Analytic perturbation techniques for the Friedrichs model revisited: Intermediate operator.

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Abstract

A techniques of an intermediate operator is developed for the Friedrichs model obtained as a finite-dimensional perturbation

$$\mathcal{P} \longrightarrow \mathcal{P}_{\varepsilon} = \mathcal{P} + \varepsilon A$$

of the momentum operator $\mathcal{P} = \frac{1}{i}\frac{d}{dx}$, defined by the corresponding operator-extension procedure. This technique permits to observe a creation of the resonance at the given point k_0 via presenting the Scattering matrix for the above pair as a product of the non-analytic at $(\varepsilon,\, k) = (0,\, k_0)$ factor $S_0^{\varepsilon}(k)$ which is the Scattering matrix to the pair $\left\{\mathcal{P}_0^{\varepsilon},\, \mathcal{P}\right\}$ of the momentum with a local intermediate operator $\mathcal{P}_0^{\varepsilon}$, and an analytic factor $S\left(\mathcal{P}+\varepsilon A,\, \mathcal{P}_0^{\varepsilon}\right)$ of both variables $(\varepsilon,\, k)$ near the point $(0,\, k_0)$ which is the Scattering matrix of the pair $\left(\mathcal{P}_{\varepsilon}, \mathcal{P}_0^{\varepsilon}\right)$. The corresponding representation is valid also for eigenfunctions of the perturbed operator.

1 Introduction

The standard technique of analytic perturbation theory is developed for additive perturbations $A_{\varepsilon} = A + \varepsilon B$ of operators with discrete spectrum, see for instance [15]. Mathematicians knew already long ago, see [4], that for perturbation of continuous spectrum the analytic perturbation procedure is convergent only for small values of the perturbation parameter ε - "below the threshold of creation of resonances ", but can't be extended beyond this limit. Probably I. Prigogine was the first physicist who really worried about this fact, and even tried to attract attention of specialists to necessity of extending the range of application of the analytic perturbation technique. Trying to reconsider the analytical perturbation techniques for continuous spectrum he suggested to replace continuous branches of spectrum by resonances which

form a discrete set in the complex plane of the spectral parameter and probably may be treated with the analytic perturbation techniques, similarly to eigenvalues.Prigogine attributed the idea of the important role of resonances in perturbation theory to H. Poincare, [30], who connected them with small denominators in celestial mechanics. Unfortunately the price of replanting of the whole perturbation problem into the resonance ground is too big, since the set of resonance states is, normally, complete only in a tiny subspace of the original space. I. Prigogine assumed that this price may be reduced, if we leave the cosy Hilbert environment and pass to the larger class of Banach or topological spaces, say via rigging the original Hilbert space, [36, 37]. But the elegant construction developed from this idea did not become practical, since the choice of the Gelfand triples of the rigged spaces is not uniquely defined by the physical content of the problem, but is rather in hands of the researcher. Even for classical model dynamical systems the choice of rigging may affect the final results essentially, just violating the basic physical requirement of uniqueness.

According to evidence from his collaborators, I.Prigogine attempted also to use the idea of an *intermediate operator* as a base for development of the analytic perturbation techniques for operators with continuous spectrum. He assumed that there may exist an operator C_{ε} in the commutant see [32, 33, 7, 36, 2] of the non-perturbed operator A_0 such that the analytic perturbation procedure is convergent for the pair $(C_{\varepsilon}, A_{\varepsilon})$.

In fact this dream by I. Prigogine was almost realized by I. Glazman when he invented in fifties his "splitting method", see [9], which totally changed the whole picture of the qualitative spectral analysis. But this discovery was never considered in context of the analytic perturbation techniques.

In Scattering problems on Quantum Neworks, in particular one on the Quantum Switch, see [19], presented as a circular quantum well Ω_0 with four straight wires Ω_s width δ, attached to it orthogonally, the idea of the intermediate operator looks quite natural. The absolutely-continuous spectrum of the Schreödinger equation on this network has a band structure, and the configuration of scattered waves is obviously different on neighboring spectral bands. This observation suggests the idea that the intermediate operator should be defined *locally* (on a certain interval of energy), but not globally, as probably Prigogine expected. In [19] the "restricted" intermediate operator for the values of energy on the certain spectral band is actually defined as a Schrödinger operator supplied with the semi-transparency conditions on the bottom sections γ_s of the wires: the partial orthogonality to all channel

vectors e_s of the open channels,for given Fermi level $E_{_F}$, in the open channels:

$$\langle u_0,\, e_{_s}^{^l}\rangle = 0,\, \langle u_s,\, e_{_s}^{^l}\rangle = 0,\ \text{if}\ \frac{\pi^2\, l^2}{\delta^2} < \frac{2mE_{_F}}{\hbar^2}, \tag{1}$$

and the continuity in the closed channels:

$$\langle u_0 - u_s,\, e_{_s}^{^l}\rangle\Big|_{\gamma_s} = 0,\ \ \langle \frac{\partial u_0}{\partial n} - \frac{\partial u_s}{\partial n},\, e_{_s}^{^l}\rangle\Big|_{\gamma_s} = 0,\ \ \text{if}\ \frac{\pi^2\, l^2}{\delta^2} > \frac{2mE_{_F}}{\hbar^2}, \tag{2}$$

which corresponds to "chopping off" the open channels. The threshold of the continuous spectrum of the Schrödinger equation on the network with the open channel "chopped-off" is higher than the threshold of the continuous spectrum of the original equation. Solutions of the homogeneous Schrödinger equation with chopped-off lower channels decrease exponentially at infinity. It appeared that the scattered waves of the original problem can be obtained via matching of these solutions in lower channels with the oscillating exponentials, subject to the above partial matching conditions (2). This matching is done based on Dirichlet-to-Neumann map of the corresponding intermediate operator (for given Fermi level) and results in Scattered waves on the open channels, with a finite-dimensional Scattering matrix. Construction of the Scattering matrix on the next spectral band, for higher Fermi level, requires another intermediate operator, because the structure of the Scattered wave on the neighboring spectral band is different. Hence for the Quantum network the construction of a system of intermediate operators on the quantum network is defined by the structure of spectral bands, see [20].

One may say, that the only essential difference of the above notion of the intermediate operator from the initial idea of Prigogine is the " locality " of the structure of the family of intermediate operators, depending on the local structure of the continuous spectrum.

Seems that for operators with Lebesgues absolute continuous spectrum (constant multiplicity) the above construction of the intermediate is not applicable. In particular the standard procedure of analytic perturbation theory suggested in [4] for the additive perturbation of the momentum operator ("Friedichs model")

$$\mathcal{P}_{_\varepsilon} = \mathcal{P} + \varepsilon A$$

is converging only for small values of the perturbation parameter ε. Extension of perturbation procedure beyond the threshold of creation of resonances

requires another techniques, the non analytic techniques of the Mathematical Scattering Theory, developed in the middle of the previous century in numerous papers by T. Kato, M. Rosenbloom, M. Birman, [14, 38, 3] and others quoted in [15]. Still the practical question on a reasonable convenient procedure of construction of scattered waves and/or Scattering matrix remains.

In this paper we describe an observation which probably permits to return to the idea of the intermediate operator again and construct the combined perturbation procedure at least for operators which are equivalent to the general Friedrichs model , with the first step $\mathcal{P} \longrightarrow \mathcal{P}_0^\varepsilon$ based on the corresponding "intermediate operator" - the solvable model $\mathcal{P}_0^\varepsilon$, constructed for local use, in a certain interval of the values of the spectral parameter p near the point p_0, where the resonance is created,followed by the analytic perturbation procedure for calculation of the spectral characteristics of $\mathcal{P}_\varepsilon$ based on spectral data of the intermediate operator $\mathcal{P}_0^\varepsilon$.

The Friedrichs model is one of most popular models in spectral theory,both solvable and representative, see for instance [12, 34, 13, 24] and special papers [5, 6, 25, 21], where it is actually used for investigation of a subtle question of complex analysis on uniqueness of interpolation in Nevanlinna class. In this paper we use a modified version of the Friedrichs model which is especially convenient for exploration of the resonance scattering because of presence of spectral characteristics of the inner Hamiltonian in it, in explicit form.

The plan of our text is the following: in the next section we describe our version of the Friedrichs Model "with inner structure" and calculate the corresponding Scattering matrix in terms of the Nevanlinna-class Weyl-function defined by the spectral characteristics of the inner Hamiltonian A. In section 3 we explore the distribution of resonances and factorize the corresponding Scattering matrix as a product of the *non-analytic factor* S_ε interpreted as a Scattering matrix of properly constructed intermediate operator $\mathcal{P}_0^\varepsilon$, and an *analytic factor* which is a Scattering matrix of the pair $\mathcal{P} \longrightarrow \mathcal{P} + \varepsilon A,\ \mathcal{P}_0^\varepsilon$. In the last section (appendix) we overview the Gohberg-Sigal theorem on the logarithmic residue.

2 Friedrichs model with an inner structure

We use in this paper the Friedrichs model with an inner structure which is obtained as a perturbation [1] $\mathcal{P}_\beta$ of the orthogonal sum $\mathcal{P} \oplus A$ of the momentum $\mathcal{P} = \frac{1}{i}\frac{d}{dx}$ in $L_2(R, E)$, $\dim E = m < \infty$ and a finite-dimensional self-adjoint operator acting in the space K, $A : K \to K, \dim K = k < \infty$. An operator of similar form may appear in quantum stochastic calculus as a self-adjoint dilation of the generator of an averaged evolution on stochastic background, see for instance [17, 22].

We construct the $\mathcal{P}_\beta$ via operator extension procedure, beginning from restriction of the momentum $\mathcal{P} \to \mathcal{P}_0$ onto the domain $D_0 = W_2^{1,0}(R, E)$ of all smooth functions taking values in E and vanishing at the at origin.The operator $\mathcal{P}_0$ is a symmetric with deficiency indices (n, n), $n = \dim$ E.The corresponding adjoint operator $\mathcal{P}_0^+$ is defined on $W_2^1(R_-) \oplus W_2^1(R_+)$ without any boundary conditions at the origin. The boundary form, see for instance [28, 11, 1] of the adjoint operator $J(u, v) = \langle \mathcal{P}_0^+ u,\, v\rangle - \langle u,\, \mathcal{P}_0^+ v\rangle = i\left[u\bar{v}(0^+) - u\bar{v}(0^-)\right]$ can be presented in terms of the corresponding symplectic variables $\xi_\pm$:

$$\xi_+^u = \frac{u(0^+) + u(0^-)}{2},\ \xi_-^u = i\left[u(0^+) - u(0^-)\right]$$

as

$$J_p(u, v) = \langle \xi_-^u,\, \xi_+^v\rangle_E - \langle \xi_+^u,\, \bar{\xi}_-^v\rangle_E \tag{3}$$

A version of operator extension theory for the finite-dimensional operator was developed in [16]. The symplectic version of the corresponding extension procedure we are going to use is described in [1, 18]. Main obstacle to the construction procedure in finite-dimensional case - the absence of the adjoint operator - is avoided via reducing of the construction onto the defect - the sum of deficiency subspaces $N_i = N$, $N_{-i} = \frac{A+iI}{A-iI}N$, $\dim N = n$ such that $N_i \cap N_{-i} = 0$. The restricted operator is defined on $D_{A_0} = \frac{1}{A-iI}K \ominus N$.Choosing an orthogonal basis $\mathbf{e}_s \in N$ and the corresponding basis $\hat{\mathbf{e}}_s = \frac{A+iI}{A-iI}\mathbf{e}_s \in N_{-i}$, we introduce a new basis in defect $D = N + N_{-i}$:

$$W_s^+ = \frac{\mathbf{e}_s + \hat{\mathbf{e}}_s}{2} = \frac{A}{A - iI}\mathbf{e}_s,\ W_s^- = \frac{\mathbf{e}_s - \hat{\mathbf{e}}_s}{2i} = -\frac{1}{A - iI}\mathbf{e}_s.$$

[1] Here β is a matrix-valued (non small,generally) parameter labelling self-adjoint extensions, see below (4).

Then the elements from the defect are uniquely presented as linear combinations : $u_d = \frac{A}{A-ipI}\eta_+ - \frac{1}{A-ipI}\eta_-$, where

$$\eta_+ = \sum_s \overset{s}{\eta_+}\mathbf{e}_s, \;\; \eta_- = \sum_s \overset{s}{\eta_-}\mathbf{e}_s.$$

The formal adjoint operator A^+ is defined on the defect as :

$$A^+\mathbf{e}_s = -i\mathbf{e}_s, \;\; A^+\hat{\mathbf{e}}_s = i\hat{\mathbf{e}}_s$$

or

$$A^+W_s^+ = W_s^-, \;\; A^+W_s^- = -W_s^+.$$

Lemma 2.1 *[28, 18] The boundary form of the formal adjoint operator is calculated in terms of symplectic variables $\overset{u}{\eta_\pm}$, $\overset{v}{\eta_\pm}$ as*

$$J_A(u,v) = \langle A^+u,\, v\rangle - \langle u,\, A^+v\rangle = \langle \overset{u}{\eta_+},\, \overset{v}{\eta_-}\rangle - \langle \overset{u}{\eta_-},\, \eta_+^v\rangle$$

and it depends on the defect part of the vectors u, v only.

Consider the orthogonal sum $P_0 \oplus A_0$ of the restricted operators and construct a Lagrangian plane $\mathcal{L}_\beta$ parametrized by the hermitian matrix B connecting the symplectic coordinates $\xi_\pm$ of the "outer" component with the symplectic coordinates $\eta_\pm$ of the "inner" components

$$B = \begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix}$$

with elements $\beta_{10} + = \beta_{01} \in C_m \times C_n, \;\; \beta_{00} \in C_m \times C_m, \; \beta_{11} \in C_1 \times C_1, .$

Theorem 2.1 *The joint boundary form $J_p(u,v) + J_A(\mathbf{u},\, \mathbf{v})$ vanishes on the Lagrangian plane $\mathcal{L}_\beta$ described by the equation :*

$$\begin{pmatrix} \xi_- \\ \eta_+ \end{pmatrix} = B \begin{pmatrix} \xi_+ \\ \eta_- \end{pmatrix}. \tag{4}$$

This Lagrangian plane defines a joint self-adjoint extension P_β of $P_0 \oplus A_0$.

The constructed operator $\mathcal{P}_\beta$ has absolutely continuous spectrum multiplicity m on the interval $(-\infty,\, \infty)$.The corresponding eigenfunctions $\mathbf{\Psi}$ have two components : in the "outer" space $L_2(R,\, E)$ and in the inner space

$K\ \mathbf{\Psi} = \{\Psi_{_0},\, \Psi_1\}$. They fulfill the adjoint homogeneous equations and the above boundary conditions (4). The symplectic coordinates $\eta_{\pm}$ of the solution are connected via analog of the Weyl function $\mathcal{M}(p) = P_{_N}\, \frac{I+pA}{A-pI}\, P_{_N}$, see for instance [29, 26]:

$$\eta_{_-} = -\mathcal{M}(p)\eta_{_+}.$$

The Weyl function belongs to Nevanlinna class (that is: it is analytic and has a positive imaginary part in upper half-plane $\Im p > 0$). Then, presenting the outer component of the "incoming" eigenfunction by the Ansatz

$$\Psi_{_0}(x,\, \nu) = \left\{ \begin{array}{ll} e^{ipx}\nu & \text{for} \quad x < 0, \\ e^{ipx} S_{_\beta}\nu & \text{for} \quad x > 0, \end{array} \right. , \tag{5}$$

and the inner component defined as

$$\Psi_{_1}(\nu) = \frac{A + iI}{A - pI}\eta_{_+}(\nu),$$

with $\eta_{_+}(\nu)$ to be found from the above boundary condition (4), we can rewrite the equation as

$$\left(\begin{array}{c} i(S\nu - \nu) \\ \eta_{_+}(\nu) \end{array} \right) = \left(\begin{array}{cc} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{array} \right) \left(\begin{array}{c} \frac{S+1}{2}\nu \\ -\mathcal{M}\ (p)\ \eta_{_+}(\nu) \end{array} \right),$$

hence $\eta_{_+}(\nu) = \beta_{_{10}} \frac{I+S}{2}\nu - \beta_{_{11}}\, \mathcal{M}\eta_{_+}(\nu)$ and

$$i\,(S-1)\,\nu = \left[\beta_{00} - \beta_{_{01}}\ \mathcal{M}\,(1 + \beta_{_{11}}\,\mathcal{M})^{^{-1}}\,\beta_{_{01}}\right]\ \frac{S+I}{2}\,\nu$$

and

$$\eta_{_\nu} = \frac{1}{I + \beta_{_{11}}\mathcal{M}}\beta_{_{10}}\frac{I+S}{2}\nu.$$

This implies the following expression for the Scattering matrix S :

Theorem 2.2 *The Scattering matrix defined as the transmission coefficient in the exterior component of the Scattered waves (5) is presented as:*

$$S_{_\beta}(p) = \frac{i + \frac{1}{2}\left[\beta_{00} - \beta_{_{01}}\,\mathcal{M}\,(1 + \beta_{_{11}}\,\mathcal{M})^{^{-1}}\,\beta_{_{10}}\right]}{i - \frac{1}{2}\left[\beta_{00} - \beta_{_{01}}\,\mathcal{M}\,(1 + \beta_{_{11}}\,\mathcal{M})^{^{-1}}\,\beta_{_{10}}\right]}, \tag{6}$$

where the denominator is the first factor and the numerator is the second. The scattered waves of the perturbed operator are defined by (5) with $S = S_{_\beta}$.

Remark 1 The perturbed momentum operator P_{β} acting in extended space $L_2(R,E)\oplus K$, is unitary equivalent to P. This unitary equivalence is defined by the corresponding wave operators, transforming the non-perturbed scattered waves into perturbed ones

$$W_{-}e^{ipx}\nu=\left(\begin{array}{c}\Psi_0\\ \Psi_1\end{array}\right),$$

$$\mathcal{P}_{\beta}=W_{-}\mathcal{P}W_{-}^{+}.$$

The same operators can transform the multiplication operator $Q: u\longrightarrow xu$ into

$$\mathcal{Q}_{\beta}=W_{-}\,\mathcal{Q}\,W_{-}^{+},$$

acting in the extended space, such that the pair $\mathcal{P}_{\beta}$, $\mathcal{Q}_{\beta}$ fulfills the same commutation relations as $\mathcal{P}$, $\mathcal{Q}$. This fact permits to introduce the corresponding non-hermitian creation and annihilation operators, coherent states and other standard objects.

Remark 2 If $B=0$, then $S=1$, which corresponds to the non-perturbed operator. But *it is impossible* to construct an analytic (with respect to the perturbation parameters β_{il}) branch of eigenfunctions $\Psi_{\nu}(p,\beta)$ of the perturbed operator for any p that coincides with the eigenfunction $e^{ipx}\nu$ of the non-perturbed operator at $B=0$. In the following section we will suggest a special perturbation procedure which permits to overcome this basic difficulty locally, near certain point $(p_0,\,0)$ in the space $(p,\,\beta)$ based on introduction of a special *intermediate operator*.

Generally the above formula 6 produces an expression for the Scattering matrix with non-trivial asymptotic at infinity

$$S_{\beta}(p)\rightarrow\frac{i+\frac{1}{2}\left[\beta_{00}-\beta_{01}\left(-PAP\right)\left[1+\beta_{11}\left(-PAP\right)\right]^{-1}\beta_{10}\right]}{i-\frac{1}{2}\left[\beta_{00}-\beta_{01}\left(-PAP\right)\left[1+\beta_{11}\left(-PAP\right)\right]^{-1}\beta_{10}\right]}$$

when $p\rightarrow\infty$.

Theorem 2.3 *The Scattering matrix has the "natural" asymptotic behavior at infinity* $S_{\beta}(p)\rightarrow I$, *if and only if*

$\left[\beta_{00} - \beta_{01}\left(-PAP\right)\left(1+\beta_{11}\left(-PAP\right)\right)^{-1}\beta_{10}\right] = 0.$ *In particular the Scattering matrix tends to unity at infinity if*

$$\beta_{11} = 0 \ \textit{and} \ \beta_{00} + \beta_{01}PAP\beta_{10} = 0. \tag{7}$$

If the boundary parameter β_{00} is selected such that this condition is fulfilled, then the corresponding Scattering matrix is represented as a finite Plashke-Potapov product [2] *with zeroes (resonances) in upper half-plane.*

Proof Introducing the notation

$$\mathcal{M}(p) = -PAP + P\frac{I+A^2}{A-pI}P := a + m(p),$$

with the Nevanlinna-class function m tending to zero at infinity, we can present the expression for the numerator of the Scattering matrix (6) as [3]

$$i + \frac{1}{2}\left[\beta_{00} - \beta_{01}\frac{a}{I+\beta_{11}a}\beta_{10}\right] +$$
$$\frac{1}{2}\left[\beta_{01}\frac{a}{I+\beta_{11}a}\beta_{10} - \beta_{01}\frac{a}{I+\beta_{11}a+\beta_{11}m}\beta_{10}\right] +$$
$$\frac{1}{2}\left[-\beta_{01}\frac{m}{I+\beta_{11}a+\beta_{11}m}\beta_{10}\right]. \tag{8}$$

Due to Hilbert identity the mid term can be presented as

$$\frac{1}{2}\left[\beta_{01}\frac{a}{I+\beta_{11}a}\beta_{11}m\,\frac{a}{I+\beta_{11}a+\beta_{11}m}\beta_{10}\right].$$

Together with the theorem condition this implies the following formula for the numerator of the Scattering matrix:

$$i + \frac{1}{2}\left[\beta_{01}\left(\frac{a}{I+\beta_{11}a}\beta_{11} - I\right)\ m\,\frac{I}{I+\beta_{11}a+\beta_{11}m}\beta_{10}\right]$$

with an expression in brackets approaching zero at infinity. Then the Scattering matrix

$$S_\beta p = \frac{i + \frac{1}{2}\left[\beta_{01}\left(\frac{a}{I+\beta_{11}a}\beta_{11} - I\right)\ m\,\frac{a}{I+\beta_{11}a+\beta_{11}m}\beta_{10}\right]}{i - \frac{1}{2}\left[\beta_{01}\left(\frac{a}{I+\beta_{11}a}\beta_{11} - I\right)\ m\,\frac{a}{I+\beta_{11}a+\beta_{11}m}\beta_{10}\right]}$$

[2] See [23, 8]

[3] In all fractions below (8) the numerator is preceding the denominator

tends to I at infinity.

In particular the Scattering matrix tends to unity at infinity if $\beta_{11} = 0$ and $\beta_{00} + \beta_{01} PAP\beta_{10} = 0$. In this case the expressions in brackets in both numerator and denominator are Nevanlinna functions and the Scattering matrix

$$S_\beta(p) = \frac{2i - \beta_{01} m \beta_{10}}{2i + \beta_{01} m \beta_{10}} = \frac{2i - \beta_{01} \frac{I+A^2}{A-pI} \beta_{10}}{2i - \beta_{01} \frac{I+A^2}{A-pI} \beta_{10}} \tag{9}$$

is a finite Blaschke-Potapov product [23, 8] with vector zeroes $p_s : S_\beta(p_s)\nu_s = 0$ in upper half-plane $\Im p_s > 0$:

$$S_\beta(p) = \prod_s \left[\frac{p - p_s}{p - \bar{p}_s} P_s + (I - P_s) \right].$$

Here P_s are orthogonal projections in E which depend on the order of factors, see the discussion in the end of the next section.

□

3 Analytic perturbation procedure and Intermediate operator

We begin with a general statement concerning resonances.

Lemma 3.1 *If the condition (7) is fulfilled, then the zeroes of the Scattering matrix resonances depend analytically on the boundary parameter β_{01} and may be found for small values of the parameter via analytical perturbation procedure.*

Proof is based on matrix version of Rouche theorem by Gohberg and Sigal, see [10] and Appendix below where the simplest version of this general fact is described. We consider here only the generic case when all eigenvalues α_s of the operator A are simple. Denoting $q_s = e_s \rangle \langle e_s$ the corresponding eigen-projections, we may present the function in the numerator of the Scattering matrix as

$$\beta_{01} P \frac{I + A^2}{A - pI} P \beta_{01} = \sum_s \frac{1 + \alpha_s^2}{\alpha_s - p} \varepsilon_s Q_s, \tag{10}$$

where $Q_s = \nu_s\rangle\langle\nu_s$ is an orthogonal projection onto the one-dimensional subspace spanned by $\beta_{01}Pe_s = \| \beta_{01}Pe_s \| \nu_s$ and $\varepsilon_s = \| \beta_{01}Pe_s \|$. We assume that $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_3, \ldots)$ is a non-zero vector. We will use ε as a perturbation parameter instead of the matrix β_{01}. Our nearest aim is: to calculate the resonance p_{s0} created from α_0 at $\varepsilon_0 = 0$, assuming that $|\varepsilon| = \max\varepsilon_s$ is small.

It is clear that an essential contribution to the above function (10) near the pole α_0 is defined by the nearest singular summand $\frac{1+\alpha_0^2}{\alpha_0 - p}\varepsilon_0 Q_0$. Planning to use the Gohberg-Sigal theorem, see Appendix, introduce two functions

$$m(p) = 2i - \sum_s \frac{1+\alpha_s^2}{\alpha_s - p}\varepsilon_s Q_s \text{ and } m_0(p) = 2i - \frac{1+\alpha_0^2}{\alpha_0 - p}\varepsilon_0 Q_0,$$

$$m_0^{-1}(p) = \frac{1}{2i}\left(I - Q_0 \frac{i\varepsilon_0 \frac{1+\alpha_0^2}{2}}{\alpha_0 + i\varepsilon_0 \frac{1+\alpha_0^2}{2} - p}\right),$$

and the ratio

$$m_0^{-1}(p)\ m(p) = I - m_0^{-1} \sum_{s\neq 0} \frac{1+\alpha_s^2}{\alpha_s - p}\varepsilon_s Q_s := I - m_0^{-1}\mathbf{m}_0.$$

Zeroes of the function m coincide with resonances. The only zero of the function m_0 sits at $\alpha_0(\varepsilon) = \alpha_0 + i\varepsilon_0 \frac{1+\alpha_0^2}{2}$. Consider a circle Σ_0 with radius δ centered at $\alpha_0(\varepsilon)$. The ratio $m_0^{-1}\mathbf{m}_0$ can be estimated on the circle $\Sigma_0 = \left\{p : |\alpha_0 + i\frac{\varepsilon_0}{2} - p| = \delta\right\}$ as

$$\| m_0^{-1}\mathbf{m}_0 \| \frac{1}{2}\left[1 + \varepsilon_0 \frac{1+\alpha_0^2}{2}\right] \sum_{s\neq 0} \frac{\varepsilon_s}{|\alpha_s - \alpha_0| - \delta}, \tag{11}$$

hence it is small for small $|\varepsilon| + \delta << \min |\alpha_0 - \alpha_s|$. Both functions m, m_0 are analytic inside the circle Σ_0, hence, due to Gohberg - Sigal theorem, the function m has zeroes inside Σ_0 with the total multiplicity $M_0 = \dim Q_0$, in particular: it has only one simple zero, if $M_0 = 1$.

We continue our reasoning assuming that $M_0 = 1$. Then the function $m^{-1} = \left[I - m_0^{-1}\mathbf{m}_0\right]^{-1} m_0^{-1}$ has only one pole $p_0(\varepsilon)$, which will be found based on integration of m^{-1} on the circle Σ_0.

Consider the left factorization of the function m at the resonance $p_0(\varepsilon)$:

$$m(p) = \left([p - p_0(\varepsilon)] \, \mathbf{P}_0^+ \, (\varepsilon) + b(\varepsilon)(I - \mathbf{P}_0^+) \right) \, \hat{\mu} \, (p) := m_0^+ \, \hat{\mu} \, (p). \qquad (12)$$

Here $\mathbf{P}_0^+$ is the orthogonal projection onto the null-space of m^+ at the point $p_0(\varepsilon)$ ("left" null-space of $m : \mathbf{P}_0^+ m \left(p_0(\varepsilon) \right) = 0$. The residue of the function m^{-1} at $p_0(\varepsilon)$ is calculated as an integral of $m^{-1} = \left[I - m_0^{-1} \mathbf{m}_0 \right]^{-1} m_0^{-1} = m_0^+ \, \hat{\mu}$ on the circle. The second representation gives the formula :

$$\frac{1}{\hat{\mu} \left(p_0(\varepsilon) \right)} \mathbf{P}_0^+(\varepsilon) = \frac{1}{2\pi} \oint_{\Sigma_0} \frac{1}{\hat{\mu(p)}} \left(\frac{\mathbf{P}_0^+(\varepsilon)}{[p - p_0(\varepsilon)]} + \frac{\left(I - \mathbf{P}_0^+(\varepsilon) \right)}{b(\varepsilon)} \right) dp. \qquad (13)$$

On the other hand the residue can be found via integration of another expression for $m^{-1}(p)$ on the circle:

$$\frac{1}{2\pi} \oint_{\Sigma_0} \left[\left(m_0(p) \right)^{-1} + \left(m_0(p) \right)^{-1} \mathbf{m}_0(p) \left(m_0(p) \right)^{-1} + \ldots \right] dp. \qquad (14)$$

The series in the integrand is converging geometrically, and each term of it is calculated by residues at the pole $\alpha_0 \varepsilon$, for instance

$$\frac{1}{2\pi} \oint_{\Sigma_0} \left(m_0(p) \right)^{-1} dp =$$

$$\frac{1}{2\pi} \oint_{\Sigma_0} \left\{ \frac{1}{2i} \left[\frac{\alpha_0 - p}{\alpha_0 + i \frac{\varepsilon_0 (1 + \alpha_0^2)}{2} - p} \right] Q_0 + \frac{1}{2i} \left[I - Q_0 \right] \right\} dp = \varepsilon_0 \frac{1 + \alpha_0^2}{2} Q_0.$$

Next terms contain derivatives of $\mathbf{m}_0$ at the pole $\alpha_0 \varepsilon = \alpha_0 + i \frac{\varepsilon_0 (1 + \alpha_0^2)}{2}$. The structure of the whole expansion is similar to the expansion by residues which arises in the standard Feynmann diagram technique, because the idea of calculation of the residue based on use of two different forms of the integrand is the same as in Feynmann case.

The analyticity of the projection $\mathbf{P}_0^+(\varepsilon)$ as a function of ε follows from the geometrical convergence of the series in the integrand of (14) which follows from the estimate (11).

The orthogonal projection $P_0^+(\varepsilon_0)$ onto the null - subspace of $m^+(p_0(\varepsilon_0))$ is calculated, up to a constant, as

$$\mathbf{P}_0^+(\varepsilon)\left(\frac{1}{\mu(p_0(\varepsilon))}\right)^+\frac{1}{\mu(p_0(\varepsilon))}\mathbf{P}_0^+(\varepsilon)=\mathrm{Const}\mathbf{P}_0^+(\varepsilon).$$

The zero $p_0(\varepsilon)$ of the function m can be obtained from comparison of the previous integral with the integral

$$\frac{1}{2\pi}\oint_{\Sigma_0}\frac{p}{\mu(p)}dp=\frac{p_0(\varepsilon)}{\mu(p_0(\varepsilon))}\mathbf{P}_0^+(\varepsilon).$$

Thus both the resonance $p_0(\varepsilon)$ and the corresponding left root-vector ν_0^+ are defined. The right root-vector can be found in a similar way.

□

Assuming that the condition of the preceding theorem are fulfilled, consider the rational representation of the Scattering matrix of the operator $\mathcal{P}_\beta$

$$S_\beta(p)=\frac{2i-\sum_s\frac{1+\alpha_s^2}{\alpha_s-p}\varepsilon_s Q_s}{2i-\sum_s\frac{1+\alpha_s^2}{\alpha_s-p}\varepsilon_s Q_s} \tag{15}$$

where Q_s is the orthogonal projection onto the subspace spanned by the vector $\beta_{01}P_N\nu_s$ obtained from the eigenvector ν_s of A via successive projections onto the deficiency subspace $N=N_i$ and then applying the linear boundary map β_{01}. If the order of Blaschke-factors is fixed, one can also present the Scattering matrix for small values of ε in form of Blashke-Potapov product with simple Blaschke factors B_s

$$\prod_s\left\{\left[\frac{p-p_s(\varepsilon)}{p-\bar{p}_s(\varepsilon)}\right]P_s+P_s^\perp\right\}=:\prod_s B_s. \tag{16}$$

Theorem 3.1 *Each Blaschke factor in (16) is an analytic function of both variables (p,ε) on the product of a small neighborhood of the origin in ε-space and a complement of a small neighborhood of the corresponding point α_s of creation of the resonance $p_s(\varepsilon)$ in p-space. In particular, selecting $B_0=S_0^\varepsilon$ as the first factor on the right, we obtain the factorization of the Scattering matrix in form of two factors:*

$$S_\varepsilon=S_\varepsilon^0\ S_0^\varepsilon, \tag{17}$$

with the left factor analytic with respect to $(\varepsilon,\, p)$ *on a small neighborhood* $(0,\, \alpha_{_0})$*, and the second - non-analytic on that neighborhood.*

The non-analyticity of the factor $S_{_0}^{\beta}$ causes the non-analyticity of the whole product and corresponds to the fact of the non-analyticity of the Scattering matrix with respect to the perturbation parameter at the "threshold of creation of resonances".

Remark: Construction of Blaschke-factors. Assume that the right factorizations of the Scattering matrix are constructed at each resonance

$$S_{_\beta}(p) = B^{^s}(p)\ \mathbf{B}_{_s}(p),\ \mathbf{B}_{_s}(p) = \frac{p - p_{_s}(\varepsilon)}{p - \bar{p}_{_s}(\varepsilon)}\mathbf{P}_{_s} + [I - \mathbf{P}_{_s}]\,. \tag{18}$$

The right Blaschke- factors $\mathbf{B}_{_s}(p)$ do not coincide with the corresponding Blaschke- factors $B_{_s}(p)$ in the above product (16) due to the non-commutativity of the factors. We suggest here the procedure of construction of the factors $B_{_s}(p)$ once the factors $\mathbf{B}_{_s}(p)$ are given.

Assume that the factors $B_{_0}$, $B_{_1}$, $B_{_2}$, ... are ordered from the right to the left such that $B_{_0}$ is the first factor from the right, $B_{_1}$ is the second factor from the right .... Denoting by $\mathbf{N}_{_s}$, $N_{_s}$ the ranges of $\mathbf{P}_{_s}$, $P_{_s}$ respectively and by $\nu_{_s}, \nu_{_s}$ any vectors from $\mathbf{N}_{_s}$, $N_{_s}$ we can write down the following chain of equations

$$\begin{gathered}
B_{_0}(p_{_0}) = \mathbf{B}_{_0}(p_{_0}),\ N_{_0} = \mathbf{N}_{_0},\ P_{_0} = \mathbf{P}_{_0},\\
B_{_1}(p_{_1})B_{_0}(p_{_1})\,\nu_{_1} = 0,\ \text{or}\ N_{_1} = B_{_0}(p_{_1})\mathbf{N}_{_1},\\
B_{_2}(p_{_2})B_{_1}(p_{_2})B_{_0}(p_{_2})\,\nu_{_2} = 0,\ \text{or}\ N_{_2} = B_{_1}(p_{_2})B_{_0}(p_{_2})\mathbf{N}_{_2},\\
B_{_l}(p_{_l})\ldots B_{_2}(p_{_l})B_{_1}(p_{_l})B_{_0}(p_{_l})\,\nu_{_l} = 0,\ \text{or}\ N_{_l} = B_{_{l-1}}(p_{_l})\ldots B_{_1}(p_{_l})B_{_0}(p_{_l})\mathbf{N}_{_l},
\end{gathered} \tag{19}$$

We obtain the chain of one-dimensional subspaces if each product $B_{_{l-1}}(p_{_l})\ldots B_{_1}(p_{_l})B_{_0}(p_{_l})$ of Blaschke factors does not degenerate on the corresponding subspace $\mathbf{N}_{_l}$:

$$B_{_{l-1}}(p_{_l})\ldots B_{_1}(p_{_l})B_{_0}(p_{_l})\nu_{_l} \neq 0. \tag{20}$$

Theorem 3.2 *The condition (20) of transformation of the rational form of the Scattering matrix (15) into the Blashke-product (16) is fulfilled for small values of the perturbation parameter* ε*.*

Proof For small values of the perturbation parameter the imaginary parts $\Im p_s$ of resonances are small, hence each term in the previous chain of equations can be re-written in form :

$$N_1 = \left[I - i\frac{2\Im p_0}{p_1 - \bar{p}_0} P_0 \right] \mathbf{N}_1,$$

$$N_2 = \left[I - i\frac{2\Im p_1}{p_2 - \bar{p}_1} P_1 \right] \left[I - i\frac{2\Im p_0}{p_2 - \bar{p}_0} P_0 \right] \mathbf{N}_2$$

$$N_l = \left[I - i\frac{2\Im p_{l-1}}{p_l - \bar{p}_{l-1}} P_{l-1} \right] \left[I - i\frac{2\Im p_{l-2}}{p_l - \bar{p}_{l-2}} P_{l-2} \right] \left[I - i\frac{2\Im p_0}{p_l - \bar{p}_0} P_0 \right] \mathbf{N}_2. \quad (21)$$

Then due to small $\Im p_0$ the operator $\left[I - i\frac{2\Im p_0}{p_1 - \bar{p}_0} P_0 \right]$ is invertible and hence N_1 is has the same dimension as $\mathbf{N}_1$. The projection P_1 exists. Then above argument may be applied to the second equation, to find P_2, and so on until all projections P_l are defined.

□

Note that the structure of each Blaschke B_s factor shows that it has a zero at $p_s(\varepsilon)$, a pole at $\bar{p}_s(\varepsilon)$ and both of them approach to the eigenvalue of the inner Hamiltonian when $\varepsilon \to 0$. The Blaschke factor is not analytic with respect to $(\varepsilon,\, p)$ near $(0, \alpha_0)$ due to convergence of the zero and the pole to the same point α_0. The Blaschke factor B_s is analytic with respect to $(\varepsilon,\, p)$ for small values of ε , if $|p - \alpha_s| > \delta > 0$. The whole Scattering matrix (18) is not analytic with respect to the perturbation parameter near each eigenvalues α_s of the "inner Hamiltonian " A due to presence of the non-analytic factor B_s. Nevertheless one may modify the perturbation procedure *locally*, eliminating, for instance, the non-analytic factor $S_0^\varepsilon := B_0$ via the "jump-start" [4]: by introducing of the *intermediate operator* $\mathcal{P}_0^\beta$, which is selected such that $S_0^\varepsilon = B_0$ is the Scattering matrix for the pair $\left(\mathcal{P}_0^\varepsilon,\, \mathcal{P}\right)$. Then the Scattering matrix can be presented as a product of the non-analytic, but explicit factor S_0^ε and the complementary analytic factor which is the Scattering matrix of the pair $\left(\mathcal{P}_\varepsilon,\, \mathcal{P}_0^\varepsilon\right)$. The construction of the intermediate operator is described below under assumption that *we know the resonance* $p_0(\varepsilon)$ *and the corresponding resonance root - vector* **exactly** [5].

[4] The term suggested by L. Faddeev

[5] If we know $p_0(\varepsilon)$ only asymptotically, as a finite power expansion in the perturbation parameter, the corresponding approximate formulae are still valid, but the statement of analyticity disappears. Practical corollaries from this observation will be discussed elsewhere.

Note that the real and imaginary parts of the resonance can be expand into convergent power series of the real perturbation parameter ε, because of analyticity of $p_0(\varepsilon)$. Hence the real and imaginary parts of the resonance are also real analytic functions of the perturbation parameter ε.

Theorem 3.3 *For given Blaschke factor*

$$B_\varepsilon = \left\{ \left[\frac{p - p_0(\varepsilon)}{p - \bar{p}_0(\varepsilon)} \right] P_0 + P_0^\perp \right\},$$

with the one-dimensional projection $P_0 = \nu_\varepsilon \rangle \langle \nu_\varepsilon$. *Consider the one-dimensional operator* A_0^ε *with the eigen-value* $\alpha_0^\varepsilon = \Re p_s(\varepsilon)$ *and the boundary matrix*

$$B = \begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & 0 \end{pmatrix}$$

with the deficiency vector $\mathbf{e}$ *of* A *mapped into* E *as*

$$\beta_{01} = \sqrt{2 \frac{\Im\, p_0(\varepsilon)}{1 + \alpha_0^2}} \nu_0(\varepsilon),$$

and the operator $\beta_{00} = -\beta_{01} \alpha \beta_{01}^+$. *Then the Scattering matrix for the pair* $\left(\mathcal{P}_0^\beta, \mathcal{P} \right)$ *coincides with* S_0^ε.

Proof. The Scattering matrix for the constructed operator is calculated as in (9):

$$S(p) = \frac{2i - \frac{1+\alpha_0^2}{\alpha_0 - p} \beta_{01} \rangle \langle \beta_{01}}{2i + \frac{1+\alpha_0^2}{\alpha_0 - p} \beta_{01} \rangle \langle \beta_{01}}.$$

Multiplying by $(p - \alpha)$ and dividing through $2i$, we transform the latter expression to

$$\frac{p - \alpha_0 - i \frac{(1+\alpha^2)}{2} |\beta_{01}|^2}{p - \alpha_0 - i \frac{(1+\alpha^2)}{2} |\beta_{01}|^2} P_0 + P_0^\perp =$$

$$P_0^\perp + \frac{p - \left(\alpha_0 + i |\beta_{01}|^2 \frac{1+\alpha_0^2}{2} \right)}{p - \left(\alpha_0 - i |\beta_{01}|^2 \frac{1+\alpha_0^2}{2} \right)} P_0^\perp,$$

Where $P_0^{\perp}$ is the orthogonal projection onto the complement of the $Q_0 E$ which coincides with the Blaschke factor B_0. Substituting here the data $\alpha_0(\varepsilon) = \Re p_0(\varepsilon)$, $|\beta_{01}|^2 \frac{1+\alpha^2}{2} \Im p_0(\varepsilon)$, we obtain, after simple transformations , the Blaschke-factor B_ε.

□

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5 Appendix

Let E_0 be proper subspace of the finite-dimensional Hilbert space E , $\mathbf{P}_0$ is an orthogonal projection onto E_0 and $\mathbf{P}_0^{\perp} = I - \mathbf{P}_0$ - the projection onto the orthogonal complement $E_0^{\perp} = E \ominus E_0$. We say that the analytic matrix-function m defined on the domain D_m has a *simple isolated right vector zero* at the point $p_0 \in D_m$ if it may be represented in a neighborhood $U_0 \subset D_\mu$ as a product

$$m(p) = \mu_0(p) \left[(p - p_0)\mathbf{P}_0 + b\mathbf{P}_0^{\perp} \right] \tag{22}$$

with some nonzero constant b, the right orthogonal projection $\mathbf{P}_0$ and an invertible near p_0 analytic matrix-function $\mu_0(p)$

$$\mu_0(p) = \mu_0(p_0) + \frac{p - p_0}{1!}\mu_0'(p_0) + \ldots, \ \ \mathrm{Ker}\, \mu_0(p_0) = 0.$$

Multiple zeroes are defined by the similar to (22) with several right factors with different projections. One can define in a similar way the left vector zero and the corresponding left projection based on the factorization

$$m(p) = \left[(p - p_0)\mathbf{P}_0^{+} + b\mathbf{P}_0^{\perp} \right] \mu_0^{l}(p) \tag{23}$$

For finite-dimensional square matrix-functions the left and right vector zeroes coincide and dim bfP_0 = dim bfP_0^{+} due to Fredholm theorem. The vectors $\mathbf{e}_0$, $\mathbf{e}_0^{+} \in \mathbf{N}_0$, $\mathbf{N}_0^{+}$ from the corresponding null-subspaces are called

respectively right and left *root-vectors*, $m(p_0)\mathbf{e}_0 = 0$, $m^+(p_0)\mathbf{e}_0^+ = 0$. For vectors $\mathbf{e}_\perp$ from the complementary subspace $\mathbf{e}_\perp \in E_{0\perp}$ we have $m(p_0)e_\perp \neq 0$. Similarly the *simple isolated vector pole* is defined : we say that the function m has a simple isolated vector pole at the point p_0 if it is represented as

$$m(p) = \mu\left[\frac{\mathbf{P}_0}{p-p_0} + b\mathbf{P}_0^\perp\right] \tag{24}$$

with a non-zero constant b and an orthogonal projection $\mathbf{P}_0$ onto proper subspace $\mathbf{N}_0 \subset E$, the complementary projection $\mathbf{P}_0^\perp$ and an analytic invertible function μ in a neighborhood U_0 of the point $p_0 \in D_m$. Similarly the left poles are defined, which coincide with right poles in finite-dimensional case. Both isolated poles and zeroes of analytic matrix-functions are called in [10] *characteristic values* of the argument μ. The logarithmic residue of the function μ at the simple isolated zero or pole is defined as an integral of the logarithmic derivative $m'(p)m^{-1}(p)$ on a simple smooth curve $\Gamma_0 \subset U_0$ in anti-clockwise ("positive") direction around the characteristic value m_0:

$$I_{m,p_0} = \frac{1}{2\pi i}\oint_{\Gamma_0} m'(p)m^{-1}(p)dp.$$

In [10] the *period* of the logarithmic derivative $m'(p)m^{-1}(p)$ on the simple cycle $\Gamma_0 \subset U_0$ containing no other characteristic points (zeroes, poles) inside

$$M_{m,p_0} = \frac{1}{2\pi i}Trace\oint_{\Gamma_0} m'(p)m^{-1}(p)dp.$$

is called the "multiplicity" of the characteristic value. The straightforward calculation of above integrals gives the following result:

$$I(m,p_0) = \frac{1}{2\pi i}\oint_{\Gamma_0} m'(p)m^{-1}(p)dp =$$

$$\frac{1}{2\pi i}\oint_{\Gamma_0} \mu(p)\mathbf{P}_0\left[(p_0-p)\mathbf{P}_0 + b\mathbf{P}_0^\perp\right]^{-1}\mu^{-1}(p)dp =$$

$$\frac{1}{2\pi i}\mu(p_0)\oint_{\Gamma_0} \mathbf{P}_0\left[(p_0-p)\mathbf{P}_0 + b\mathbf{P}_0^\perp\right]^{-1}dp\mu^{-1}(p_0) = \mu(p_0)\,\mathbf{P}_0\,\mu^{-1}(p_0),$$

and

$$M_{m,p_0} = \pm\mathrm{dim}\mathbf{P}_0,$$

where the sign $\pm$ is defined by the type of the characteristic value : plus for zero, minus for pole.

In generic situation considered in the text above the simplest version of the Gohberg-Sigal theorem is used with that all poles (zeroes) simple (first order).

Theorem 5.1 *If two finite square matrices* m, m_0 *depend analytically on the parameter* p *in the disc* D *radius* δ *centered at the point* p_0, *and* m_0 *has an only characteristic point* P_0 *at the center of the the disc with the multiplicity* M_0, *and on the circle* $\Sigma_0 = \{p : |p - p_0| = \delta\}$ *both functions have no characteristic values and the inequality*

$$max_{p\in\Sigma_0} \parallel m_0^{-1}(p)\,[m(p) - m_0(p)] \parallel < 1$$

is fulfilled, then the total multiplicity M_1 *of characteristic values of the function* m *inside the circle* Σ_0 *is equal to the multiplicity* M_0 *of the characteristic value of the function* m_0.

Proof of much more general statement concerning analytic functions with multiple poles and zeroes may be found in [10]. We actually need in the above text , section 3, a partial statement concerning the case when $M_0 = 1$.

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