mathrm{d}\theta}{(1-n\sin^2\theta)\sqrt{1-m\sin^2\theta}} \\ &= \int_0^x \frac{\mathrm{d}t}{(1-nt^2)\sqrt{(1-t^2)(1-mt^2)}}\ . \end{aligned} \tag{16}$$

The special cases for which $\phi = \frac{1}{2}\pi$ (and $x = 1$) are found to be particularly important, and they are called the Complete Elliptic Integrals [Milne-Thomson, §**17.3**].

The Complete Elliptic Integral of the First Kind is:

$$\begin{aligned} K(m) \stackrel{\text{def}}{=} F\left(\tfrac{1}{2}\pi|m\right) &\stackrel{\text{def}}{=} \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-m\sin^2\theta}} \\ &= \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-mt^2)}}\ . \end{aligned} \tag{17}$$

The Complete Elliptic Integral of the Second Kind is:

$$E(m) \stackrel{\text{def}}{=} E\left(\tfrac{1}{2}\pi|m\right) = \int_0^{\pi/2} \sqrt{1-m\sin^2\theta}\ \mathrm{d}\theta \ = \ \int_0^1 \sqrt{\frac{1-mt^2}{1-t^2}}\ \mathrm{d}t\ . \tag{18}$$

The complete elliptic integrals $K(m)$ and $E(m)$ can efficiently be computed to high precision, by constructing arithmetic-geometric means [Milne-Thomson. §**17.6.3** & **17.6.4**].

2 Surface Area of 3-dimensional ellipsoid

For a surface defined by $z = z(x,y)$ in rectangular Cartesian coordinates xyz, the standard formula for surface area is:

$$\text{Area} \ = \ \int\int \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ \mathrm{d}x\ \mathrm{d}y. \tag{19}$$

On the ellipsoid (1),

$$\frac{\partial z}{\partial x} = \frac{-c^2 x}{a^2 z}\ , \qquad \frac{\partial z}{\partial y} = \frac{-c^2 y}{b^2 z}\ . \tag{20}$$

Consider the octant for which $x,\ y,\ z$ are all non–negative. Then the surface area for that octant is

$$\begin{aligned}
S &= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{1+\frac{c^4x^2}{a^4z^2}+\frac{c^4y^2}{b^4z^2}}\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\frac{\frac{z^2}{c^2}+\frac{c^2}{a^2}\frac{x^2}{a^2}+\frac{c^2}{b^2}\frac{y^2}{b^2}}{z^2/c^2}}\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\frac{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{c^2}{a^2}\frac{x^2}{a^2}+\frac{c^2}{b^2}\frac{y^2}{b^2}}{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\frac{1-\left(1-\frac{c^2}{a^2}\right)\frac{x^2}{a^2}-\left(1-\frac{c^2}{b^2}\right)\frac{y^2}{b^2}}{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\ \mathrm{d}y\ \mathrm{d}x\ . \qquad (21)
\end{aligned}$$

Hence, if two semi-axes (a and b) are fixed and the other semi-axis c increases, then the surface area increases.

Denote

$$\delta \ = \ 1-\frac{c^2}{a^2}, \qquad \epsilon = 1-\frac{c^2}{b^2}\ , \qquad (22)$$

and then (21) becomes

$$S \ = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\frac{1-\delta\frac{x^2}{a^2}-\epsilon\frac{y^2}{b^2}}{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\ \mathrm{d}y\ \mathrm{d}x\ . \qquad (23)$$

For a general ellipsoid, the coordinate axes can be named so that $a \geq b \geq c > 0$, and then $1 > \delta \geq \epsilon \geq 0$.

2.1 Legendre's series expansion for ellipsoid area

In 1788, Legendre converted this double integral to a convergent series [Legendre 1788] [Legendre 1825, pp. 350–351].

Replace the variables of integration (x, y) by (ϕ, θ), where $\cos\theta = z/c$, so that $\frac{x^2}{a^2}+\frac{y^2}{b^2} = \sin^2\theta$, or

$$\frac{x^2}{(a\ \sin\theta)^2}+\frac{y^2}{(b\ \sin\theta)^2} \ = \ 1\ . \qquad (24)$$

Then let $\cos\phi = y/(b\ \sin\theta)$ so that $\sin\phi = x/(a\ \sin\theta)$, or

$$x = a\ \sin\theta\ \sin\phi, \quad y = b\ \sin\theta\ \cos\phi. \qquad (25)$$

Differentiating x in (25) with respect to ϕ (with constant θ), we get that $\mathrm{d}x = a\cos\phi\,\sin\theta\;\mathrm{d}\theta$; and differentiating the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\sin^2\theta$ with respect to θ (with constant x), we get that $2y\,\mathrm{d}y = 2b^2\sin\theta\,\cos\theta\;\mathrm{d}\theta$. Thus the element of area in (23) becomes

$$\mathrm{d}x\,\mathrm{d}y \;=\; ab\,\sin\theta\,\cos\theta\;\mathrm{d}\theta\;\mathrm{d}\phi, \tag{26}$$

and the area S of the ellipsoid octant becomes [Legendre 1825, p.350]

$$\begin{aligned}
&\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\sqrt{\frac{1-\delta\sin^2\theta\;\sin^2\phi-\epsilon\left(\sin^2\theta-\frac{x^2}{a^2}\right)}{1-\sin^2\theta}}\;ab\,\sin\theta\,\cos\theta\;\mathrm{d}\phi\,\mathrm{d}\theta\\
&\qquad = ab\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\sin\theta\,\sqrt{1-(\delta\,\sin^2\phi+\epsilon\,\cos^2\phi)\;\sin^2\theta}\;\mathrm{d}\theta\;\mathrm{d}\phi\;. \qquad (27)
\end{aligned}$$

Thus,

$$S \;=\; ab\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\sin\theta\,\sqrt{1-p\,\sin^2\theta}\;\mathrm{d}\theta\;\mathrm{d}\phi\;, \tag{28}$$

where p is a function of ϕ:

$$p \;=\; \delta\,\sin^2\phi+\epsilon\,\cos^2\phi \;=\; \epsilon+(\delta-\epsilon)\sin^2\phi\;. \tag{29}$$

Hence, as ϕ increases from 0 to $\frac{1}{2}\pi$, p increases from $\epsilon\ge 0$ to $\delta<1$.

Define

$$\mathcal{I}(m) \;\stackrel{\mathrm{def}}{=}\; \int_0^{\pi/2}\sin\theta\,\sqrt{1-m\,\sin^2\theta}\;\mathrm{d}\theta\;. \tag{30}$$

Clearly, $\mathcal{I}(m)$ is a decreasing function of m (for $m\le 1$). That integral can be expressed explicitly. For $m\in(0,1)$,

$$\mathcal{I}(m) \;=\; \frac{1}{2}+\frac{1-m}{4\sqrt{m}}\log\left(\frac{1+\sqrt{m}}{1-\sqrt{m}}\right). \tag{31}$$

Expand the integrand in (30) as a power series in p and integrate for θ from 0 to $\frac{1}{2}\pi$, to get a series expansion for $\mathcal{I}(p)$:

$$\begin{aligned}
\mathcal{I}(p) \;&=\; \int_{\theta=0}^{\pi/2}\sin\theta\,(1-p\,\sin^2\theta)^{1/2}\;\mathrm{d}\theta\\
&=\; \int_0^{\pi/2}\sin\theta\,\left(1-\tfrac{1}{2}p\sin^2\theta+\frac{\frac{1}{2}\frac{-1}{2}}{2!}p^2\sin^4\theta-\frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}}{3!}p^3\sin^6\theta+\cdots\right)\mathrm{d}\theta\\
&=\; \int_0^{\pi/2}\sin\theta\;\mathrm{d}\theta\;-\;\tfrac{1}{2}p\int_0^{\pi/2}\sin^3\theta\;\mathrm{d}\theta\;-\;\tfrac{1\cdot 1}{2\cdot 4}p^2\int_0^{\pi/2}\sin^5\theta\;\mathrm{d}\theta\\
&\qquad -\;\tfrac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}p^3\int_0^{\pi/2}\sin^7\theta\;\mathrm{d}\theta\;-\;\tfrac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}p^4\int_0^{\pi/2}\sin^9\theta\;\mathrm{d}\theta\;-\cdots\;. \qquad (32)
\end{aligned}$$

Define

$$s_k \;\stackrel{\mathrm{def}}{=}\; \int_0^{\pi/2}\sin^k\theta\;\mathrm{d}\theta \;=\; \begin{cases}\dfrac{\pi\cdot 1\cdot 3\cdot 5\ldots(k-1)}{2\cdot 2\cdot 4\cdot 6\ldots k}\;, & (\text{even } k\ge 2)\;,\\[2ex] \dfrac{2\cdot 4\cdot 6\ldots(k-1)}{3\cdot 5\cdot 7\ldots k}\;, & (\text{odd } k\ge 3)\;,\end{cases} \tag{33}$$

with $s_0 = \frac{1}{2}\pi$ and $s_1 = 1$. For all $k > -1$ [Dwight, §**854.1**],

$$s_k \; = \frac{\sqrt{\pi}\;\Gamma\big(\frac{1}{2}(1+k)\big)}{2\Gamma\big(1+\frac{1}{2}k\big)} \; . \tag{34}$$

In particular,

$$s_1 = 1, \; s_3 \; = \; \tfrac{2}{3} \; , \; s_5 \; = \; \tfrac{2\cdot 4}{3\cdot 5} \; , \; s_5 \; = \; \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} \; , \ldots \; , \tag{35}$$

and hence

$$\mathcal{I}(p) \; = \; 1 - \tfrac{1}{1\cdot 3}p - \tfrac{1}{3\cdot 5}p^2 - \tfrac{1}{5\cdot 7}p^3 - \tfrac{1}{7\cdot 9}p^4 - \cdots \; . \tag{36}$$

Therefore, the surface area of the ellipsoid is

$$\begin{aligned} A \; &= \; 8ab \int_0^{\pi/2} \mathcal{I}(p)\; \mathrm{d}\phi \\ &= \; 8ab \int_0^{\pi/2} \left(1 - \tfrac{1}{1\cdot 3}p - \tfrac{1}{3\cdot 5}p^2 - \tfrac{1}{5\cdot 7}p^3 - \tfrac{1}{7\cdot 9}p^4 - \cdots\right) \mathrm{d}\phi \\ &= 4\pi ab\big(1 - \tfrac{1}{1\cdot 3}P_1 - \tfrac{1}{3\cdot 5}P_2 - \tfrac{1}{5\cdot 7}P_3 - \tfrac{1}{7\cdot 9}P_4 - \cdots\big)\; \mathrm{d}\phi \; , \end{aligned} \tag{37}$$

where

$$\begin{aligned} P_1 \; &= \; \frac{2}{\pi}\int_0^{\pi/2} (\delta \sin^2\phi + \epsilon\cos^2\phi)\; \mathrm{d}\phi \; = \; \tfrac{1}{2}\delta + \tfrac{1}{2}\epsilon, \\ P_2 \; &= \; \frac{2}{\pi}\int_0^{\pi/2} (\delta \sin^2\phi + \epsilon\cos^2\phi)^2\; \mathrm{d}\phi \; = \; \tfrac{1\cdot 3}{2\cdot 4}\delta^2 + \tfrac{1\cdot 1}{2\cdot 2}\delta\epsilon + \tfrac{1\cdot 3}{2\cdot 4}\epsilon^2, \\ P_3 \; &= \; \frac{2}{\pi}\int_0^{\pi/2} (\delta \sin^2\phi + \epsilon\cos^2\phi)^3\; \mathrm{d}\phi \\ &\qquad = \; \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\delta^3 + \tfrac{1\cdot 3\cdot 1}{2\cdot 4\cdot 2}\delta^2\epsilon + \tfrac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4}\delta\epsilon^2 + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\epsilon^3, \quad \textit{et cetera}. \end{aligned} \tag{38}$$

Legendre gave [Legendre 1825, p.51] a generating function for the P_k:

$$\begin{aligned} \frac{1}{\sqrt{(1-\delta z)(1-\epsilon z)}} \; &= \; (1-\delta z)^{-1/2}(1-\epsilon z)^{-1/2} \\ &= \; \left(1 + \tfrac{1}{2}\delta z + \tfrac{1\cdot 3}{2\cdot 4}\delta^2 z^2 + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\delta^3 z^3 + \cdots\right)\left(1 + \tfrac{1}{2}\epsilon z + \tfrac{1\cdot 3}{2\cdot 4}\epsilon^2 z^2 + \cdots\right) \\ &= \; 1 + P_1 z + P_2 z^2 + P_3 z^3 + P_4 z^4 + \cdots \; . \end{aligned} \tag{39}$$

Infinite series had been used by mathematicians since the 13th century in India and later in Europe, but very little attention had been given to convergence. Consequently much nonsense had been published, resulting from the use of infinite series which did not converge. From 1820 onwards, Cauchy developed the theory of infinite series, and he stressed the importance of convergence [Grabiner, Chapter 4]. In 1825, Legendre carefully explained that his series (37) for the area does converge [Legendre 1825, p.351].

All terms after the first in Legendre's series (37) are negative, and hence the partial sums of that series decrease monotonically towards the surface area.

I have searched many books on elliptic integrals and elliptic functions, and I have not found any later reference to Legendre's series (37) for the surface area of a general ellipsoid.

Derrick H. Lehmer stated (in 1950) a different infinite series for the surface area, in terms of the eccentricities

$$\alpha \;=\; \sqrt{1-\frac{b^2}{a^2}}. \quad \beta \;=\; \sqrt{1-\frac{c^2}{a^2}}\;. \tag{40}$$

The surface area is

$$\begin{aligned} S(a,b,c) \;&=\; 4\pi ab\left[1-\frac{1}{6}(\alpha^2+\beta^2)-\frac{1}{120}(3\alpha^4+2\alpha^2\beta^2+3\beta^4)-\cdots\right] \\ &=\; 4\pi ab\sum_{\nu=0}^{\infty}\frac{(\alpha\beta)^\nu}{1-4\nu^2}\;P_\nu\left(\frac{\alpha^2+\beta^2}{2\alpha\beta}\right)\;, \end{aligned} \tag{41}$$

where $P_\nu(x)$ is the Legendre polynomial of degree ν [Lehmer, (6)].

Philip Kuchel and Brian Bulliman constructed (in 1988) a more complicated series expansion for the area [Kuchel & Bulliman].

2.2 Bounds for ellipsoid area

As ϕ increases from 0 to $\frac{1}{2}\pi$, then $\delta\,\sin^2\phi+\epsilon\,\cos^2\phi=(\delta-\epsilon)\sin^2\phi+\epsilon$ increases from ϵ to δ. Hence, for all values of ϕ, the integrand in (27) lies between the upper and lower bounds

$$\begin{aligned} &\sin\theta\,\sqrt{1-\epsilon\,\sin^2\theta}\;\geq \\ &\qquad \sin\theta\,\sqrt{1-(\delta\,\sin^2\phi+\epsilon\,\cos^2\phi)\;\sin^2\theta}\;\geq\;\sin\theta\,\sqrt{1-\delta\,\sin^2\theta}\;. \end{aligned} \tag{42}$$

Accordingly, for all values of ϕ, the integral over θ in (27) lies between the upper and lower bounds

$$\mathcal{I}(\epsilon)\geq\int_{\theta=0}^{\pi/2}\sin\theta\,\sqrt{1-(\delta\,\sin^2\phi+\epsilon\,\cos^2\phi)\;\sin^2\theta}\;\mathrm{d}\theta\geq\;\mathcal{I}(\delta)\;. \tag{43}$$

How close are those bounds? As λ increases from 0 to 1, $\mathcal{I}(\lambda)$ decreases from $\int_{\theta=0}^{\pi/2}\sin\theta\;\mathrm{d}\theta=1$ to $\int_{\theta=0}^{\pi/2}\sin\theta\sqrt{1-\sin^2\theta}\;\mathrm{d}\theta=\int_{\theta=0}^{\pi/2}\sin\theta\;\cos\theta\;\mathrm{d}\theta=\frac{1}{2}$. Therefore, the upper bound in (43) is not more than twice the upper bound.

Integrating these bounds over $\phi=0$ to $\phi=\frac{1}{2}\pi$, we get upper and lower bounds for the double integral in (27):

$$\frac{\pi}{2}\geq\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\sin\theta\,\sqrt{1-(\delta\,\sin^2\phi+\epsilon\,\cos^2\phi)\;\sin^2\theta}\;\mathrm{d}\theta\;\mathrm{d}\phi\;\geq\;\frac{\pi}{4}\;. \tag{44}$$

Thus, we get the following bounds for the surface area $A=8S$ of a general ellipsoid with semi-axes $a\geq b\geq c$:

$$4\pi ab\;\geq\;8ab\,\mathcal{I}(\epsilon)\;\geq\;A\;\geq\;8ab\mathcal{I}(\delta)\;\geq\;2\pi ab. \tag{45}$$

The extreme upper bound $4\pi ab$ is attained for the sphere with $a=b=c$, and the extreme lower bound $2\pi ab$ is attained with $c=0$, when the ellipsoid collapses into a double-sided ellipse with semi-axes a and b.

2.3 Legendre's explicit formula for ellipsoid area

In 1825, Legendre constructed [Legendre 1825, pp.352–359] an explicit expression for the area of a general ellipsoid, in terms of Incomplete Elliptic Integrals. In Milne-Thomson's notation, with δ and ϵ as in (22),

$$A = 2\pi c^2 + \frac{2\pi ab}{\sqrt{\delta}}\left[(1-\delta)F(\sqrt{\delta}|\epsilon/\delta) + \delta\ E(\sqrt{\delta}|\epsilon/\delta)\right] . \tag{46}$$

Note that Legendre's formula does not hold for a sphere, and for a near-sphere some rapidly-convergent series should be used for $F/\sqrt{\delta}$, or else Legendre's series (37) should be used for the area.

Legendre's proof of his elliptic integral formula for area is long and complicated, and that formula has been very little known.

In 1953, Frank Bowman published [Bowman] an obscure derivation of the formula for the area (without mentioning Legendre). In 1958, Albert Eagle used his interesting version of elliptic functions to derive the formula for the area [Eagle, p.281 (12)]. He commented [Eagle, §**10.31$\frac{1}{2}$**]:

> The formula (12) giving the solution of this really difficult problem was actually given by Legendre in his *Traité des Fonctions elliptiques* in 1832,[1] obtained by means which only involved working with trig elliptic integrals, and not with elliptic functions as we now understand them. I think it is only appropriate that I should add a word here of admiration for the extraordinary mathematical ability of Legendre for solving such a difficult problem by means which it would be hopelessly beyond my abilities to accomplish. And yet I could, I suppose, if I tried, re-write all the steps of my solution in terms of the old trig integrals. But I am not making the attempt to do so!
>
> I should also like to add a word of tribute to Legendre for the immense industry and labour he put into the calculation of his extensive tables of the First and Second Incomplete Elliptic Integrals. He laboured on elliptic integrals and their calculation for something like 40 years. And before his death in 1833, at the age of 81,[2] he was a competent enough mathematician to realize that the new ideas that had very recently been put forth by Abel and Jacobi were right; and that the integrals he had spent his life considering were only the inverses of the functions that he should have been considering.

But the elliptic integral formula remained almost unknown. For example, in 1979 Stuart P. Keller asserted that "Except for the special cases of the sphere, the prolate spheroid and the oblate spheroid, no closed form expression exists for the surface area of the ellipsoid. This situation arises because of the fact that it is impossible to carry out the integration in the expression for the surface area in closed form for the most general case of three unequal axes"[Keller, p.310].

In 1989 Derek Lawden published a clear proof [Lawden, pp.100–102] of the formula (without mentioning Legendre), and in 1994 Leo Maas derived the elliptic integral formula [Maas], which he credits to Legendre. In 1999 an incorrect

[1] Rather, in 1825.
[2] Rather, at the age of 91.

version of the formula was published [Wolfram, p.976], without proof or references.

The surface area of an ellipsoid was represented by Bille Carlson in terms of his very complicated function R_G [Carlson 1977, p.271]. In 2003, Edward Neumann gave a much simpler representation [Neumann, (1.4)] of Carlson's function:

$$R_G(x,y,z) \;=\; \frac{1}{4}\int_0^\infty \left[\frac{\dfrac{x}{t+x}+\dfrac{y}{t+y}+\dfrac{z}{t+z}}{\sqrt{(t+x)(t+y)(t+z)}}\right] t\ \mathrm{d}t\ . \tag{47}$$

But neither representation provides any clear way of computing the area.

3 Ellipsoid in n dimensions

Consider an n-dimensional ellipsoid ($n \geq 3$) centred at the coordinate origin, with rectangular Cartesian coordinate axes along the semi-axes $a_1,\ a_2,\dots,a_n$,

$$\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\cdots+\frac{x_n^2}{a_n^2} \;=\; 1. \tag{48}$$

The area of the $(n-1)$-dimensional surface is given by a generalization of (23):

$$A \;=\; 2^n \int\limits_{x_1=0}^{a_1}\ \int\limits_{x_2=0}^{a_2\sqrt{1-\frac{x_1^2}{a_1^2}}} \cdots \int\limits_{x_{n-1}=0}^{a_{n-1}\sqrt{1-\frac{x_1^2}{a_1^2}-\frac{x_2^2}{a_2^2}-\cdots-\frac{x_{n-2}^2}{a_{n-2}^2}}}$$

$$\sqrt{\frac{1-\delta_1\dfrac{x_1^2}{a_1^2}-\delta_2\dfrac{x_2^2}{a_2^2}-\cdots-\delta_{n-1}\dfrac{x_{n-1}^2}{a_{n-1}^2}}{1-\dfrac{x_1^2}{a_1^2}-\dfrac{x_2^2}{a_2^2}-\cdots-\dfrac{x_{n-1}^2}{a_{n-1}^2}}}\quad \mathrm{d}x_{n-1}\cdots\mathrm{d}x_2\ \mathrm{d}x_1, \tag{49}$$

where (cf. (22)):

$$\delta_i \;\stackrel{\text{def}}{=}\; 1-\frac{a_n^2}{a_i^2}, \qquad (i=1,2,\dots,n-1). \tag{50}$$

As in 3 dimensions, if any semi-axis increases while the others are fixed, then the area increases.

Label the axes so that $a_1 \geq a_2 \geq \cdots \geq a_n$, and then $1 > \delta_1 \geq \delta_2 \geq \cdots \geq \delta_{n-1} \geq 0$.

Scale each variable as $x_i = a_i z_i$, so that (52) reduces to integration over the unit n-sphere

$$z_1^2+z_2^2+\cdots+z_n^2 = 1. \tag{51}$$

Thus, the area of the ellipsoid is

$$A \;=\; 2^n a_1 a_2\cdots a_{n-1} \int\limits_{z_1=0}^{1}\ \int\limits_{z_2=0}^{\sqrt{1-z_1^2}} \cdots \int\limits_{z_{n-1}=0}^{\sqrt{1-z_1^2-z_2^2-\cdots-z_{n-2}^2}}$$

$$\sqrt{\frac{1-\delta_1 z_1^2-\delta_2 z_2^2-\cdots-\delta_{n-1} z_{n-1}^2}{1-z_1^2-z_2^2-\cdots-z_{n-1}^2}}\, \mathrm{d}z_{n-1}\cdots \mathrm{d}z_2\, \mathrm{d}z_1\ . \quad (52)$$

3.1 Bounds for the area

The unit ball in n dimensions has volume [Smith & Vamanamurthy]

$$\Omega_n \;=\; \frac{\pi^{n/2}}{\Gamma\left(1+\frac{1}{2}n\right)}\ . \quad (53)$$

Thus,

$$\Omega_2 \;=\; 2\pi,\ \Omega_3 \;=\; \tfrac{4}{3}\pi,\ \Omega_4 \;=\; \tfrac{1}{2}\pi^2,\ \Omega_5 \;=\; \tfrac{8}{15}\pi^2,\ \Omega_6 \;=\; \tfrac{1}{6}\pi^3\ \textit{et cetera}. \quad (54)$$

For each $j = 1,\ 2,\ \cdots\ ,n$, scaling the j-th semi-axis from 1 to a_j multiplies the volume by a_j, and hence the ellipsoid (48) has the volume

$$V \;=\; \Omega_n a_1 a_2 a_3 \cdots a_n\ . \quad (55)$$

3.1.1 Carlson's bounds

The surface area of an n-dimensional ellipsoid was represented by Bille Carlson in terms of his very complicated function R [Carlson 1966, (4.1)], which is defined by an integral representation [Carlson 1966, (2.1)]. From that, he proved the inequalities

$$\left(\prod_{i=1}^{n} a_i\right)^{-1/n} \;<\; \frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i} \;<\; \frac{A}{nV} \;<\; \sqrt{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i^2}}\ , \quad (56)$$

where "$a_1, \cdots, a_n$ are positive, finite and not all equal" [Carlson 1966, (4.3)].

Therefore,

$$\Omega_n a_1 a_2 \cdots a_n \left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \;<\; A$$
$$\;<\; \Omega_n a_1 a_2 \cdots a_n \sqrt{\frac{n}{a_1^2}+\frac{n}{a_2^2}+\cdots+\frac{n}{a_n^2}}\ . \quad (57)$$

3.1.2 Refined bounds

In terms of $b_i = 1/a_i$, (56) becomes

$$\sqrt[n]{\prod_{i=1}^{n} b_i} \;<\; \frac{1}{n}\sum_{i=1}^{n} b_i \;<\; \frac{A}{nV} \;<\; \sqrt{\frac{1}{n}\sum_{i=1}^{n} b_i^2}\ . \quad (58)$$

Thus, $A/(nV)$ is bounded by the root-mean-square $r = (b_1^2 + b_2^2 + \cdots + b_n^2)/n)^{1/2}$ of the b_i and their arithmetic mean $m = (b_1 + b_2 + \cdots + b_n)/n$, which is greater than the geometric mean of the b_i .

For positive numbers b_i,

$$(b_1 + b_2 + \cdots + b_n)^2 \;=\; b_1^2 + b_2^2 + \cdots + b_n^2 + \sum_{i \neq j} b_i b_j \;>\; b_1^2 + b_2^2 + \cdots + b_n^2 \;, \tag{59}$$

and so

$$\sum_{i=1}^{n} b_i \;>\; \sqrt{\sum_{i=1}^{n} b_i^2} \;, \tag{60}$$

and hence

$$\frac{1}{n} \sum_{i=1}^{n} b_i \;>\; \frac{1}{\sqrt{n}} \sqrt{\frac{1}{n} \sum_{i=1}^{n} b_i^2} \;. \tag{61}$$

Therefore,

$$\sqrt{n} \;>\; \frac{\sqrt{\frac{1}{n} \sum_{i=1}^{n} b_i^2}}{\frac{1}{n} \sum_{i=1}^{n} b_i} \;. \tag{62}$$

Thus, Carlson's inequalities give simple upper and lower bounds for the area of an n-dimensional ellipsoid, whose ratio is less than $\sqrt{n}$.

The variance of the inverse semi-axes is

$$\begin{aligned} v \;&\stackrel{\text{def}}{=}\; \frac{1}{n} \left[(b_1 - m)^2 + (b_2 - m)^2 + \cdots + (b_n - m)^2 \right] \\ &=\; \frac{1}{n} \left[b_1^2 + b_2^2 + \cdots + b_n^2 - 2m(b_1 + b_2 + \cdots + b_n) + nm^2 \right] \\ &=\; \frac{1}{n} \left[b_1^2 + b_2^2 + \cdots + b_n^2 - nm^2 \right] \;=\; r^2 - m^2 \;. \end{aligned} \tag{63}$$

Therefore, the ratio of Carlson's upper and lower bounds for $A/(nV)$ is expressed in terms of the mean m of the inverse semi-axes and their standard deviation $s = \sqrt{v}$:

$$\frac{\sqrt{\frac{1}{n} \sum_{i=1}^{n} b_i^2}}{\frac{1}{n} \sum_{i=0}^{n} b_i} \;=\; \frac{r}{m} \;=\; \sqrt{\frac{r^2}{m^2}} \;=\; \sqrt{1 + \frac{v}{m^2}} \;=\; \sqrt{1 + \left(\frac{s}{m}\right)^2} \;. \tag{64}$$

3.1.3 Examples

For a near-sphere with a_1/a_n not much larger than 1, (64) shews that the ratio of Carlson's upper and lower bounds is much closer to 1 than to the $\sqrt{n}$ in (62).

For example, consider ellipsoids in which the semi-axes are in arithmetic progression, from $a_1 = 1$ to a_n. Table 1 gives, for each n and a_n, Ω_n and Carlson's lower and upper bounds for the area and their ratio, followed by $\sqrt{n}$.

Table 1. Lower and upper bounds for surface area

n	a_n	Ω_n	**Low Bound**	**Up Bound**	**Ratio**	$\sqrt{n}$
4	2	$4{\cdot}9348 \times 10^{0}$	$6{\cdot}251 \times 10^{1}$	$6{\cdot}465 \times 10^{1}$	1·0343	2
10	2	$2{\cdot}5502 \times 10^{0}$	$8{\cdot}124 \times 10^{2}$	$8{\cdot}327 \times 10^{2}$	1·0250	3·162
20	13	$2{\cdot}5807 \times 10^{-2}$	$1{\cdot}961 \times 10^{14}$	$2{\cdot}708 \times 10^{14}$	1·3813	4·472
256	3	$1{\cdot}1195 \times 10^{-152}$	$1{\cdot}547 \times 10^{-78}$	$1{\cdot}627 \times 10^{-78}$	1·0515	16
256	36	$1{\cdot}1195 \times 10^{-152}$	$4{\cdot}996 \times 10^{147}$	$8{\cdot}277 \times 10^{147}$	1·6568	16
256	100	$1{\cdot}1195 \times 10^{-152}$	$3{\cdot}428 \times 10^{254}$	$7{\cdot}804 \times 10^{254}$	2·2768	16

In each of these examples, including that with $a_{256} = 100$, the ratio of the bounds is substantially smaller than $\sqrt{n}$.

Bounds for area of spheroids If the semi-axes of an ellipsoid are $a_1 = a_2 = \cdots = a_{n-1}$ and $a_n \neq a_1$ (with $n > 2$), then we may call that a spheroid, oblate if $a_1 > a_n$ and prolate if $a_1 < a_n$. That spheroid has volume (55)

$$V \;=\; \Omega_n a_1^{n-1} a_n \;. \tag{65}$$

Carlson's bounds in (57) become

$$\begin{aligned} m \;&=\; \frac{(n-1)b_1 + b_n}{n} \;=\; b_1 \left(1 + \frac{(b_n/b_1) - 1}{n}\right), \\ r \;&=\; \sqrt{\frac{(n-1)b_1^2 + b_n^2}{n}} \;=\; b_1 \sqrt{1 + \frac{(b_n/b_1)^2 - 1}{n}} \;. \end{aligned} \tag{66}$$

Hence, for a sequence of spheroids with fixed semi-axes a_1 and a_n as $n \to \infty$,

$$m \;=\; \frac{1}{a_1}\left(1 + \frac{(a_1/a_n) - 1}{n}\right) \quad \text{and} \quad r \;=\; \frac{1}{a_1}\left(1 + O\left(n^{-1}\right)\right) \;. \tag{67}$$

Therefore, the area is asymptotically

$$A \;\sim\; nV/a_1 \;=\; n\Omega_n a_1^{n-2} a_n \;. \tag{68}$$

However, Carlson's representation of the area does not provide any clear way of computing that area.

4 Reduction to Abelian Integral

The infinite series for surface area of a 3-dimensional ellipsoid has been generalized to n-dimensional ellipsoids by using n-dimensional spherical coordinates — but that proves to be very complicated [Lehmer, (11)].

For 3-dimensional ellipsoids, Lawden used an ingenious method to reduce the double integral (23) to a single integral [Lawden, pp.100-102]. For n-dimensional ellipsoids, Lawden's method is here generalized to reduce the $(n-1)$-dimensional integral (52) to an Abelian integral. That Abelian integral can readily be computed.

The multiple integral in (52) can be rewritten as

$$\begin{aligned} A \;=&\; \\ & 2a_1 a_2 \cdots a_{n-1} \int\limits_{z_1=-1}^{1} \; \int\limits_{z_2=-\sqrt{1-z_1^2}}^{\sqrt{1-z_1^2}} \cdots \int\limits_{z_{n-1}=-\sqrt{1-z_1^2-z_2^2-\cdots-z_{n-2}^2}}^{\sqrt{1-z_1^2-z_2^2-\cdots-z_{n-2}^2}} \\ & \qquad h^{-1}\, \mathrm{d}z_{n-1} \cdots \mathrm{d}z_2\, \mathrm{d}z_1 \;, \end{aligned} \tag{69}$$

where

$$h \;\stackrel{\text{def}}{=}\; \sqrt{\frac{1 - z_1^2 - z_2^2 - \cdots - z_{n-1}^2}{1 - \delta_1 z_1^2 - \delta_2 z_2^2 - \cdots - \delta_{n-1} z_{n-1}^2}} \;, \tag{70}$$

with $h^{-1} \geq 1$. In the $(n-1)$-dimensional subspace with $z_n = 0$, the value of h ranges from 0 on that $(n-1)$-sphere $z_1^2 + z_2^2 + \cdots + z_{n-1}^2 = 1$ to $h = 1$ at the centre.

At any point in the unit $(n-1)$-ball which is the domain of integration in (52), any $(n-1)$-dimensional element

$$\mathrm{d}v \ = \ \mathrm{d}z_1 \,\mathrm{d}z_2 \,\cdots \mathrm{d}z_{n-1} \tag{71}$$

gets enlarged by the factor $1/h$, to give an element of surface area of the ellipsoid:

$$a_1 a_2 \cdots a_{n-1} h^{-1} \ \mathrm{d}z_1 \,\mathrm{d}z_2 \,\cdots \mathrm{d}z_{n-1} \ = \ h^{-1} \ \mathrm{d}x_1 \,\mathrm{d}x_2 \,\cdots \mathrm{d}x_{n-1} \ . \tag{72}$$

Thus

$$A \ = \ 2 a_1 a_2 \cdots a_{n-1} \int \frac{\mathrm{d}v}{h} \ , \tag{73}$$

integrated over the unit $(n-1)$-ball.

The points on the unit n-sphere with that value of h satisfy (70), and hence

$$h^2 (1 - \delta_1 z_1^2 - \delta_2 z_2^2 - \cdots - \delta_{n-1} z_{n-1}^2) \ = \ 1 - z_1^2 - z_2^2 - \cdots - z_{n-1}^2 \ . \tag{74}$$

Therefore, those points on the n-sphere project onto points in the $(n-1)$-sphere (with $z_n = 0$), forming the $(n-1)$-dimensional ellipsoid

$$\frac{z_1^2}{\left(\dfrac{1-h^2}{1-\delta_1 h^2}\right)} + \frac{z_2^2}{\left(\dfrac{1-h^2}{1-\delta_2 h^2}\right)} + \cdots + \frac{z_{n-1}^2}{\left(\dfrac{1-h^2}{1-\delta_{n-1} h^2}\right)} \ = \ 1 \ . \tag{75}$$

(Ellipsoids of this type with different values of h are not confocal.)

This ellipsoid with parameter h has the semi-axes

$$f_j \ = \ \sqrt{\frac{1-h^2}{1-\delta_j h^2}} \qquad (j = 1,\, 2, \cdots,\, n-1), \tag{76}$$

and hence its $(n-1)$-dimensional measure is

$$w(h) \ = \ \Omega_{n-1} g(h), \tag{77}$$

where

$$g(h) \ \stackrel{\mathrm{def}}{=} \ f_1 f_2 \cdots f_{n-1} \ = \ \frac{(1-h^2)^{(n-1)/2}}{\sqrt{(1-\delta_1 h^2)(1-\delta_2 h^2)\cdots(1-\delta_{n-1} h^2)}} \ . \tag{78}$$

Consider the $(n-1)$-dimensional measure $\mathrm{d}w$ of the shell bounded by the $(n-1)$-dimensional ellipsoids (75) with parameters h and $h + \mathrm{d}h$. This is the projection of a strip of the n-dimensional sphere (51) (with parameter ranging from h to $h + \mathrm{d}h$), and hence the element of area for that strip is $\mathrm{d}A \ = \ 2 a_1 a_2 \cdots a_{n-1} \mathrm{d}v/h$. It follows that the surface area between the point $z_n = 1$ (with $h = 1$) and the unit $(n-1)$-sphere where $z_n = 0$ (with $h = 0$) is given by

$$S \ = \ \int \frac{\mathrm{d}w}{h} \ = \ \Omega_{n-1} \int \frac{\mathrm{d}g(h)}{h} \ = \ \Omega_{n-1} \int_1^0 \frac{1}{h} \frac{\mathrm{d}g(h)}{\mathrm{d}h} \ \mathrm{d}h \ . \tag{79}$$

Denote $z = -h^2$ and define

$$r(z) \stackrel{\text{def}}{=} (1+\delta_1 z)(1+\delta_2 z)\cdots(1+\delta_{n-1}z), \tag{80}$$

and

$$p(z) \stackrel{\text{def}}{=} f_1^2 f_2^2 \cdots f_{n-1}^2 = \frac{(1+z)^{n-1}}{r(z)}, \tag{81}$$

so that

$$g(h) = \sqrt{p(-h^2)}, \tag{82}$$

and

$$\frac{\mathrm{d}g}{\mathrm{d}h} = \frac{\mathrm{d}g}{\mathrm{d}p} \cdot \frac{\mathrm{d}p}{\mathrm{d}z} \cdot \frac{\mathrm{d}z}{\mathrm{d}h} = \frac{-h}{\sqrt{p}} \frac{\mathrm{d}p}{\mathrm{d}z}. \tag{83}$$

Hence,

$$\frac{1}{h}\frac{\mathrm{d}g(h)}{\mathrm{d}h} = \frac{-1}{\sqrt{p(z)}} \frac{\mathrm{d}p(z)}{\mathrm{d}z}. \tag{84}$$

It follows from (81) and (80) that

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{(n-1)(1+z)^{n-2}}{r} - \frac{(1+z)^{n-1}}{r^2} \frac{\mathrm{d}r}{\mathrm{d}z}, \tag{85}$$

and

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}z} &= \delta_1(1+\delta_2 z)(1+\delta_3 z)\cdots(1+\delta_{n-1}z) \\
&\quad +(1+\delta_1 z)\delta_2\cdots(1+\delta_{n-1}z) + (1+\delta_1 z)(1+\delta_2 z)\cdots(1+\delta_{n-2}z)\delta_{n-1} \\
&= r\left(\frac{\delta_1}{1+\delta_1 z} + \frac{\delta_2}{1+\delta_2 z} + \cdots + \frac{\delta_{n-1}}{1+\delta_{n-1}z}\right),
\end{aligned} \tag{86}$$

and hence

$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}z} &= \frac{(1+z)^{n-2}}{r(z)}\left(n-1-(1+z)\sum_{i=1}^{n-1}\frac{\delta_i}{1+\delta_i z}\right) \\
&= \frac{(1+z)^{n-2}}{r(z)}\sum_{i=1}^{n-1}\left(1-\frac{\delta_i(1+z)}{1+\delta_i z}\right) = \frac{(1+z)^{n-2}}{r(z)}\sum_{i=1}^{n-1}\frac{1-\delta_i}{1+\delta_i z}.
\end{aligned} \tag{87}$$

It follows from (84) that

$$\begin{aligned}
\frac{1}{h}\frac{\mathrm{d}g(h)}{\mathrm{d}h} &= \frac{-1}{\sqrt{p(z)}}\frac{\mathrm{d}p(z)}{\mathrm{d}z} \\
&= \frac{-(1+z)^{(n-3)/2}}{\sqrt{r(z)}}\left(\frac{1-\delta_1}{1+\delta_1 z} + \cdots + \frac{1-\delta_{n-1}}{1+\delta_{n-1}z}\right) \\
&= \frac{-(1-h^2)^{(n-3)/2}}{\sqrt{(1-\delta_1 h^2)(1-\delta_2 h^2)\cdots(1-\delta_{n-1}h^2)}} \times \\
&\quad \left(\frac{1-\delta_1}{1-\delta_1 h^2} + \frac{1-\delta_2}{1-\delta_2 h^2} + \cdots + \frac{1-\delta_{n-1}}{1-\delta_{n-1}h^2}\right).
\end{aligned} \tag{88}$$

This last expression is continuous on [0,1] for $n \geq 3$, and hence it follows from (79) that the surface area of the ellipsoid is given explicitly as

$$\begin{aligned} A &= 2a_1 a_2 \cdots a_{n-1} \Omega_{n-1} \int_1^0 \frac{1}{h} \frac{\mathrm{d}g}{\mathrm{d}h} \, \mathrm{d}h \\ &= \frac{2a_1 \cdots a_{n-1} \pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \int_0^1 \sqrt{\frac{(1-h^2)^{n-3}}{(1-\delta_1 h^2)(1-\delta_2 h^2)\cdots(1-\delta_{n-1}h^2)}} \times \\ & \left(\frac{1-\delta_1}{1-\delta_1 h^2} + \frac{1-\delta_2}{1-\delta_2 h^2} + \cdots + \frac{1-\delta_{n-1}}{1-\delta_{n-1} h^2} \right) \mathrm{d}h \ , \end{aligned} \tag{89}$$

which is an Abelian integral.

For $n = 2$ this gives the circumference of the ellipse

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} \ = \ 1 \ , \tag{90}$$

with $\delta_1 = 1 - (a_2/a_1)^2$, as

$$A \ = \ \frac{4a_2^2}{a_1} \int_0^1 \frac{\mathrm{d}h}{(1-\delta_1 h^2)\sqrt{(1-h^2)(1-\delta_1 h^2)}} \ , \tag{91}$$

which is an elliptic integral. Indeed, it was shewn in (9) that the circumference of that ellipse is $4a_1 E(\delta_1)$.

For $n = 3$, writing the semi-axes as a, b, c as in (1), and writing δ and ϵ as in (22), this becomes:

$$A \ = \ \frac{2\pi c^2}{ab} \int_0^1 \frac{a^2 + b^2 - (a^2 + b^2 - 2c^2)h^2}{\big((1-\delta h^2)(1-\epsilon h^2)\big)^{3/2}} \, \mathrm{d}h \ , \tag{92}$$

and this reduces to the formula (46) in standard elliptic intehrals.

For odd $n = 2k+1$ (with $k > 0$) the integrand in (89) is smooth, with the polynomial factor $(1-h^2)^{(n-3)/2} = (1-h^2)^{k-1}$, and hence this Abelian integral can readily be evaluated by Romberg integration. But for even $n = 2k$ (with $k > 1$), the integrand has the factor $(1-h^2)^{(n-3)/2} = [(1+h)(1-h)]^{k-3/2}$. The factor $(1-h)^{k-3/2}$ makes the $(k-1)$-th derivative of the integrand unbounded as $h \nearrow 1$, so that direct numerical integration would be inefficient.

In order to prevent that near-singularity in the integrand for even $n \geq 4$, make the substitution $h = 1 - x^2$ so that $1 - h^2 = x^2(2 - x^2)$, and (89) becomes the Abelian integral

$$\begin{aligned} A \ = \ & \frac{4a_1 a_2 \cdots a_{n-1} \pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \int_0^1 x^{n-2} \times \\ & \sqrt{\frac{(2-x^2)^{n-3}}{(1-\delta_1(1-x^2)^2)(1-\delta_2(1-x^2)^2)\cdots(1-\delta_{n-1}(1-x^2)^2)}} \times \\ & \left(\frac{1-\delta_1}{1-\delta_1(1-x^2)^2} + \cdots + \frac{1-\delta_{n-1}}{1-\delta_{n-1}(1-x^2)^2} \right) \mathrm{d}x \ . \end{aligned} \tag{93}$$

This integrand is smooth for all $n \geq 2$, and hence this Abelian integral can readily be computed by Romberg integration.

4.1 Examples

A program has been written (in THINK Pascal) to compute A by this formula. Table 2 gives the actual areas for those ellipsoids for which upper and lower bounds had been given in Table 1.

Table 2. Areas with lower and upper bounds

n	a_n	**Lower Bound**	**Computed Area**	**Upper Bound**
4	2	$6{\cdot}251 \times 10^{1}$	$6{\cdot}39222065129 \times 10^{1}$	$6{\cdot}465 \times 10^{1}$
10	2	$8{\cdot}124 \times 10^{2}$	$8{\cdot}29183467191 \times 10^{2}$	$8{\cdot}327 \times 10^{2}$
20	13	$1{\cdot}961 \times 10^{14}$	$2{\cdot}58913095038 \times 10^{14}$	$2{\cdot}708 \times 10^{14}$
256	3	$1{\cdot}547 \times 10^{-78}$	$1{\cdot}62645053874 \times 10^{-78}$	$1{\cdot}627 \times 10^{-78}$
256	36	$4{\cdot}996 \times 10^{147}$	$8{\cdot}18623897626 \times 10^{147}$	$8{\cdot}277 \times 10^{147}$
256	100	$3{\cdot}428 \times 10^{254}$	$7{\cdot}56178774520 \times 10^{254}$	$7{\cdot}804 \times 10^{254}$

5 Equal semi-axes

If any of the semi-axes a_i share common values, then the corresponding pairs of factors under the square root sign in (89) (or in (93)) can be taken outside the square root sign. If the value of a_j occurs with odd multiplicity $2m+1$ (amongst $a_1, \cdots, a_{n-1}$), then a single factor $(1-\delta_j h^2)$ will remain in the denominator under the square root, with the factor $(1-\delta_j h^2)^m$ in the denominator of the integrand.

In particular, for odd $n = 2k+1$, if only one value of a_j has odd multiplicity then select that as a_n. The square root term then reduces to 1, since all factors in the denominator, and also $(1-h^2)^{2k-2}$ in the numerator, can be taken outside the square root sign. In such a case the integrand in (89) is a rational function of h^2, and thatcan be integrated in terms of elementary functions. If only 3 values of a_j have odd multiplicities then select the smallest as a_n and the others as a_1 and a_2, so that the integrand becomes a rational function of h^2 divided by $\sqrt{(1-\delta_1 h^2)(1-\delta_2 h^2)}$. Thus, in that case the integral is an elliptic integral.

For even n, if all values of a_j have even multiplicity then select the smallest value as a_n, so that $\delta_{n-1} = 0$. After all pairs of factors have been taken outside the square root the integrand becomes a rational function of h^2 multiplied by $\sqrt{1-h^2}$, and that can be integrated in terms of elementary functions by the substitution $h = \sin\theta$. If only 2 values of a_j have odd multiplicities then select the smaller as a_n and the other as a_1, so that the integrand becomes a rational function of h^2 multiplied by $\sqrt{(1-h^2)/(1-\delta_1 h^2)}$. Thus, in that case the integral is an elliptic integral.

5.1 Biaxial ellipsoids

Consider biaxial ellipsoids of the type $B_{p,q}$, with p semi-axes equal to β and q semi-axes equal to γ, so that spheroids are of type $B_{n-1,1}$. Denote $n = p+q$, and then

$$\beta \;=\; a_1 \;=\; a_2 \;=\cdots=\; a_p \neq a_{p+1} \;=\; a_{p+2} \;=\cdots=\; a_n \;=\; \gamma\,, \qquad (94)$$

and

$$\delta_1 = \delta_2 = \cdots = \delta_p = 1 - \frac{\gamma^2}{\beta^2}, \quad \delta_{p+1} = \delta_{p+2} = \cdots = \delta_{n-1} = 0\ . \quad (95)$$

Hereafter, abbreviate δ_1 to δ. Thus, $1-\delta = \gamma^2/\beta^2$.

If either (or both) of β and γ has even multiplicity $p = 2m$ then choose that as a_1, so that the ellipsoid is of type $B_{2m,q}$. With $n = 2m+q$. the formula (89) becomes

$$\begin{aligned} A &= \frac{2\beta^{2m}\gamma^{q+1}\pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \int_0^1 \sqrt{\frac{(1-h^2)^{2m+q-3}}{(1-h^2)^{2m}}} \left(\frac{2m}{\beta^2(1-\delta h^2)} + \frac{q-1}{\gamma^2}\right) \mathrm{d}h \\ &= \frac{2\beta^{2m}\gamma^{q-1}\pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \int_0^1 \frac{(1-h^2)^{(n-3)/2}}{(1-\delta h^2)^m} \left(\frac{2m\gamma^2}{\beta^2(1-\delta h^2)} + q - 1\right) \mathrm{d}h. \quad (96) \end{aligned}$$

For odd $q = 2k+1$, the factor $(1-h^2)^{(n-3)/2}$ becomes the polynomial factor $(1-h^2)^{m+k-1}$. Thus, the area for ellipsoids of type $B_{2m,2k+1}$ is

$$A = \frac{4\beta^{2m}\gamma^{2k}\pi^{m+k}}{(m+k)!} \int_0^1 \frac{(1-h^2)^{m+k-1}}{(1-\delta h^2)^m} \left(\frac{m\gamma^2}{\beta^2(1-\delta h^2)} + k\right) \mathrm{d}h. \quad (97)$$

The integrand is a rational function of h^2, and hence this integral can be expressed in terms of elementary functions. The detailed working is given in Appendix A.

For even $q = 2k$,

$$(1-h^2)^{(n-3)/2} = \frac{(1-h^2)^{m+k-1}}{\sqrt{1-h^2}}\ . \quad (98)$$

Thus, for biaxial ellipsoids of type $B_{2m,2k}$, the formula (89) reduces to

$$\begin{aligned} A = {} & \frac{2\beta^{2m}\gamma^{2k}\pi^{m+k-1}\sqrt{\pi}}{\Gamma\left(m+k+\frac{1}{2}\right)} \times \\ & \int_0^1 \frac{(1-h^2)^{m+k-1}}{(1-\delta h^2)^m\sqrt{1-h^2}} \left(\frac{2m\gamma^2}{\beta^2(1-\delta h^2)} + 2k - 1\right) \mathrm{d}h. \quad (99) \end{aligned}$$

The integrand is a rational function of h^2 and $\sqrt{1-h^2}$, which can be integrated in terms of elementary functions. The detailed working is given in Appendix B.

Thus, for biaxial ellipsoids with one (or both) semi-axes having even multiplicity, the surface area can be expressed in terms of elementary functions.

If neither of β and γ have even multiplicity then both have odd multiplicities $2m+1$ and $2k+1$, and $n = 2k+2m+2$ is even. Thus, for ellipsoids of type $B_{2m+1,2k+1}$ the formula (89) becomes

$$\begin{aligned} A &= \frac{2\beta^{2m+1}\gamma^{2k+2}\pi^{(2m+2k+1)/2}}{\Gamma\left(\frac{2m+2k+3}{2}\right)} \times \\ & \qquad \int_0^1 \sqrt{\frac{(1-h^2)^{2m+2k-1}}{(1-\delta h^2)^{2m+1}}} \left(\frac{2m+1}{\beta^2(1-\delta h^2)} + \frac{2k}{\gamma^2}\right) \mathrm{d}h \\ &= \frac{2\beta^{2m+1}\gamma^{2k}\pi^{m+k}\sqrt{\pi}}{\Gamma\left(m+k+1+\frac{1}{2}\right)} \times \\ & \int_0^1 \frac{(1-h^2)^{m+k}}{(1-\delta h^2)^m\sqrt{(1-\delta h^2)(1-h^2)}} \left(\frac{(2m+1)\gamma^2}{\beta^2(1-\delta h^2)} + 2k\right) \mathrm{d}h, \quad (100) \end{aligned}$$

which is an elliptic integral.

6 Capacity

James Clerk Maxwell gave (in 1873) the first extensive mathematical treatment of electrostatic capacity [Maxwell], in his epoch-making monograph[3] **A Treatise on Electricity and Magnetism**. He defined *Capacity of a Conductor* in §50 [Maxwell v.1, p.48]:

> If one conductor is insulated while all the surrounding conductors are kept at the zero potential by being put in communication with the earth, and if the conductor, when charged with a quantity E of electricity, has a potential V, the ratio of E to V is called the Capacity of the conductor. If the conductor is completely enclosed within a conducting vessel without touching it, then the charge on the inner conductor will be equal and opposite to the charge on the inner surface of the outer conductor, and will be equal to the capacity of the inner conductor multiplied by the difference of the potentials of the two conductors.

Maxwell then considered the *General Theory of a System of Conductors*: "Let $A_1,\ A_2,\ \ldots A_n$ be any number of conductors of any form. Let the charge or total quantity of electricity on each of them be $E_1,\ E_2,\ \ldots E_n$, and let their potentials be $V_1,\ V_2,\ \ldots V_n$ respectively." [Maxwell v.1, p.89]. He explained how

> we should obtain n equations[4] of the form [Maxwell v.1, p.90].

$$
\begin{array}{rcl}
E_1 & = & q_{11}V_1 \ldots + q_{1s}V_s \ldots + q_{1n}V_n, \\
- & - & - \quad - \quad - \quad - \quad - \quad - \\
E_r & = & q_{r1}V_1 \ldots + q_{rs}V_s \ldots + q_{rn}V_n, \\
- & - & - \quad - \quad - \quad - \quad - \quad - \\
E_n & = & q_{n1}V_1 \ldots + q_{ns}V_s \ldots + q_{nn}V_n.
\end{array} \tag{101}
$$

> The coefficients in these equations $\cdots$ may be called Coefficients of Induction.
>
> Of these q_{11} is numerically equal to the quantity of electricity on A_1 when A_1 is at potential unity and all the other bodies are at potential zero. This is called the Capacity of A_1. It depends on the form and position of all the conductors in the system.

Maxwell remarked that "The mathematical determination of the coefficients $\cdots$ of capacity from the known forms and positions of the conductors is in general difficult." [Maxwell v.1, p.90]. He explained how to compute an upper bound for the capacity [Maxwell v.1, p.117], and he explained the method of

[3] My copy was formerly in the library of the Wellington Philosophical Society, and then in the library of the Royal Society of New Zealand.

[4] This is a noteworthy early example of a system of linear algebraic equations, written with double and single subscripts. GJT

J. W. Strutt (later Lord Rayleigh) for calculating upper and lower bounds for capacity "in the very important case in which the electrical action is entirely between two conducting surfaces S_1 and S_2, of which S_2 completely surrounds S_1 and is kept at potential zero" [Maxwell v.1, p.118].

6.1 Capacity of spheres

Maxwell considered "two concentric spherical surfaces of radii a and b, of which b be the greater, be maintained at potentials A and B respectively," and he shewed that the potential at radius r between those spheres is of the form

$$V = C_1 + C_2 r^{-1} . \tag{102}$$

From this he found the capacity of the enclosed sphere:

$$C = \frac{ab}{b-a} . \tag{103}$$

[Maxwell v.1, p.153].

Fix the radius a of the inner sphere and let the radius b of the outer sphere increase without bound. As $r \nearrow \infty$ the potential is

$$V = C_1 + C_2 r^{-1} \to C_1 . \tag{104}$$

and as $b \nearrow \infty$ the capacity converges to the limit

$$C = \frac{a}{1-(a/b)} \searrow a . \tag{105}$$

[Maxwell v.1, p.153].

Thus, we say that a sphere of radius a, in a 3-dimensional universe otherwise empty, has capacity equal to a.

And similar results hold (110) in space of more than 3 dimensions.

6.2 Capacity of coaxial cylinders

Maxwell considered *Two Infinite Coaxal Cylindric Surfaces*. "Let the radius of the outer surface of a conducting cylinder be a, and let the radius of the inner surface of a hollow cylinder, having the same axis with the first, be b. Let their potentials be A and B respectively". In any plane orthogonal to the cylinder axis the potential satisfies the 2-dimensional Laplace equation, and Maxwell shewed that the potential at radius r between those cylinders is of the form

$$V = C_1 + C_2 \log r . \tag{106}$$

From this he found that the capacity of a length l of the inner cylinder is

$$C = \frac{l}{2\log(b/a)} . \tag{107}$$

[Maxwell v.1, p.155].

Fix the radius a of the inner cylinder and let the radius b of the outer cylinder increase without bound. As the radius $r \nearrow \infty$ the potential (106) increases (or decreases) unboundedly, and as $b \nearrow \infty$ the capacity (107) of a length l of the inner cylinder converges to 0.

Thus, in a 2-dimensional potential field, the concept of capacity does not apply to an isolated finite body in an otherwise empty universe.

Accordingly, for the capacity of isolated figures we shall consider figures with dimension $n \geq 3$.

7 Ellipsoid capacity in n dimensions

Consider the n-dimensional ellipsoid (48).

7.1 Bounds for the capacity

The electrostatic capacity C of an n-dimensional ellipsoid was represented by Bille Carlson in terms of his very complicated function R [Carlson 1966, (4.2)], which is defined by an integral representation [Carlson 1966, (2.1)]. From that, he obtained the inequalities

$$\left(\prod_{i=1}^{n} a_i\right)^{1-2/n} < \frac{C}{n-2} < \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^{n-2} , \tag{108}$$

where $a_1, \cdots, a_n$ are positive, finite and not all equal [Carlson 1966, (4.9)].

Taking the $(n-2)$-th roots in (108), this becomes

$$\left(\prod_{i=1}^{n} a_i\right)^{1/n} < \left(\frac{C}{n-2}\right)^{1/(n-2)} < \frac{1}{n}\sum_{i=1}^{n} a_i \, . \tag{109}$$

Thus, $[C/(n-2)]^{1/(n-2)}$ lies between the geometric mean of the semi-axes a_i and their arithmetic mean.

In particular, for an n-sphere of radius r,

$$C = (n-2)r^{n-2} \, . \tag{110}$$

Carlson's inequalities give simple upper and lower bounds for the capacity of an n-dimensional ellipsoid. But if $a_n \searrow 0$ (with fixed $a_1,\ a_2, \ldots,\ a_{n-1}$), the geometric mean converges to 0 but the arithmetic mean converges to $(a_1 + \cdots + a_{n-1})/n$. Hence, for each n there is no upper bound to the ratio of Carlson's upper and lower bounds for capacity.

Carlson's representation of the capacity does not provide any clear way of computing that capacity.

8 Capacity as Hyperelliptic Integral

For a 3-dimensional ellipsoid, the capacity C can be expressed by a well-known elliptic integral [Pólya & Szegő, (6.4)]:

$$\frac{1}{C} = \frac{1}{2}\int_0^{\infty} \frac{\mathrm{d}u}{\sqrt{(a_1^2+u)(a_2^2+u)(a_3^2+u)}} \, . \tag{111}$$

For an n-dimensional ellipsoid with $n \geq 3$, this generalizes to the integral

$$\frac{1}{C} \;=\; \frac{1}{2}\int_0^\infty \frac{\mathrm{d}u}{\sqrt{(a_1^2+u)(a_2^2+u)\cdots(a_n^2+u)}} \;, \tag{112}$$

which is an elliptic integral if $n = 3$ or 4, and is a hyperelliptic integral if $n \geq 5$.

As a check on the coefficient of the integral, applying this formula to the n-sphere of radius r, we get

$$\frac{1}{C} \;=\; \frac{1}{2}\int_0^\infty (r^2+u)^{-n/2}\,\mathrm{d}u \;=\; \frac{1}{2}\left[\frac{(r^2+u)^{1-n/2}}{1-n/2}\right]_0^\infty \;=\; \frac{r^{2-n}}{n-2} \;, \tag{113}$$

in agreement with (110).

8.1 Evaluation of integral

The hyperelliptic integrand in (112) is not suitable for numerical integration.

Define

$$\mu_j \;\stackrel{\text{def}}{=}\; 1 - \frac{a_j^2}{a_1^2} \;, \tag{114}$$

so that each $\mu_j < 1$. If the ellipsoid semi-axes are labelled so that

$$a_1 \;\geq a_2 \;\geq a_3 \;\geq\; \cdots \;\geq\; a_n \;>\; 0, \tag{115}$$

then

$$0 \;=\; \mu_1 \;\leq\; \mu_2 \;\leq\; \mu_3 \;\leq\; \cdots \;\leq\; \mu_n \;<\; 1 \;. \tag{116}$$

Then, with $y = u/a_1^2$ we get

$$\begin{aligned}
\frac{1}{C} \;&=\; \frac{1}{2a_1^n}\int_0^\infty \left[\left(1+\frac{u}{a_1^2}\right)\left(\frac{a_2^2}{a_1^2}+\frac{u}{a_1^2}\right)\cdots\left(\frac{a_n^2}{a_1^2}+\frac{u}{a_1^2}\right)\right]^{-1/2}\,\mathrm{d}u \\
&=\; \frac{1}{2a_1^{n-2}}\int_0^\infty \left[(1+y)\left(\frac{a_2^2}{a_1^2}+y\right)\cdots\left(\frac{a_n^2}{a_1^2}+y\right)\right]^{-1/2}\,\mathrm{d}y \\
&=\; \frac{1}{2a_1^{n-2}}\int_0^\infty \left[(1+y)\,(1+y-\mu_2)\cdots(1+y-\mu_n)\right]^{-1/2}\,\mathrm{d}y \;.
\end{aligned} \tag{117}$$

Now substitute $1+y = 1/x^2$, so that $\mathrm{d}y = -2\mathrm{d}x/x^3$, and we get the hyperelliptic integral

$$\frac{1}{C} \;=\; \frac{1}{a_1^{n-2}}\int_0^1 \frac{x^{n-3}}{\sqrt{(1-\mu_2x^2)(1-\mu_3x^2)\cdots(1-\mu_nx^2)}}\,\mathrm{d}x \;. \tag{118}$$

For even $n = 2q \geq 4$ we could substitute $1+y = 1/z$, so that $\mathrm{d}y = -\mathrm{d}z/z^2$, and we get the simpler hyperelliptic integral

$$\frac{1}{C} \;=\; \frac{1}{2a_1^{2q-2}}\int_0^1 \frac{z^{q-2}}{\sqrt{(1-\mu_2z)(1-\mu_3z)\cdots(1-\mu_{2q}z)}}\,\mathrm{d}z \;. \tag{119}$$

8.1.1 Capacity of 3-dimensional ellipsoid

In particular, for a 3-dimensional ellipsoid (not a sphere), substituting $z = x\sqrt{\mu_3}$,

$$\begin{aligned}
\frac{1}{C} &= \frac{1}{a_1}\int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-\mu_2 x^2)(1-\mu_3 x^2)}} \\
&= \frac{1}{a_1}\int_0^{\sqrt{\mu_3}} \frac{\mathrm{d}z}{\sqrt{\left(1-\left(\frac{\mu_2}{\mu_3}\right)z^2\right)(1-z^2)}} \\
&= \frac{1}{a_1\sqrt{\mu_3}}\int_0^{\sqrt{\mu_3}} \frac{\mathrm{d}z}{\sqrt{\left(1-\left(\frac{\mu_2}{\mu_3}\right)z^2\right)(1-z^2)}} \\
&= \frac{1}{\sqrt{a_1^2-a_3^2}}\int_0^{\sqrt{\mu_3}} \frac{\mathrm{d}z}{\sqrt{(1-mz^2)(1-z^2)}} = \frac{K(\sqrt{\mu_3}\,|\,m)}{\sqrt{a_1^2-a_3^2}}\,, \qquad (120)
\end{aligned}$$

where $m = \mu_2/\mu_3$ and $K(x|m)$ is Legendre's Incomplete Elliptic Integral of the First Kind (13). Therefore, the capacity is

$$C = \frac{\sqrt{a_1^2-a_3^2}}{K\left(\arcsin\sqrt{1-\dfrac{a_3^2}{a_1^2}}\;\middle|\;\dfrac{a_1^2-a_2^2}{a_1^2-a_3^2}\right)}\,. \qquad (121)$$

8.2 Examples

The integrand in (118) is smooth on the interval [0,1], and hence the integral can be evaluated readily by Romberg integration, to give the capacity C.

A program has been written (in THINK Pascal) to compute C by this formula. For the same ellipsoids whose surface areas have been calculated in Table 2, Table 3 gives the computed capacity, with Carlson's lower and upper bounds.

Table 3. Capacities with lower and upper bounds

n	a_n	Lower Bound	Capacity	Upper Bound
4	2	$4{\cdot}216 \times 10^{0}$	$4{\cdot}40659279 \times 10^{0}$	$4{\cdot}5$
10	2	$1{\cdot}697 \times 10^{2}$	$1{\cdot}81082801 \times 10^{2}$	$2{\cdot}050 \times 10^{2}$
20	13	$8{\cdot}854 \times 10^{14}$	$2{\cdot}14341967 \times 10^{15}$	$2{\cdot}931 \times 10^{16}$
256	3	$6{\cdot}832 \times 10^{73}$	$8{\cdot}30807716 \times 10^{73}$	$7{\cdot}352 \times 10^{78}$
256	36	$1{\cdot}991 \times 10^{298}$	$1{\cdot}15971902 \times 10^{299}$	$1{\cdot}847 \times 10^{324}$
256	100	$4{\cdot}269 \times 10^{404}$	$5{\cdot}99217774 \times 10^{405}$	$1{\cdot}099 \times 10^{435}$

9 Capacity for repeated semi-axes

If any of the semi-axes $a_1,\ a_2,\ \ldots,\ a_n$ share common values, then the corresponding pairs of factors under the square root sign in (112) can be taken outside the square root sign. If the value of a_j occurs with odd multiplicity

$2m+1$, then the factor a_j^2+u will remain in the denominator under the square root, with the factor $(a_j^2+u)^m$ in the denominator of the integrand.

Hence, if no value of a_j has odd multiplicity then the integrand in (112) is the reciprocal of a monic polynomial in u of degree $n/2$. If only one value of a_j has odd multiplicity then label it as a_1, and the integrand in (112) is

$$\frac{1}{\mathcal{P}(u)\sqrt{a_1^2+u}} ,$$

where $\mathcal{P}$ is a monic polynomial of degree $(n-1)/2$. If only 2 values of a_j have odd multiplicity then label them as a_1 and a_2, and the integrand in (112) is

$$\frac{1}{\mathcal{Q}(u)\sqrt{(a_1^2+u)(a_2^2+u)}} ,$$

where $\mathcal{Q}$ is a monic polynomial of degree $(n-2)/2$. Therefore, if no more than 2 values of a_j have odd multiplicity then the hyperelliptic integral giving the capacity can be reduced to elementary functions of the semi-axes.

If only 3 values of a_j have odd multiplicity then label them as a_1, a_2 and a_3, and the integrand in (112) is

$$\frac{1}{\mathcal{R}(u)\sqrt{(a_1^2+u)(a_2^2+u)(a_3^2+u)}} ,$$

where $\mathcal{R}$ is a monic polynomial of degree $(n-3)/2$. If only 4 values of a_j have odd multiplicity then label them as a_1, a_2, a_3 and a_4, and the integrand in (112) is

$$\frac{1}{\mathcal{S}(u)\sqrt{(a_1^2+u)(a_2^2+u)(a_3^2+u)(a_4^2+u)}} ,$$

where $\mathcal{S}$ is a monic polynomial of degree $(n-4)/2$. Therefore, if only 3 or 4 values of a_j have odd multiplicity then the hyperelliptic integral (112) giving the capacity reduces to an elliptic integral of the semi-axes.

In particular, the capacity of a general ellipsoid in 3 or 4 dimensions is given by an elliptic integral, which reduces to elementary functions if any semi-axis is repeated.

9.1 Capacity of biaxial ellipsoids

In particular, consider biaxial ellipsoids of type $B_{p,q}$, which are defined as ellipsoids with p semi-axes equal to β and q semi-axes equal to γ, where (unlike (94)) we take $\beta > \gamma$. Denote $n = p+q$, and then

$$\beta \;=\; a_1 \;=\; a_2 \;=\; \cdots \;=\; a_p > a_{p+1} \;=\; a_{p+2} \;=\cdots\;=\; a_n \;=\; \gamma\ , \tag{122}$$

and

$$\mu_1 \;=\cdots=\; \mu_p \;=\; 0, \quad \mu_{p+1} \;=\cdots=\; \mu_n \;=\; 1-\frac{\gamma^2}{\beta^2}\ . \tag{123}$$

Hereafter, abbreviate μ_n to μ, where $0 \le \mu < 1$.

Then, (118) becomes

$$\frac{1}{C} \;=\; \frac{1}{\beta^{p+q-2}}\int_0^1 \frac{x^{p+q-3}}{\sqrt{(1-\mu x^2)^q}}\,\mathrm{d}x\ . \tag{124}$$

Substitute $x\sqrt{\mu} = z$ and then $z = \sin\theta$, and this becomes

$$\begin{aligned} \frac{1}{C} &= \frac{1}{(\beta\sqrt{\mu})^{p+q-2}} \int_0^{\sqrt{\mu}} \frac{z^{p+q-3}}{(1-z^2)^{q/2}}\, \mathrm{d}z \\ &= \frac{1}{(\beta^2-\gamma^2)^{(p+q-3)/2}} \int_0^{\arcsin\sqrt{\mu}} \frac{\sin^{p+q-3}\theta}{\cos^{q-1}\theta}\, \mathrm{d}\theta\ . \end{aligned} \tag{125}$$

This can be integrated explicitly in terms of elementary functions — the full working is given in Appendix C.

9.2 Evaluation of integrals

A general elliptic integral can be reduced to expressions involving Legendre's standard elliptic integrals. Legendre's Incomplete Elliptic Integrals of the First and Second Kinds can be computed efficiently [Reinsch & Raab], and the Complete Integrals $K(m)$ and $E(m)$ can be computed very efficiently [Milne-Thomson §**17.6.3** & §**17.6.4**].

But for a general elliptic integral (e.g. (100)), reduction to the standard elliptic integrals is an extremely complicated operation [Milne-Thomson §**17.1**]. It is usually simplest to evaluate elliptic integrals directly by Romberg integration, possibly after performing some substitution (e.g. $x = \sin\theta$) to make the integrand smooth.

And similarly for complicated integrals which could be expressed in terms of elementary functions, as in the Appendices.

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Appendices

A Surface area of biaxial ellipsoids $B_{2m,2k+1}$

Here, $m \geq 1$ and $k \geq 0$.

Denote

$$J_{m,k} \;=\; \int_0^1 \frac{(1-h^2)^{m+k-1}}{(1-\delta h^2)^{m+1}} \, \mathrm{d}h \;, \tag{126}$$

where $\delta = 1 - (\gamma/\beta)^2$, and then the integral in (97) can be represented as

$$m(\gamma/\beta)^2 J_{m,k} + k J_{m-1,k+1} \;. \tag{127}$$

The integrand in (126) is

$$L \;=\; \left(\frac{1-h^2}{1-\delta h^2}\right)^{m+1} (1-h^2)^{k-2} \;. \tag{128}$$

This will be converted to partial fractions.

Denote $x = 1/(1-\delta h^2)$, so that

$$\frac{1-h^2}{1-\delta h^2} \;=\; \frac{1+(\delta-1)x}{\delta} \;=\; \frac{1+u}{\delta} \;, \tag{129}$$

where $u = (\delta-1)x$, and

$$1-h^2 \;=\; (1-1/\delta)(1+1/u) \;=\; \frac{(\delta-1)(1+u)}{\delta u} \;. \tag{130}$$

Thus, the integrand is

$$\begin{aligned}
L \;&=\; \left(\frac{1+u}{\delta}\right)^{m+1} \left(\frac{\delta-1}{\delta}\right)^{k-2} \left(\frac{1+u}{u}\right)^{k-2} \\
&=\; \frac{(\delta-1)^{k-2}(1+u)^{m+k-1}}{\delta^{m+k-1} u^{k-2}} \\
&=\; \frac{(\delta-1)^{k-2}}{\delta^{m+k-1}} \, u^{2-k} \sum_{q=0}^{m+k-1} \binom{m+k-1}{q} u^q \\
&=\; \frac{(\delta-1)^{k-2}}{\delta^{m+k-1}} \sum_{q=0}^{m+k-1} \binom{m+k-1}{q} u^{q-k+2} \\
&=\; \frac{1}{\delta^{m+k-1}} \sum_{q=0}^{m+k-1} \binom{m+k-1}{q} \frac{(\delta-1)^q}{(1-\delta h^2)^{q-k+2}} \\
&=\; \frac{1}{\delta^{m+k-1}} \sum_{q=0}^{m+k-1} \binom{m+k-1}{q} (\delta-1)^q (1-\delta h^2)^{k-q-2} \;.
\end{aligned} \tag{131}$$

This partial fraction expansion of L can then be integrated, to give

$$J_{m,k} = \frac{1}{\delta^{m+k-1}} \sum_{q=0}^{m+k-1} \binom{m+k-1}{q} (\delta-1)^q \int_0^1 (1-\delta h^2)^{k-q-2} \, \mathrm{d}h \;. \tag{132}$$

If $k > 1$ and $q < k-1$ then $k-q-2 \geq 0$, and so the integrand is a polynomial in h^2 of degree $k-q-2$; but for $q \geq k-1$ the integrand is the reciprocal of a polynomial in h^2 of degree $q+k-2$. Thus, we need to consider

$$P_g \stackrel{\text{def}}{=} \int_0^1 (1-\delta h^2)^g \; \mathrm{d}h \; , \tag{133}$$

for integer g.

Denote $X = 1-\delta h^2$, so that $\mathrm{d}X = -2\delta h \; \mathrm{d}h$.

Integrating by parts,

$$\begin{aligned} P_g &= \int_0^1 X^g \; \mathrm{d}h \;=\; hX^g\Big|_0^1 - \int h \; \mathrm{d}(X^g) \\ &= (1-\delta)^g - g\int hX^{g-1}\mathrm{d}X \;=\; (1-\delta)^g + 2g\int_0^1 \delta h^2 X^{g-1} \; \mathrm{d}h \\ &= (1-\delta)^g + 2g\int_0^1 \left[X^{g-1} - (1-\delta h^2)X^{g-1}\right] \; \mathrm{d}h \\ &= (1-\delta)^g + 2g\int_0^1 \left[X^{g-1} - X^g\right] \; \mathrm{d}h \\ &= (1-\delta)^g + 2gP_{g-1} - 2gP_g \; , \end{aligned} \tag{134}$$

and hence

$$(2g+1)P_g \;=\; 2gP_{g-1} + (1-\delta)^g \; . \tag{135}$$

Thus we get the reduction formulæ

$$P_g \;=\; \frac{2gP_{g-1} + (1-\delta)^g}{2g+1} \tag{136}$$

for all g, and

$$P_{g-1} \;=\; \frac{(2g+1)P_g - (1-\delta)^g}{2g} \tag{137}$$

for all nonzero g.

For integrating the polynomials with $g \geq 0$, start with $P_0 = 1$, and then apply the reduction formula (136) successively for $g = 1,\; 2,\; 3,\; \cdots$, to generate the integrals P_g for $g \geq 0$.

If $\beta > \gamma > 0$ then $0 < \delta < 1$, but if $\gamma > \beta > 0$ then $\delta < 0$. For integrating the rational functions with $g < 0$, start with

$$P_{-1} \;=\; \int_0^1 \frac{\mathrm{d}h}{1-\delta h^2} \;=\; \begin{cases} \dfrac{1}{2\sqrt{\delta}} \log\left(\dfrac{1+\sqrt{\delta}}{1-\sqrt{\delta}}\right) & (\beta > \gamma,\; \delta > 0), \\ \dfrac{\arctan(\sqrt{-\delta})}{\sqrt{-\delta}} & (\beta < \gamma,\; \delta < 0) \; . \end{cases} \tag{138}$$

Then apply the reduction formula (137) successively for $g = -1,\; -2,\; -3, \cdots$, to generate the integrals $P_{-2},\; P_{-3},\; P_{-4},\; \cdots$.

Thus, the integral $J_{m,k}$ has been constructed in terms of elementary functions.

Then the integral in (97) can be computed (127) from $J_{m,k}$ and $J_{m-1,k+1}$, to give explicitly the surface area for a biaxial ellipsoid of type $B_{2m,2k+1}$, in terms of elementary functions of its semi-axes β and γ.

B Surface area of biaxial ellipsoids $B_{2m,2k}$

Here, $m \geq 1$ and $k \geq 1$.

Denote

$$H_{m,k} \;=\; \int_0^1 \frac{(1-h^2)^{m+k-1}}{(1-\delta h^2)^{m+1}} \, \frac{\mathrm{d}h}{\sqrt{1-h^2}} \;, \tag{139}$$

where $\delta = 1 - (\gamma/\beta)^2$, and then the integral in (99) can be represented as

$$2m(\gamma/\beta)^2 H_{m,k} + (2k-1) H_{m-1,k+1} \;. \tag{140}$$

This integrand for $H_{m,k}$ is $L/\sqrt{1-h^2}$, where L is the integrand (128) for $J_{m,k}$. Corresponding to the sum (131) of integrals for $J_{m,k}$, we get the sum of integrals

$$H_{m,k} \;=\; \frac{1}{\delta^{m+k-1}} \sum_{q=0}^{m+k-1} \binom{m+k-1}{q} (\delta-1)^q \int_0^1 \frac{\mathrm{d}h}{(1-\delta h^2)^{q-k+2}\sqrt{1-h^2}} \;. \tag{141}$$

Corresponding to P_j in (133), we consider

$$U_j \;\stackrel{\mathrm{def}}{=}\; \int_0^1 \frac{\mathrm{d}h}{(1-\delta h^2)^j \sqrt{1-h^2}} \;, \tag{142}$$

and the substitution $h = \sin\psi$ converts this to

$$U_j \;=\; \int_0^{\pi/2} \frac{\mathrm{d}\psi}{(1-\delta\sin^2\psi)^j} \;. \tag{143}$$

In (141), if $k > 1$ and $q < k-1$ then $q - k + 2 \leq 0$, and so the integrand is a polynomial in $\sin^2\psi$. For $q = 0$ to $k-2$, the integral in (141) is

$$\begin{aligned} U_{q-k+2} \;&=\; \int_0^{\pi/2} (1-\delta\sin^2\psi)^{k-q-2} \;\mathrm{d}\psi \\ &=\; \int_0^{\pi/2} \sum_{r=0}^{k-q-2} \binom{k-q-2}{r} (-\delta)^r \sin^{2r}\psi \;\mathrm{d}\psi \\ &=\; \sum_{r=0}^{k-q-2} \binom{k-q-2}{r} (-\delta)^r \int_0^{\pi/2} \sin^{2r}\psi \;\mathrm{d}\psi \\ &=\; \sum_{r=0}^{k-q-2} \binom{k-q-2}{r} (-\delta)^r s_{2r} \;, \end{aligned} \tag{144}$$

(cf. (33)). In particular, $U_0 = \frac{1}{2}\pi$.

But for $q \geq k-1$ the integrand is the reciprocal of a polynomial in $\sin^2\psi$, and so we need to consider U_j in (142) for positive integer j.

For $j \geq 1$, from (143) we get

$$U_j \;=\; \int_0^{\pi/2} \frac{\mathrm{d}\psi}{(1-\delta+\delta\cos^2\psi)^j} \;=\; \int_0^{\pi/2} \frac{\sec^{2j}\psi}{((1-\delta)\sec^2\psi+\delta)^j} \;\mathrm{d}\psi \;. \tag{145}$$

Substitute $f = \tan\psi$ so that $\mathrm{d}f = \sec^2\psi\ \mathrm{d}\psi = (1+f^2)\ \mathrm{d}\psi$, and we get

$$
\begin{aligned}
U_j &= \int_0^\infty \frac{(1+f^2)^j}{(1+f^2)[(1-\delta)(1+f^2)+\delta]^j}\ \mathrm{d}f \\
&= \int_0^\infty \frac{(1+f^2)^{j-1}}{(1+(1-\delta)f^2)^j}\ \mathrm{d}f \ = \ \int_0^\infty \frac{(1+f^2)^{j-1}}{(1+\eta f^2)^j}\ \mathrm{d}f\ . \qquad (146)
\end{aligned}
$$

where

$$
\eta \ \stackrel{\text{def}}{=}\ 1-\delta \ = \ \left(\frac{\gamma}{\beta}\right)^2 \ > \ 0\ . \qquad (147)
$$

For $j=1$, substitute $x = f\sqrt{\eta}$, and then

$$
U_1 \ = \ \int_0^\infty \frac{\mathrm{d}f}{1+\eta f^2} \ = \ \frac{1}{\sqrt{\eta}}\int_0^\infty \frac{\mathrm{d}x}{1+x^2} \ = \ \frac{\pi\beta}{2\gamma}\ . \qquad (148)
$$

The integrand in (146) corresponds to L in (128) for $m = j-1$ and $k=1$, with $-h^2$ replaced by f^2 and δ replaced by η. From (131), the partial fraction expansion of the integrand in (144) is

$$
\frac{(1+f^2)^{j-1}}{(1+\eta f^2)^j} \ = \ \frac{1}{\eta^{j-1}}\sum_{q=0}^{j-1}\binom{j-1}{q}\frac{(\eta-1)^q}{(1+\eta f^2)^{q+1}}\ . \qquad (149)
$$

Therefore, for $j>1$,

$$
\begin{aligned}
U_j &= \int_0^\infty \frac{1}{\eta^{j-1}}\sum_{q=0}^{j-1}\binom{j-1}{q}\frac{(\eta-1)^q}{(1+\eta f^2)^{q+1}}\ \mathrm{d}f \\
&= \eta^{1-j}\sum_{q=0}^{j-1}\binom{j-1}{q}(\eta-1)^q\int_0^\infty \frac{\mathrm{d}f}{(1+\eta f^2)^{q+1}}\ . \qquad (150)
\end{aligned}
$$

Substitute $x = f\sqrt{\eta}$ and then $x = \tan\theta$, and we get

$$
\begin{aligned}
\int_0^\infty \frac{\mathrm{d}f}{(1+\eta f^2)^{q+1}} &= \frac{1}{\sqrt{\eta}}\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)^{q+1}} \ = \ \frac{\beta}{\gamma}\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)^{q+1}} \\
&= \frac{\beta}{\gamma}\int_0^{\pi/2}\cos^{2q}\theta\ \mathrm{d}\theta \ = \ \frac{\beta}{\gamma}\int_0^{\pi/2}\sin^{2q}\theta\ \mathrm{d}\theta \\
&= \frac{\beta s_{2q}}{\gamma} \ = \ \frac{\beta}{\gamma}\cdot\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2q-1)}{(2q)}\ , \qquad (151)
\end{aligned}
$$

in view of (33).

Thus, the integral $H_{m,k}$ has been constructed in terms of elementary functions.

Then the integral in (99) can be computed (140) from $H_{m,k}$ and $H_{m-1,k+1}$, to give explicitly the surface area for a biaxial ellipsoid of type $B_{2m,2k}$, in terms of elementary functions of its semi-axes β and γ.

C Capacity for biaxial ellipsoids

The indefinite integral

$$G_{m,k} \stackrel{\text{def}}{=} \int \frac{\sin^m x}{\cos^k x}\, \mathrm{d}x \ . \tag{152}$$

has the reduction formula [Dwight, §**452.9**] for $k \neq 1$:

$$G_{m,k} = \frac{\sin^{m-1} x}{(k-1)\cos^{k-1} x} - \left(\frac{m-1}{k-1}\right) G_{m-2,k-2} \ . \tag{153}$$

Applied to the definite integral

$$I_{m,k} \stackrel{\text{def}}{=} \int_0^{\arcsin\sqrt{\mu}} \frac{\sin^m x}{\cos^k x}\, \mathrm{d}x \ . \tag{154}$$

this becomes

$$I_{m,k} = \frac{\mu^{(m-1)/2}}{(k-1)(1-\mu)^{(k-1)/2}} - \left(\frac{m-1}{k-1}\right) I_{m-2,k-2} \ . \tag{155}$$

C.1 Odd $q = 2j+1$

Denote $m = p+q-3$, and let $q = 2j+1$ so that $m = p+2j-2$. The formula (125) requires the integral $I_{m,q-1} = I_{p+2j-2,2j}$. We shall compute $I_{p,2}$, and then apply the reduction formula (155) $j-1$ times, to compute successively $I_{p+2,4}$, $I_{p+4,6}$, $I_{p+6,8}$, $\ldots$, $I_{p+2j-2,2j}$.

As our starting value, we require

$$I_{p,2} = \int_0^w \frac{\sin^p x}{\cos^2 x}\, \mathrm{d}x \ . \tag{156}$$

For a prolate spheroid with the larger semi-axis β having multiplicity $p = 1$, this becomes

$$I_{p,2} = \int_0^w \frac{\sin x\, \mathrm{d}x}{\cos^2 x} = \left.\frac{1}{\cos x}\right|_0^w = \frac{1}{\cos w} - 1 = \frac{1}{\sqrt{1-\mu}} - 1 \ . \tag{157}$$

But for $p > 1$ we need to compute

$$I_{p-2,0} = \int_0^w \sin^{p-2} x\, \mathrm{d}x \ , \tag{158}$$

and then apply the reduction formula once to get $I_{p,2}$, and then proceed as above to compute $I_{p+2j-2,2j}$.

C.1.1 Integrating powers of sines

For non-negative integer h, define

$$\varsigma_h = \int_0^w \sin^h x\, \mathrm{d}x \ , \tag{159}$$

where $w = \arcsin\sqrt{\mu}$. In particular,

$$\begin{aligned}
\varsigma_0 &= w = \arcsin\sqrt{\mu}, \\
\varsigma_1 &= 1 - \cos w = 1 - \sqrt{1-\mu} = \frac{\mu}{1+\sqrt{1-\mu}} . \qquad (160)
\end{aligned}$$

For $h > 1$, integrating (159) by parts, we get the reduction formula

$$\begin{aligned}
\varsigma_h &= \left(\frac{h-1}{h}\right)\varsigma_{h-2} - \frac{\sin^{h-1} w \ \cos w}{h} \\
&= \left(\frac{h-1}{h}\right)\varsigma_{h-2} - \frac{\mu^{(h-1)/2}\sqrt{1-\mu}}{h} . \qquad (161)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varsigma_2 &= \tfrac{1}{2}\left(\varsigma_0 - \sqrt{\mu}\sqrt{1-\mu}\right) = \tfrac{1}{2}\left(\arcsin\sqrt{\mu} - \sqrt{\mu(1-\mu)}\right) , \\
\varsigma_3 &= \tfrac{1}{3}\left(2\varsigma_1 - \mu\sqrt{1-\mu}\right) = \tfrac{1}{1\times 3}\left(2 - (2+\mu)\sqrt{1-\mu}\right) , \\
\varsigma_4 &= \tfrac{1}{4}\left(3\varsigma_2 - \mu\sqrt{\mu}\sqrt{1-\mu}\right) = \tfrac{1}{2\times 4}\left(3\arcsin\sqrt{\mu} - (3+2\mu)\sqrt{\mu(1-\mu)}\right) , \\
\varsigma_5 &= \tfrac{1}{5}\left(4\varsigma_3 - \mu^2\sqrt{1-\mu}\right) = \tfrac{1}{3\times 5}\left(8 - (8+4\mu+3\mu^2)\sqrt{1-\mu})\right) , \\
\varsigma_6 &= \tfrac{1}{6}\left(5\varsigma_4 - \mu^2\sqrt{\mu}\sqrt{1-\mu}\right) \\
&= \tfrac{1}{4\times 6}\left(15\arcsin\sqrt{\mu} - (15+10\mu+4\mu^2)\sqrt{\mu(1-\mu)}\right), \ \textit{et cetera}. \qquad (162)
\end{aligned}$$

Thus, the definite integral ς_h in (158) can be expressed in term of elementary functions, for all integers $h \geq 0$.

C.2 Even $q = 2j$

Denote $m = p + q - 3$, and let $q = 2j$ so that $m = p + 2j - 3$. The formula (125) requires the integral $I_{m,q-1} = I_{p+2j-3,2j-1}$. We shall compute $I_{p-1,1}$, and then apply the reduction formula (155) $j - 1$ times, to compute successively $I_{p+1,3}$, $I_{p+3,5}$, $I_{p+5,7}$, $\ldots$, $I_{p+2j-3,2j-1}$.

As our starting value, we require

$$I_{p-1,1} = \int_0^w \frac{\sin^{p-1} x}{\cos x}\ \mathrm{d}x . \qquad (163)$$

C.2.1 Odd $p = 2g + 1$

For odd $p = 2g + 1$, this becomes

$$\begin{aligned}
I_{2g,1} &= \int_0^w \sin^{2g} x \ \frac{\mathrm{d}x}{\cos x} = \int_0^w (1-\cos^2 x)^g \ \frac{\mathrm{d}x}{\cos x} \\
&= \int_0^w \sum_{r=0}^{g} \binom{g}{r} (-\cos^2 x)^r \ \frac{\mathrm{d}x}{\cos x} = \sum_{r=0}^{g} \binom{g}{r} (-1)^r \int_0^w \cos^{2r} x \ \frac{\mathrm{d}x}{\cos x} \\
&= \int_0^w \frac{\mathrm{d}x}{\cos x} + \sum_{r=1}^{g} \binom{g}{r} (-1)^r \int_0^w \cos^{2r-1} x \ \mathrm{d}x . \qquad (164)
\end{aligned}$$

Now,

$$\int_0^w \frac{\mathrm{d}x}{\cos x} \;=\; \int_0^w \sec x \;\mathrm{d}x \;=\; \log\left(\frac{1+\tan(w/2)}{1-\tan(w/2)}\right) \;, \tag{165}$$

and

$$\begin{aligned}\left(\frac{1+\tan(w/2)}{1-\tan(w/2)}\right)^2 \;&=\; \left(\frac{\cos(w/2)+\sin(w/2)}{\cos(w/2)-\sin(w/2)}\right)^2 \\ &=\; \frac{1+\sin w}{1-\sin w} \;=\; \frac{1+\sqrt{\mu}}{1-\sqrt{\mu}} \;.\end{aligned} \tag{166}$$

Therefore,

$$\int_0^w \frac{\mathrm{d}x}{\cos x} \;=\; \frac{1}{2}\log\left(\frac{1+\sqrt{\mu}}{1-\sqrt{\mu}}\right) \;. \tag{167}$$

Next, substitute $x = \frac{1}{2}\pi - y$, and we get

$$\begin{aligned}\int_0^w \cos^k x \;\mathrm{d}x \;&=\; \int_{(\pi/2)-w}^{\pi/2} \sin^k y \;\mathrm{d}y \\ &=\; \int_0^{\pi/2} \sin^k x \;\mathrm{d}x \;-\; \int_0^{(\pi/2)-w} \sin^k x \;\mathrm{d}x \;=\; s_k - \lambda_k \;.\end{aligned} \tag{168}$$

Here, s_k is the important special case (34) of ς_k with $\mu = 1$.

For non-negative integer h, define

$$\lambda_h \;=\; \int_0^v \sin^h x \;\mathrm{d}x \;, \tag{169}$$

where

$$v \;=\; \tfrac{1}{2}\pi - w \;=\; \arccos\sqrt{\mu} \;=\; \arcsin\sqrt{1-\mu} \;, \tag{170}$$

so that

$$\begin{aligned}\lambda_0 \;&=\; v \;=\; \arcsin\sqrt{1-\mu}, \\ \lambda_1 \;&=\; 1-\cos v \;=\; 1-\sqrt{\mu} \;.\end{aligned} \tag{171}$$

The reduction formula (161) for ς_h is to be converted by interchanging sin w ($=\sqrt{\mu}$) and cos w ($=\sqrt{1-\mu}$), giving the reduction formula for λ_h with $h > 1$

$$\begin{aligned}\lambda_h \;&=\; \left(\frac{h-1}{h}\right)\lambda_{h-2} - \frac{\cos^{h-1} w \;\sin w}{h} \\ &=\; \left(\frac{h-1}{h}\right)\lambda_{h-2} - \frac{(1-\mu)^{(h-1)/2}\sqrt{\mu}}{h} \;.\end{aligned} \tag{172}$$

Staring from λ_0 and λ_1 as given in (170), this reduction formula can generate successively $\lambda_2,\ \lambda_3,\ \lambda_4,\ \cdots$.

Thus, we compute the required integral

$$I_{2g,1} \;=\; \frac{1}{2}\log\left(\frac{1+\sqrt{\mu}}{1-\sqrt{\mu}}\right) + \sum_{r=1}^{g}\binom{g}{r}(-1)^r(s_{2r-1}-\lambda_{2r-1}) \;. \tag{173}$$

And then we apply the reduction formula (155) $j-1$ times, to compute successively $I_{p+1,3},\ I_{p+3,5},\ I_{p+5,7},\ \ldots\ ,\ I_{p+2j-3,2j-1}$.

C.2.2 Even $p = 2g$

As our starting value, we require

$$I_{p-1,1} \;=\; I_{2g-1,1} \;=\; \int_0^w \frac{\sin^{2g-1} x}{\cos x}\, \mathrm{d}x \;. \tag{174}$$

Then substituting $c = \cos\, x$ we get

$$\begin{aligned}
I_{p-1,1} &= \int_0^w \frac{\sin^{2(g-1)} x\ (\sin\, x\ \mathrm{d}x)}{\cos\, x} \\
&= \int_0^w \frac{(1-\cos^2 x)^{g-1}\ (\sin\, x\ \mathrm{d}x)}{\cos\, x} \\
&= \int_{\cos\, w}^1 (1-c^2)^{g-1}\ \frac{\mathrm{d}c}{c} \;=\; \int_{\sqrt{1-\mu}}^1 \sum_{r=0}^{g-1} \binom{g-1}{r} (-c^2)^r\ \frac{\mathrm{d}c}{c} \\
&= \sum_{r=0}^{g-1} \binom{g-1}{r} (-1)^r \int_{\sqrt{1-\mu}}^1 c^{2r-1}\ \mathrm{d}c \\
&= \int_{\sqrt{1-\mu}}^1 \frac{\mathrm{d}c}{c} + \sum_{r=1}^{g-1} \binom{g-1}{r} (-1)^r \int_{\sqrt{1-\mu}}^1 c^{2r-1}\ \mathrm{d}c \\
&= -\log\sqrt{1-\mu} + \sum_{r=1}^{g-1} \binom{g-1}{r} (-1)^r \frac{1-\left(\sqrt{1-\mu}\right)^{2r}}{2r} \\
&= \frac{1}{2}\left[\log\left(\frac{1}{1-\mu}\right) + \sum_{r=1}^{g-1} \binom{g-1}{r} (-1)^r \frac{1-(1-\mu)^r}{r}\right] \;.
\end{aligned} \tag{175}$$

Having computed $I_{p-1,1}$, we then apply the reduction formula (155) $j-1$ times, to compute successively $I_{p+1,3},\; I_{p+3,5},\; I_{p+5,7},\; \ldots,\; I_{p+2j-3,2j-1}$.