Direct and Inverse Boundary Uniqueness Theorems for Analytic Functions Smooth in the Closed Unit Disc

Jeong Eun Lee
email: jlee111@math.auckland.ac.nz
Department of Mathematics
University of Auckland
New Zealand

April 1, 2004

1 Introduction

We call an analytic function $f(z)$ infinitely smooth on the open unit disc Ω if it is continuous on $\overline{\Omega}$ and all derivatives exist and are bounded $|f^{(l)}(z)| \le C_l$, $l = 1, 2, \cdots, \infty$. The function $f(z)$ is called finitely smooth $f(z) \in C^N(\Omega)$ if $|f^{(l)}(z)| \le C_l$, $l = 1, 2, \cdots, N$ for $z \in \Omega$.
L. Beurling and L. Carleson have shown that the following boundary uniqueness theorem is true for finitely smooth analytic functions, see [1] and references therein:

Theorem 1.1 *The set of boundary zeroes A on the unit circle $\partial\Omega$, of maximal multiplicity N, of a finitely smooth analytic function $f \in C^N(\Omega)$ is fully characterized by the following conditions*

$$\sum_i |\Delta_i| \ln \frac{1}{|\Delta_i|} < \infty \,, \qquad\qquad \sum_i |\Delta_i| = 2\pi \,. \tag{1}$$

on the complementary intervals $\{\Delta_i\}$ on $\partial\Omega$ of the zeroes A.

If the condition (1) is not fullfilled then A is a *set of uniqueness* of all of the classes $C^N(\Omega)$. This means that any two functions f_1, $f_2 \in C^N(\Omega)$ which, along with all the derivatives up to order N, coincide on A are identically equal in the unit disc Ω. On the other hand, if condition (1) is fullfilled this implies zeroes of infinite order may exist on the boundary and consequently two *distinct* functions may coincide on A.

Generally, the characterisation of the set of uniqueness of infinitely smooth analytic functions in the unit disc depends on the rate of growth of the coefficients C_p where $f^{(p)} \leq C_p$ for all p. If $C_p \leq (1+\delta)^p p!$ then the corresponding function admits an analytic continuation in the larger disc $|z| < 1+\delta$. Hence a function becomes analytic not only inside Ω but also outside of Ω. Such functions cannot have zeroes of infinite multipicity by the *classical inner uniqueness theorem* for analytic functions. However, if the C_p grow faster than $(1+\delta)^p p!$ then there may be zeroes of infinite multiplicity of the function f on the boundary of the unit circle. Even a single point on the boundary of the unit disc can be the set of uniquness of an infinitely smooth analytic function, i.e. if two functions, along with all derivatives, coincide at a point then they coincide on the whole unit disc.

The class of functions such that a set of uniqueness is a single point on $\partial\Omega$ is called a boundary quasi-analytic class.

In this paper we consider the Gevrey class which is characterized by $C_p = p^{\alpha p}$ where $|f^{(p)}| \leq C_p$ and $\alpha > 1$. The exact description of the set of uniqueness in terms of the lengths of the complimentary intervals $\{\Delta_i\}$ is impossible and another characterisation is necessary which is given in terms of the positions of intervals with small lengths [2].

Our aim is the derivation of simple sufficient conditions on the set $A \subset \partial\Omega$ which guarentee uniqueness of continuation of the function from A in the Gevrey class. Furthermore, we define a new extended class, based on the Gevrey class, and give a definition of boundary quasi-analyticity for this class.

The starting point of this work is based on the analysis developed in [3] and Jensen's formula.

2 The Direct Uniqueness Theorem

In this report we consider only functions $f(z)$ which are analytic on the open unit disc $\Omega = \{z : |z| < 1\}$ and infinitely differentiable ($f \in C^\infty$) on the colosure of Ω. We say that a function $f(z)$ having these properties belongs to the Gevrey class, G_α ($\alpha \geq 1$), if the derivatives of $f(z)$ satisfy $\sup_{z\in\Omega} |f^{(p)}(z)| \leq Cd^p p^{\alpha p}$. Below we assume, without loss of generality, that $C = d = 1$.

Assuming that $f(z)$ has a zero of infinite order on $\partial\Omega$ at a we observe the following inequality based on the Taylor expansion for each N

$$\left| f(z) - \sum_{l=0}^{N} \frac{(z-a)^l}{l!} f^{(l)}(a) \right| \leq \frac{|z-a|^{N+1}}{(N+1)!} \sup_{z\in\Omega} \left| f^{(N+1)}(z) \right| .$$

Since a is a zero of infinite order all derivatives of f at a equal zero and the function $f(z)$ can estimated as

$$|f(z)| \leq \frac{|z-a|^{N+1}}{(N+1)!} \sup \left| f^{(N+1)} \right| \leq \frac{|z-a|^{N+1}}{(N+1)!} (N+1)^{\alpha(N+1)} \tag{2}$$

for all N. We approximate the right side of (2), writing $p = N+1$ and using the asymptotic formulae for the factorial $p! \approx e^{-p} p^p$ (Stirlings formula)

$$|f(z)| \leq \frac{|z-a|^p}{p!} p^{\alpha p} \approx \frac{p^{\alpha p}}{e^{-p} p^p} |z-a|^p . \tag{3}$$

Denoting the right hand side by

$$T(p,z) = \frac{p^{\alpha p}}{e^{-p} p^p} |z-a|^p = e^{(\alpha-1)p\ln p + p + p\ln|z-a|} \tag{4}$$

we find the derivative

$$\frac{dT(p,z)}{dp} = ((\alpha-1)(\ln p + 1) + 1 + \ln|z-a|) e^{(\alpha-1)p\ln p + p + p\ln|z-a|} .$$

This is zero at the critical point

$$p_c = e^{\frac{\alpha}{1-\alpha}} |z-a|^{\frac{1}{1-\alpha}} \tag{5}$$

so that the mininum of $T(p,z)$ is

$$T_{\min}(z) = e^{(1-\alpha)|z-a|^{\frac{1}{1-\alpha}}} e^{\frac{\alpha}{1-\alpha}} = e^{-c|z-a|^{\frac{1}{1-\alpha}}}$$

where

$$c = (-1+\alpha)e^{\frac{\alpha}{1-\alpha}}$$

is a real constant depending on α.

The critical value p_c of the parameter p may not be an integer, however $|p_c - \hat{p}_c| < 1$ for the nearest integer $\hat{p}_c$. Since the derivative of $T(p,z)$ with respect to p is zero at p_c, the ratio of the value of $T(p_c,z)$ and $T(\hat{p}_c,z)$ can be estimated

$$\frac{T(\hat{p}_c,z)}{T(p_c,z)} = 1 + \mathrm{o}(1)$$

and hence $T(\hat{p}_c,z)$ is relatively bounded by $T(p_c,z) = T_{\min}(z)$.

For any positive real α we can easily see that $T_{\min}(z) \leq 1$ on Ω, hence the following estimates are true near the zero, $z=a$, of f on the boundary

$$|f(z)| \leq e^{-c|z-a|^{-\frac{1}{\alpha-1}}}. \tag{6}$$

We define $\ln_- |f(z)| = \ln \frac{1}{|f(z)|}$ which is always positive since $|f(z)| < 1$

$$\ln_- |f(z)| = \ln \frac{1}{|f(z)|} \geq \frac{c}{|z-a|^{\frac{1}{\alpha-1}}}. \tag{7}$$

Let A be the zero set of infinite order of $f(z)$ on $\partial\Omega$ and denote by $\{\Delta_i = (a_i, a_{i+1})\}$ the set of complementary intervals of A. Then due to (6) the following estimate is satisfied

$$|f(z)| \leq e^{-c \cdot \mathrm{dist}(z,A)^{-\frac{1}{\alpha-1}}}$$

where $\mathrm{dist}(z,A)$ is the distance between z and the set A. The values of the function $f(z)$ exist pointwise on the unit circle since $f(z)$ is infinitely differentiable on the closed disc $\overline{\Omega}$. For $|z| = 1$ on the unit circle the direct distance from z to A can be replaced by the distance along the arc of the unit circle.

Let us consider the interval $\Delta_i = (a_i, a_{i+1})$ and, using (7), estimate $|f(z)|$ on Δ_i in the following way

$$\begin{aligned} \ln_- |f(z)| &\geq \frac{c}{|z-a_i|^{\frac{1}{\alpha-1}}} \quad \text{for} \quad |z-a_i| < \frac{|\Delta_i|}{2} \\ \ln_- |f(z)| &\geq \frac{c}{|z-a_{i+1}|^{\frac{1}{\alpha-1}}} \quad \text{for} \quad |z-a_{i+1}| < \frac{|\Delta_i|}{2}\,. \end{aligned}$$

The integral of $\ln_- |f(z)|$ over the unit circle $\partial\Omega$ is equal to the sum of integrals over the complementary intervals so we have

$$\begin{aligned} \int_{\partial\Omega} \ln_- |f(z)|\, ds &= \sum_i \int_{\Delta_i} \ln_- |f(z)|\, ds \\ &\geq \sum_i \left[\int_{a_i}^{\frac{a_i+a_{i+1}}{2}} \frac{c}{|z-a_i|^{\frac{1}{\alpha-1}}}\, ds + \int_{\frac{a_i+a_{i+1}}{2}}^{a_{i+1}} \frac{c}{|z-a_{i+1}|^{\frac{1}{\alpha-1}}}\, ds \right]. \end{aligned}$$

Then

$$\int_{\partial\Omega} \ln_- |f(z)|\, ds \geq 2 \sum_i \int_{a_i}^{\frac{a_i+a_{i+1}}{2}} \frac{c}{|z-a_i|^{\frac{1}{\alpha-1}}}\, ds\,. \tag{8}$$

Let us consider two cases separately.

$\underline{1 < \alpha \leq 2}$ In this case the estimate (8), along with Jensen's inequality

$$\int \ln_- |f(\varphi)|\, d\varphi < \infty\,,$$

gives the following

$$\infty > \int_{\partial\Omega} \ln_- |f(z)|\, ds \geq 2 \sum_i \int_{a_i}^{\frac{a_i+a_{i+1}}{2}} \frac{c}{|z-a_i|^{\frac{1}{\alpha-1}}} ds = \infty$$

contradiction. This implies the absence of zeroes of infinite multiplicity of functions in the Gevrey class with $1 < \alpha \leq 2$. Consequently, every single point is a set of uniqueness for G_α with $1 < \alpha \leq 2$.
The classes of functions with this property are called *quasi-analytic,* i.e. G_α with $1 < \alpha \leq 2$ are quasi-analytic classes on the unit disc.

$\underline{\alpha > 2}$ In this case the estimate (8) gives the following

$$\infty > \int_{\partial\Omega} \ln_- |f(z)|\, ds \geq c\frac{\alpha - 1}{\alpha - 2} \sum_i \left(\frac{|\Delta_i|}{2}\right)^{\frac{2\alpha-3}{\alpha-1}} .$$

If the set A satisfies this inequality then, keeping in mind that the exponent $\frac{2\alpha-3}{\alpha-1}$ is greater than 1, the integral in condition (1) is finite and hence the set A is not a set of uniqueness. Consequently, there may exist zeroes of infinite multiplicity in the Gevrey class if $\alpha > 2$.
This is only a partial characterisation of the set of uniqueness. The precise description was found by S. Khruschev and is based on conditions on the locations of complementary intervals of different lengths [2].

3 The Extended Class $G_{2,F}$

We've observed the quasi-analyticity of the Gevrey class G_2. Now we construct an extended class $G_{2,F}$ which is also quasi-analytic. We have established above that for $a \in A$, a zero of infinite order on $\partial\Omega$,

$$|f(z)| \leq \frac{|z-a|^p}{p!} \sup_{z\in\partial\Omega} |f^{(p)}|$$

for each p. Recalling (3,4) we say that $f \in G_{2,F}$ if for any $a \in A$

$$f \leq e^{p(\ln p + \ln|z-a| + F(p))} .$$

The function $F(p)$ will be choosen such that

$$\lim_{p\to\infty} F(p) = \infty\,, \quad \lim_{p\to\infty} pF^{'}(p) = 0\,, \quad \lim_{p\to\infty} p^2 F^{''}(p) = 0\,. \tag{9}$$

In other words, an infinitely smooth function f belongs to the class $G_{2,F}$ if

$$\left|f^{(p)}\right| \leq p^{2p} e^{p(F(p)-1)}$$

for all positive intergers p where F satisfies (9).
Let us denote the distance from z to a along the arc of the unit circle by $s = |z-a|$ for $|z| = 1$. We also define the function

$$T(s,p) = e^{p(\ln p + \ln s + F(p))} .$$

The upper bound of $|f(z)|$ can be estimated by minimising $T(s,p)$ with respect to p. Differentiating $T(p)$

$$\frac{dT(s,p)}{dp} = \ln p + \ln s + F(p) + 1 + pF'(p)$$

we get

$$\frac{1}{s} = pe^{F(p)+1+pF'(p)} \tag{10}$$

at the critical point so that $|f|$ is estimated as

$$|f(z)| \leq e^{-p(1+pF'(p))}.$$

The function f is infinitely differentiable so it may be evaluated pointwise on $\partial\Omega$. Let us consider the integral of $\ln_- |f|$ over one complementary interval, $\Delta_i = (a_i, a_{i+1})$, of the set A. This integral can be estimated as

$$\int_{\Delta_i} \ln_- |f|\, ds \geq \int_{\Delta_i} p(1 + pF'(p))\, ds\,.$$

Using the symmetry about the midpoint of the complementary interval, as done in the previous section, we see that we need only consider the integral

$$\int_0^{\frac{|\Delta_i|}{2}} \ln_- |f|\, ds \geq \int_0^{\frac{|\Delta_i|}{2}} p(1 + pF'(p))\, ds\,. \tag{11}$$

For simplicity we change variables from s to p in the integral on the right hand side, where s and p are related by equation (10). This gives us

$$ds = \frac{1}{p^2}(-1 + 2pF' + p^2F'')e^{-(F+1+pF')}\, dp\,.$$

The endpoint $s = \frac{|\Delta_i|}{2}$ corresponds to $\hat{p}_i$ where $\frac{2}{|\Delta_i|} = \hat{p}_i e^{F(\hat{p}_i)+1+\hat{p}_iF'(\hat{p}_i)}$. It is clear from (10) that at $s = 0$, $p \to \infty$. Then (11) can be rewritten

$$\int_0^{\frac{|\Delta_i|}{2}} \ln_- |f|\, ds \geq \int_{\hat{p}_i}^{\infty} \frac{1}{p}(1 + pF')(1 - 2pF' - p^2F'')e^{-(F+1+pF')}\, dp\,.$$

Recalling condition (9) we see that for large p the integral simplifies to

$$\int_0^{\frac{|\Delta_i|}{2}} \ln_- |f|\, ds \geq \int_{\hat{p}_i}^{\infty} \frac{e^{-F}}{p}\, dp\,. \tag{12}$$

The integral of $\ln_- |f|$ over $\partial\Omega$ is the sum of the integral over each complementary interval of A.

$$\int_{\partial\Omega} \ln_- |f|\, ds \geq 2\sum_i \int_0^{\frac{|\Delta_i|}{2}} p(1 + pF'(p))\, ds \geq 2\sum_i \int_{\hat{p}_i}^{\infty} \frac{e^{-F}}{p}\, dp\,.$$

The function $F(p)$ can be any function satisfying (9). Supposing

$$F(p) = \ln(\ln p)^{\beta}$$

with $0 < \beta < 1$. We obtain a quasi-analytic class because (12) becomes

$$\int_0^{\frac{|\Delta_i|}{2}} \ln_- |f|\, ds \geq \int_{\hat{p}_i}^{\infty} \frac{dp}{p(\ln p)^{\beta}} = \infty\,.$$

Since the integral over one complementary interval is divergent the integral over $\partial\Omega$ is divergent. Consequently, by Jensen's inequality, we again get a contradiction so that $G_{2,\beta}$ is a class of boundary quasi-analytic functions for $0 < \beta \leq 1$.

For $\beta > 1$ the integral

$$\int_{\hat{p}_i}^{\infty} \frac{dp}{p(\ln p)^{\beta}} < \infty$$

is convergent so we cannot derive quasi-analyticity of this class as previously. We suppose that this class is not quasi-analytic but verification of this statement needs construction of an example.

The set A of zeroes of infinite order for functions in the class $G_{2,F}$ is characterized by the condition

$$\sum_i \int_{p_i}^{\infty} \frac{e^{-F}}{p}\, dp < \infty\,,$$

Jensen's inequality, and from (10)

$$\frac{2}{|\Delta_i|} = p_i\, e^{F(p_i)+1+p_i F'(p_i)}\,.$$

4 Application of the Uniqueness Theorem to the Spectral Theory of the Non-Selfadjoint Schrödinger Operator : some prospects

In $L_2(0,\infty)$ we consider the differential equation of Schrödinger type along with a boundary condition

$$-y'' + q(x)\, y - \lambda\, y = f(x), \quad y'|_0 = hy|_0 \;. \tag{13}$$

This is a typical non-homogeneous equation which, assuming some weak conditions on the real function q (such as $q \to 0$, as $x \to \infty$) has a square-integrable solution $y(x)$ with square integrable first and second derivatives for all values of λ off the real axis and for any $f(x) \in L_2(0,\infty)$. The operator defined by the left hand side of the equation has continuous spectrum along the semiaxis $[0,\infty)$ and may also have a countable number of discrete eigenvalues on the negative semiaxis.
If the function $q(x)$ is real and converges rapidly enough to 0 as $x \to \infty$, precisely if

$$\int_0^\infty x|q(x)|\, dx < \infty \tag{14}$$

then the number of negative eigenvalues of the self-adjoint spectral problem is finite.
However, the spectrum becomes more sophisticated if the potential q is complex. Differential equations of Schrödinger type with complex potential and complex boundary conditions are often used in the nuclear physics but the corresponding spectral problem (13) in this case is non-selfadjoint. In such a case we may have an infinite number of eigenvalues in the upper half-plane. Moreover, since the operator is non-selfadjoint, the eigenvectors are not orthogonal to each other, hence the Bessel inequality cannot be applied to prove the convergence of eigenvector expansions. For this reason non-selfadjoint operators with an infinite number of eigenvalues are very incovenient to use. The non-selfadjoint opertors with finite numbers of eigenvalues are more convenient since the convergence of the eigenvector expansion is guarenteed. It appears that the answer to the question of the number of discrete eigenvalues for complex q can be given in terms of the number of zeroes of some analytic function.

Let us consider the spectral problem (13) for $f(x) = 0$ with a complex rapidly decreasing potential (14) and boundary condition specified by a complex constant h. We consider the Schrödinger equation

$$-y'' + q(x)\,y = \lambda\,y = k^2\,y$$

or

$$-y''(x) - k^2\,y(x) = -q(x)\,y(x) \tag{15}$$

which has an exponentially decreasing solution (the Jost solution) given by the integral equation

$$y(k,x) = e^{ikx} - \int_x^\infty \frac{\sin k(s-x)}{k}\,q(s)\,y(s)\,ds.$$

This equation can be solved via iteration and the solution $y(k,x)$ is square integrable for $\Im k > 0$. Then $y(k,x)$ will be an eigenfunction if it satisfies the boundary condition

$$D(k) = y'(k,0) - hy(k,0) = 0\,.$$

The Jost function $D(k)$ is an analytic function for $\Im k > 0$ and is as smooth as the Fourier transform of the potential q

$$D(k) \approx \int_0^\infty e^{-ikx} q(x) dx = \mathcal{F}\{q\}(k)\,.$$

In particular, if the potential is decreasing as $\frac{1}{(1+|x|)^l}$ at infinity, then the function D has continuous derivatives $D^{(s)}$ up to order $s \leq l-1$. The set A of boundary zeroes of the function D on the real axis of the variable k is characterized by the Beurling-Carleson condition for the upper half-plane: the set of all finite complementary intervals $\{\Delta_i\}$ of A fulfills the condition

$$\sum_i |\Delta_i| \ln \frac{1}{|\Delta_i|} < \infty\,.$$

This means that the accumulation of eigenvalues k_s from the upper half-plane to the real axis may occur only on the set A of boundary (real) zeroes of the function D characterized by the above condition. In the case where the potential is decreasing more rapidly, for instance $|q(x)| \leq e^{-x^\beta}$, $\beta < 1$, then the function D is infinitely differentiable and has smoothness of the Fourier transform

$$\int_0^\infty e^{inx} e^{-x^\beta} \, dx \, .$$

Then the accumulation of the eigenvalues occurs on a set, similar to one described by the condition $\sum |\Delta_i|^{\frac{2\alpha-3}{\alpha-1}} < \infty$ on the finite set of complementary intervals. In particular the phenomenon of boundary quasi-analyticity appears if $\beta \geq \frac{1}{2}$. In this case the set of boundary zeroes of infinite multiplicity is empty and the total number of eigenvalues is finite. The case of quasi-analyticity beyond G_2 in $G_{2,\beta}$ should be investigated.

References

[1] L. Carleson, "Sets of uniqueness for functions analytic in the unit disc", *Acta Mathematica,* **87**, 325–345 (1952).

[2] S. V. Hruščev, "Sets of uniqueness for the Gevrey classes", *Ark. Mat.,* **15** no. 2, 253–304 (1977).

[3] B. S. Pavlov, "Dilation theory and spectral analysis of non-seladjoint differential operators", in *Twelve papers in analysis. A translation of Mathematical programming and related questions,* American Mathematical Society Translations, Ser. 2, **115**, 103–142 (1980).