Comparisons of integrators on a diverse collection of restricted three-body test problems

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Abstract

In earlier work, we presented a collection of 20 two-dimensional symmetric periodic orbits for the restricted three-body problem and compared the ubiquitous matlab integrators `ode45` and `ode113` on the orbits.

The current work greatly extends this. We present a diverse collection of 28 symmetric and asymmetric, stable and unstable periodic orbits for the two- and three-dimensional problems and compare the performance of eight general purpose integrators on the orbits.

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1 Introduction

The restricted three-body (R3B) problem has many important applications in astronomy and astrophysics. It is used to model the orbits of celestial bodies including the Trojan asteroids and comets of Jupiter, the Saturnian satellites Telesto, Calypso, Helene, Janus and Epimetheus, the interplanetary cloud in the Earth's orbit, and Near Earth Asteroids. It is also used to model the gamma-ray flux from binary stars and the stability of barred galaxies, and to find orbits for lunar probes and robotic observatories. References for these applications include [1], [2], [4], [10], [12], [20], [21], [22], [28], [29], [30, 31].

In earlier work [25], we presented a collection of 20 two-dimensional symmetric periodic orbits for the restricted three-body problem and compared the matlab integrators `ode45` and `ode113` on the orbits. The current work greatly extends this. We present a diverse collection of 28 orbits and compare the performance of the Taylor series integrator `ATOMFT`

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[5], the Adams integrators `DIVA` [13, 14], `STEP` [24] and `ode113`, the explicit Runge-Kutta integrators `RKSUITE` [3], `DXRK8` [15, 16] and `ode45`, and the extrapolation integrator `ODEX` [8], pp. 440-443, on the orbits.

The orbits are divided into five groups G1 to G5, with G1 to G4 containing six problems each and G5 four problems. The orbits in G1 are two-dimensional and symmetric and those in G2 two-dimensional and asymmetric. G3 and G4 contain three-dimensional symmetric and asymmetric orbits respectively. G5 contains one orbit for each of the four combinations of symmetric and asymmetric orbits in two and three dimensions. The period T ranges two orders of magnitude across the 28 problems and μ ranges three orders of magnitude from 0.000953875, representing the Sun-Jupiter system, to 0.5, representing an equi-mass binary star. Orbits can be symmetric about different axises and in G3 we have two symmetries.

Unstable orbits have important applications such as transfer orbits between stable orbits. Instability can also be used to explain the small number of observed celestial bodies following a particular orbit. Hence, within each of G1 to G4, the orbits range from those that are stable or nearly so to those for which the magnitude λ_M of the largest eigenvalue of the monodromy matrix is $O(10^5)$. G5 contains problems for which λ_M is $O(10^{10})$ to $O(10^{12})$.

The equations of motion, μ, the period T and the initial conditions for the orbits are specified in §2. This is followed in §3 by a brief description of the integrators and in §4 by a summary of detailed comparisons between the integrators. We end in §5 with a discussion of our work. Unless stated otherwise, the testing of `ATOMFT`, `DIVA`, `STEP`, `DXRK8`, `RKSUITE` and `ODEX` was in double precision on a single processor of a SGI 3400, and the testing of `ode45` and `ode113` was in Matlab 6 on a IBM xSeries 350 processor.

2 Problems

We solve the R3B problem in rotating, three-dimensional, rectangular Cartesian coordinates with the origin at the centre of mass of the two massive bodies, the $x - y$ plane in their plane of motion, and the x-axis along their line of centres. The distance between the two massive bodies is constant and is used as the unit of distance. The units of time and mass are chosen so the period of the massive bodies is 2π and the gravitational constant is one. The equations of motion for the particle of infinitesimal mass are (see for example [23], p. 126)

$$\begin{aligned} x'' &= x + 2y' - (1-\mu)\frac{x - x_1}{r_1^3} - \mu\frac{x - x_2}{r_2^3}, \\ y'' &= y - 2x' - (1-\mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3}, \\ z'' &= - (1-\mu)\frac{z}{r_1^3} - \mu\frac{z}{r_2^3}, \end{aligned} \tag{1}$$

where the prime operator denotes differentiation with respect to t, $x_1 = -\mu$, $x_2 = 1 - \mu$, $r_1 = \sqrt{(x - x_1)^2 + y^2 + z^2}$, $r_2 = \sqrt{(x - x_2)^2 + y^2 + z^2}$.

The above equations of motion can be written as an autonomous Hamiltonian system with three degrees of freedom. The Hamiltonian function is

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + yp_x - xp_y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \tag{2}$$

where $p_x = x' - y$, $p_y = y' + x$ and $p_z = z'$ are the momenta.

The two-dimensional R3B problem has the particle restricted to the $x - y$ plane - the third equation in (1) is omitted and z and p_z are identically zero.

As we did in [25], we had planned to use known orbits or those that were easily found from known orbits by continuation on the initial conditions. This approach worked well for G1 because symmetric two-dimensional orbits are often used in applications and many such orbits are known. Far fewer asymmetric two-dimensional orbits and even fewer three-dimensional orbits are known and we found it necessary to do extensive searching using continuation on μ, T and the initial conditions to find realistic orbits that satisfied the criteria listed in §1.

On numerous occasions we continued along a family of orbits with the aim of finding an orbit with a large value of λ_M only to have λ_M reach a global maximum on the family less than the required value. This difficulty was compounded by the need to use quadruple precision on some orbits, slowing the calculations fifty-fold, and the need to use small continuation steps when λ_M was large.

As part of this searching, we found many new families of orbits that bifurcated horizontally or vertically from known orbits. Some of these new families will be reported on elsewhere.

Tables 1, 2, 3, 4 and 5 list λ_M, μ, the period T and the initial conditions for G1 to G5 respectively. The problems are numbered one to 28 from the first problem in G1 to the last one in G5. An accurate integration in quadruple precision using the values in Tables 1 to 4 produces a global error of 10^{-15} or less after one period. Because of the larger value of λ_M, the same accurate integration using the values in Table 5 produces a larger global error but the error is sufficiently small that the values may be used for testing at limiting precision in IEEE double precision arithmetic.

3 Integrators

The integrators `ATOMFT`, `DIVA`, `DXRK8`, `ODEX`, `RKSUITE`, `STEP`, `ode45` and `ode113` are all general purpose integrators that control the local error.

`ATOMFT` is a variable-order, variable-stepsize (VOVS) Taylor series integrator for initial value problems of q-th and mixed order, $q > 0$. It consists of a pre-processor and a run-time library; the pre-processor transforms Fortran-like statements of the problem into a form suitable for the run-time library.

`DIVA` is a VOVS Adams integrator for q-th order and mixed order problems. It uses PECE mode on the first step, PEC mode for the rest of the start phase and PECE mode thereafter. The predictor and corrector are orders k and $k+1$ respectively, $k \leq 19$, where

Stable	9.538750000000000000000E-04	1.8192800516658157075219E+02
	-1.0261711714064448847729E+00	0
	0	3.7457956802558677672400E-02
9.4	9.538750000000000000000E-04	1.839500615.585268282519E+02
	-1.0330714782716820682649E+00	0
	0	5.1669725611366658195694E-02
108	9.538750000000000000000E-04	1.8372395797886763954176E+02
	-1.0507532233536824163092E+00	0
	0	8.6543196563733586492512E-02
994	9.538750000000000000000E-04	1.7726426949775685978445E+02
	-9.6415219186103326593158E-01	0
	0	-8.6379899696573913544020E-02
9990	9.538750000000000000000E-04	1.7690634445593856580951E+02
	-8.7592115629666651894565E-01	0
	0	-2.69215.5603183756399622E-01
61800	9.538750000000000000000E-04	1.7634456526875210873364E+02
	-6.1280761087144776091354E-01	0
	0	-9.0058960403222809217922E-01

Table 1: G1: Problems 1 (top) to 6 (bottom). The entry for each problem has three rows: row 1 is λ_M, μ, T; row 2 is $x(0)$, $x'(0)$; row 3 is $y(0)$, $y'(0)$.

k may differ from equation to equation. STEP is a VOVS Adams integrator for first order problems. It uses PECE mode on every step with the predictor order k and the corrector order $k+1$, where $k \leq 12$ and k is the same for all equations. STEP uses Møller's [17, 18] technique, also known as compensated summation, to reduce the round-off error when the local error tolerance TOL is less than $100u$, where u is the unit round-off.

DXRK8 is a fixed-order, variable-stepsize explicit Runge-Kutta (ERK) integrator that advances the solution using an order eight formula and controls the stepsize using embedded formulae of orders three and five. Twelve derivative evaluations are performed on each accepted step. RKSUITE is a fixed-order, variable-stepsize ERK integrator that comes with a 2-3, a 4-5 and a 7-8 pair. We used the 7-8 pair because the order eight formulae in it is the same as in DXRK8, thus permitting a direct comparison between DXRK8 and RKSUITE. The 7-8 pair uses thirteen derivative evaluations per step.

ODEX is a VOVS extrapolation integrator for first order problems. ode45 is a variable-stepsize ERK integrator in matlab that uses a 4-5 pair; ode113 is a VOVS Adams integrator in matlab that has features in common with STEP.

During an integration, DIVA and DXRK8 will recommend increasing TOL if the round-off error is deemed too large. This recommendation was followed in all cases. The default value for the optional inputs to all integrators was used whenever possible.

Stable	4.5000000000000000000000E-01	1.6863215300001330934600E+01
	-1.9025491946311694457820E+00	5.5001351839464263773728E-04
	0	1.4418914030192800012617E+00
12.5	4.5000000000000000000000E-01	1.6863983336193011082966E+01
	-1.9025292693506234803410E+00	1.1771017102159842531746E-03
	0	1.4418396489878533982504E+00
112	4.5000000000000000000000E-01	1.6870716137484340375065E+01
	-1.9023548528836989348361E+00	3.2898003683750466906103E-03
	0	1.4413869682798902928257E+00
1010	4.5000000000000000000000E-01	1.6932002067177152309702E+01
	-1.9007883901669330866102E+00	9.5988284556044120518366E-03
	0	1.4373494388213475419716E+00
9740	4.5000000000000000000000E-01	1.7616071235202744734222E+01
	-1.8861067930944052490757E+00	2.4136971354071783433815E-02
	0	1.4020054862358416896635E+00
99900	4.5000000000000000000000E-01	2.1196636029465948546825E+01
	-1.8716014560426429956669E+00	2.2752183961406589052022E-02
	0	1.3762000025635070005351E+00

Table 2: G2: Problems 7 (top) to 12 (bottom). The entry for each problem has three rows: row 1 is λ_M, μ, T; row 2 is $x(0)$, $x'(0)$; row 3 is $y(0)$, $y'(0)$.

4 Comparisons

Our first comparisons were on G1 to G4 with TOL $= 10^{-i}$, $i = 3, \ldots, 12$, and computational effort measured by the number of derivative evaluations[1]. The accuracy of a numerical solution was measured by the L_2 norm of the absolute global error in the position, velocity, or position and velocity after one period. We denote the norms by E_r, E_v and E_{rv} respectively.

Figure 1 contains $\log_{10} - \log_{10}$ graphs of the number of derivative evaluations against E_r for `DIVA`, `STEP` and `ODEX` on Problems 13 to 18. We observe `ODEX` is less efficient than the two Adams integrators on all six problems. This conclusion holds for the remaining 18 problems in G1 to G4 and for E_v. The two Adams integrators were of similar efficiency on most problems in G1 to G4, the only marked exceptions being on Problems 13 and 14 (this is one of the reasons we included the graphs for these problems). For Problems 16, 17 and 18, `ode113`, `DIVA` and `STEP` all showed poor proportionality between TOL and E_r at lax tolerances, possibly a reflection of the size of λ_M. The poor tolerance proportionality also occurred for the last three problems in G1 and G4 and for the last problem in G2. If E_v is used in place of E_r for the comparisons, the efficiency of `DIVA` relative to `STEP` increases on some problems, but the increase is small and the conclusions for E_r hold in the main.

There were some differences in performance between `ode113` and the other two Adams

[1]One derivative evaluation of (1) expressed as a first order system requires little more effort than one derivative evaluation of (1).

Stable	5.0000000000000000000000E-01	1.3077797466816870736170E+00
	-5.2076703604238487405786E-01	0
	0	1.2425099279759854223274E-01
	5.6799832576692780821836E-01	0
9.93	9.5387500000000000000000E-04	1.1283547199825733366571E+02
	-1.0672983098502719644011E+00	0
	0	1.0223703288918266831851E-01
	0	8.2674871199349638965076E-04
99.3	9.5387500000000000000000E-04	1.1277912006688051541137E+02
	-1.1306987310713130386259E+00	0
	0	2.5013853845120814873904E-01
	0	2.3101533427609815.52716E-01
1010	9.5387500000000000000000E-04	1.1215834062852009365652E+02
	-1.2023992264193873176451E+00	0
	0	3.6945103474729912446937E-01
	0	1.6749906279613435233719E-01
9980	9.5387500000000000000000E-04	1.1186680230293044160169E+02
	-1.2040699043255095898055E+00	0
	0	3.68951127621015.5737530E-01
	0	1.4940366649919138769637E-01
101000	9.5387500000000000000000E-04	1.1117429105575848734830E+02
	-1.2027961230322259158372E+00	0
	0	3.6258145835612408366108E-01
	0	1.2647613326954293834722E-01

Table 3: G3: Problems 13 (top) to 18 (bottom). The entry for each problem has four rows: row 1 is λ_M, μ, T; row 2 is $x(0)$, $x'(0)$; row 3 is $y(0)$, $y'(0)$; row 4 is $z(0)$, $z'(0)$.

integrators on individual problems, but these differences were not large and in general the performance of `ode113` was similar to that for `DIVA` and `STEP`.

Figure 2 contains $\log_{10} - \log_{10}$ graphs of the the number of derivative evaluations against E_v for `DXRK8`, `RKSUITE` and `ODEX`. We observe `ODEX` is less efficient than the two ERK integrators on all six problems at lax tolerances, but the difference in efficiency is less than in Figure 1. At severe tolerances, `ODEX` is often of similar efficiency to the ERK integrators. Hairer, Nørsett and Wanner [8], p. 241, reached a similar conclusion after comparing `ODEX` and the ERK integrator `DOPRI8` on a collection of six problems, and attributed the gain in relative efficiency to the automatic order increase of `ODEX`.

The matlab integrator `ode45` is of similar efficiency to `RKSUITE` and `DXRK8` at lax tolerances and less efficient at severe tolerances. This latter observation is expected given `ode45` is only order five.

The two ERK integrators, with one notable exception, are of similar efficiency. The exception occurs on Problem 13 at severe tolerances - `DXRK8` performs many more derivative evaluations for just a small reduction in E_v. This loss in efficiency is possibly due to round-off error and does not occur for other problems in G1 to G4. Our conclusions for `DXRK8`, `RKSUITE` and `ODEX` hold in general for the other 18 problems in G1 to G4 and for

1.02	4.43133000000000000000000E-01	8.7237941467857077947016E+00
	2.7592387919212607765345E-01	-3.8544588083131450745684E-01
	3.9790646063876171172607E-02	-1.3122457112773673984206E+00
	0	9.9985810788596915320697E-04
14.2	4.43133000000000000000000E-01	8.7297812504029016997629E+00
	2.7490178094665688315093E-01	-3.8486710823596128894445E-01
	4.0071814971345043098483E-02	-1.2928074999498534991355E+00
	0	2.1781017620839337171865E-01
125	3.00000000000000000000000E-01	1.5533259017037799751676E+01
	5.4944344127337319578647E-01	4.6278776590359944380314E-01
	4.15.5144916412447679883E-01	-4.0021100595516750029843E-01
	0	-1.1594979510433699745854E+00
1070	3.00000000000000000000000E-01	1.7109149003743900993750E+01
	3.4948887745007082856087E-01	-1.4676088566486073355799E-01
	1.6765473781628668588153E-01	4.7587318938140909233709E-03
	0	-1.2129381251790282994002E+00
9840	3.00000000000000000000000E-01	1.8568410627277975991358E+01
	4.0417290307606231728916E-01	1.8223713740119355744899E-01
	4.5126674732824550432777E-01	-3.0996327403074238846653E-01
	0	-1.0941838439929243781521E+00
82200	3.00000000000000000000000E-01	1.515.5072653119103391 80E+01
	5.3504657599173190294557E-01	-6.6442428392340831198227E-01
	7.8506089720727347106219E-02	-1.4414342312096458880566E+00
	0	-3.8496074468933003275540E-02

Table 4: G4: Problems 19 (top) to 24 (bottom). The entry for each problem has four rows: row 1 is λ_M, μ, T; row 2 is $x(0)$, $x'(0)$; row 3 is $y(0)$, $y'(0)$; row 4 is $z(0)$, $z'(0)$.

E_r.

As noted in §3, `ATOMFT` transforms the right hand side of (1) before doing an integration. Hence it is difficult to use the number of derivative evaluations when comparing the efficiency of `ATOMFT` with the other integrators. A limited amount of insight into the performance of `ATOMFT` is possible by comparing E_r (or E_v or E_{rv}) as a function of TOL. Figure 3 contains the graphs of E_r against TOL on Problems 13 to 18 for `DXRK8`, `ODEX` and `ATOMFT`. We observe `ATOMFT` is intermediate to `DXRK8` and `ODEX` in performance.

As well as comparing integrators using the number of derivative evaluations, we did extensive comparisons using the CPU time. Figure 4 gives comparisons on three processors: a 500 MHz SGI processor, a 700MHz IBM xSeries 350 processor and a 2.2 GHz Dell PE2650 processor. The three plots in the left column of Figure 4 contain graphs of the CPU time for `RKSUITE`, `DXRK8`, `ODEX` and `ATOMFT` and the three plots in the right column the CPU time for `STEP`, `DIVA`, `ODEX` and `ATOMFT`. All times are relative to that for `DXRK8`. For completeness we have included the CPU time for `DXRK8` in the left column of plots - this appears as a row of $\times$. Within each plot, the left hand set of four graphs is for the twelve problems in G1 and G2 and the right hand set for the twelve problems in G3 and G4. The graphs within a set give the (relative) CPU time for TOL $= 10^{-3i}$, $i = 1, 2, 3, 4$.

1.89E+12	8.694643256000000000000000E-03	1.823756312895309902853330E+02
	-1.425534287000000000000000E+00	0
	0	7.901890508987566371721700E-01
1.66E+11	4.500000000000000000000000E-01	4.866345323644151100551200E+01
	-2.285760117914508420667800E+00	-1.890195983108327145033200E-01
	0	1.831134337145086108400600E+00
1.29E+10	1.8189215.5000000000000000E-02	1.121005136710740000000000E+02
	-1.1424762580566175552553E+00	0
	0	2.9816698472909148895102E-01
	0	2.3811532798095939834912E-01
9.20E+09	3.0000000000000000000000E-01	3.9706646443953677114283E+01
	6.090115.589054353093178E-01	-5.2280137640329554260196E-01
	3.0592526125420560229669E-01	-8.6065636600909318539309E-01
	0	1.7385760000000000000000E-04

Table 5: G5: Problems 25 (top) to 28 (bottom). For each problem row 1 is λ_M, μ, T; row 2 is $x(0)$, $x'(0)$; row 3 is $y(0)$, $y'(0)$. For Problems 27 and 28, row 4 is $z(0)$, $z'(0)$.

If the vertical scaling in Figure 4 is ignored, the general appearance of the graphs varies little among the three processors. This was unexpected - we had thought the differences in hardware and Fortran compilers would have produced greater variation in the graphs. We also observe from Figure 4 that `ATOMFT` uses seven to fifteen times the CPU time of `DXRK8` for TOL $= 10^{-3}$ but no more than than four times as much for TOL $= 10^{-12}$, a reduction probably due to the variable-order scheme used by `ATOMFT`.

Figures 5 and 6 contain $\log_{10} - \log_{10}$ graphs of the CPU time against E_{rv} for Problems 13 to 18. Figure 5 has the graphs for `RKSUITE`, `DXRK8` and `ODEX` and Figure 6 for `DIVA`, `STEP` and `ODEX`. From Figure 5 we observe `DXRK8` requires less CPU time than `RKSUITE` to achieve the same global error. We also observe the CPU time required by `ODEX` is usually between that of `DXRK8` and `RKSUITE` and becomes less relative to that for `DXRK8` as TOL is reduced. Similar conclusions hold for `STEP`, `DIVA` and `ODEX` (Figure 6) except `ODEX` requires less CPU time relative to the other two integrators. The observations from Figures 5 and 6 suggest `DXRK8` requires the least CPU time of the five integrators for the tolerances considered.

This completes our comparisons on G1 to G4 for TOL $= 10^{-i}, i = 3, \ldots, 12$. We now consider TOL $= 10^{-2}$ and TOL $= 10^{-i}$, $i = 12, 13, 14, 15$. We have included the results for TOL $= 10^{-12}$ to provide continuity with the previous results.

When TOL was 10^{-2}, all integrators completed their integrations. `DXRK8` flagged possible stiffness on Problems 1, 2, 4, 5 and 14. Since we were using the default values for integrator options, we continued with the integration on these problems.

As a general statement, the performance of the integrators for TOL $= 10^{-i}, i = 12, \ldots, 15$, was affected noticeably by round-off error. This caused poor tolerance proportionality, some failed integrations and some recommendations to increase TOL.

`RKSUITE` completed all 24 integrations with TOL $= 10^{-12}$, 22 with TOL $= 10^{-13}$ and

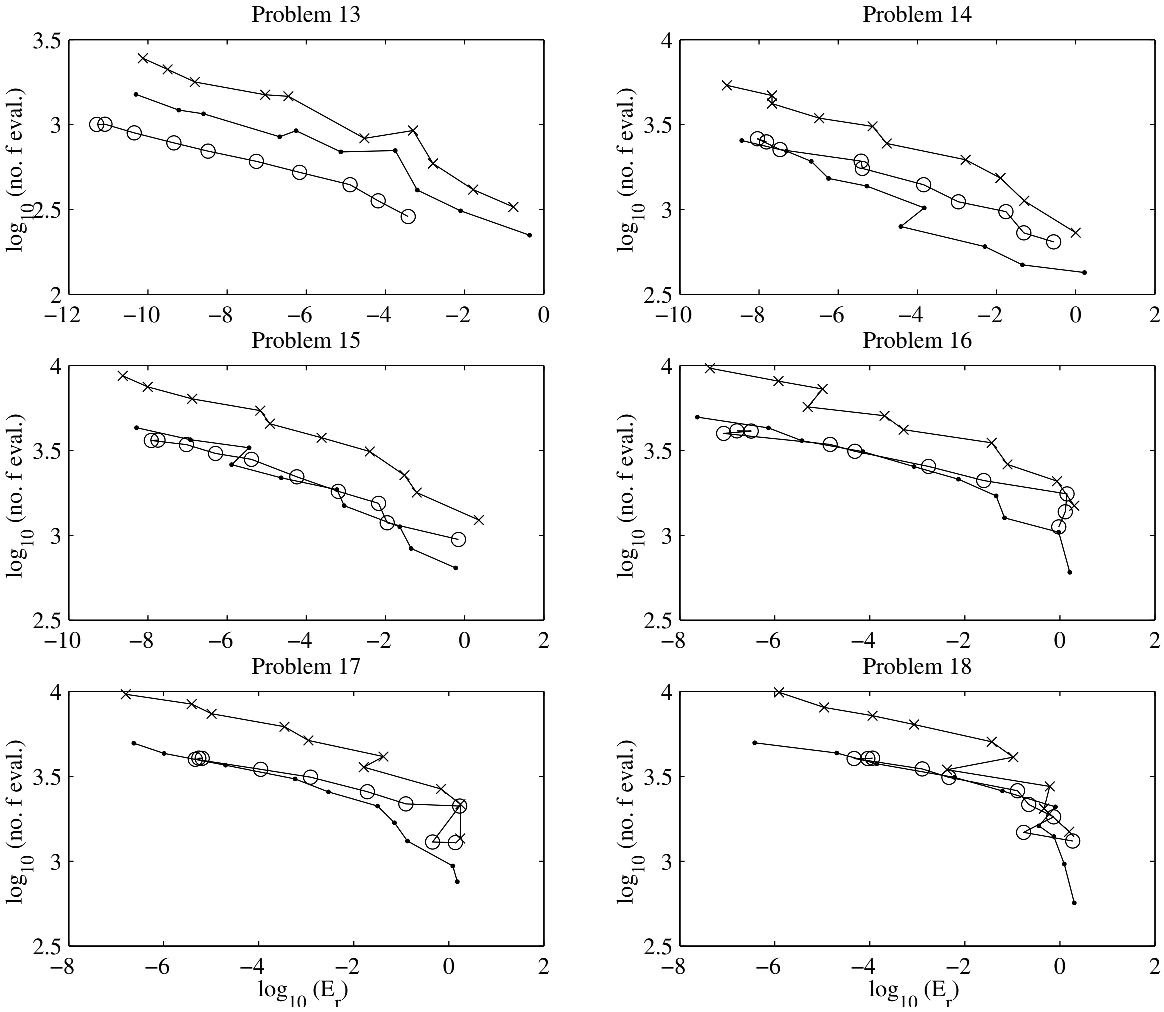

Figure 1: $\log_{10} - \log_{10}$ graphs of the number of derivative evaluations against E_r for Problems 13 to 18: `DIVA` (·–·), `STEP` (∘–∘), `ODEX` (×–×).

15 with TOL $= 10^{-14}$, with all failed integrations occurring because the stepsize became too small. `RKSUITE` failed to start all integrations for TOL $= 10^{-15}$ because TOL was smaller than the minimum allowed by the integrator.

When integrating Problems 7 to 13 with TOL $= 10^{-15}$, `DXRK8` recommended an increase in TOL partway through the integrations. All other integrations with TOL $= 10^{-i}$, $i = 12, \ldots, 15$, were completed without comment from `DXRK8`.

`DIVA` made more recommendations than `DXRK8` to increase the tolerance, doing so on 11, 12 and 16 problems for TOL $= 10^{-i}$, $i = 12, 13, 14$, respectively. The recommendation was made on all 24 problems when TOL $= 10^{-15}$.

`STEP`, `ODEX`, `ATOMFT` and `ode113` completed all integrations for tolerances of 10^{-12}, 10^{-13}, 10^{-14} and 10^{-15} without setting error flags. We did not test `ode45` for these tolerances because it is unlikely an order five method would be used at such severe tolerances.

The final set of comparisons was on G5. Because λ_n is very large for the problems in

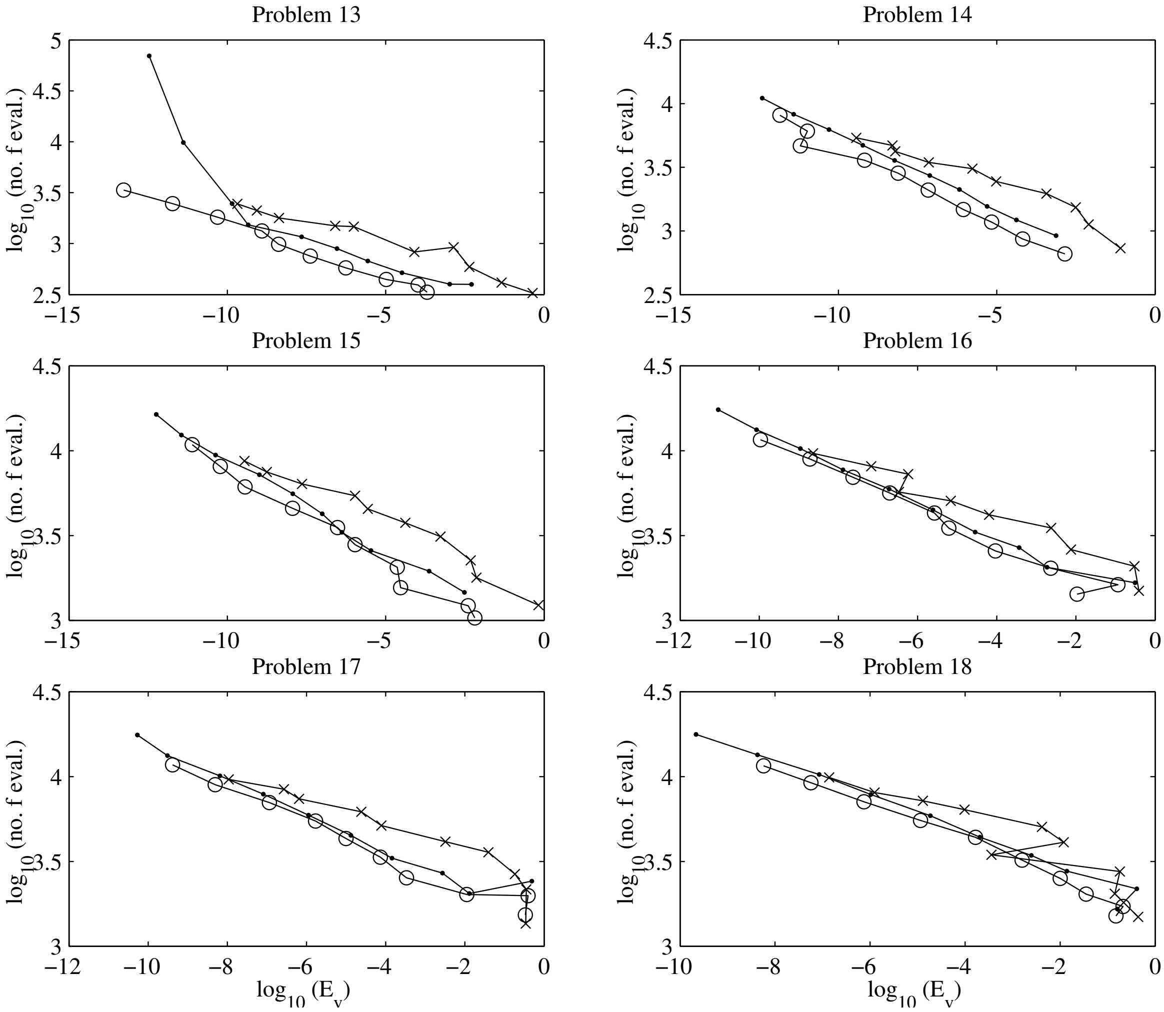

Figure 2: $\log_{10} - \log_{10}$ graphs of the number of derivative evaluations against E_v for Problems 13 to 18: `DXRK8` (·–·), `RKSUITE` (∘–∘), `ODEX` (×–×).

G5, we used $\mathrm{TOL} = 10^{-i}$, $i = 10, \ldots, 15$.

For each of the six tolerances on Problems 25 and 26, the numerical solution calculated by `RKSUITE`, `DXRK8`, `STEP`, `DIVA` and `ODEX` was unstable in the sense it spiralled outwards with t from the massive bodies. We re-attempted the integrations in quadruple precision using `RKSUITE`[2] and the same tolerances and obtained bounded solutions, suggesting the unstable solutions were caused by a lack of arithmetic precision. `ATOMFT` was more successful on Problems 25 and 26 as seen by the values of E_{rv} in Table 6.

`RKSUITE`, `DXRK8`, `STEP`, `DIVA` and `ODEX` fared better on Problems 27 and 28 than on Problems 25 and 26, possibly because λ_M is one to two orders of magnitude smaller. Tables 7 and 8 give E_{rv} and the number of derivative evaluations. The number of evaluations for `ATOMFT` is omitted because the derivative function differs markedly from that for the

[2] `RKSUITE` was chosen because it was the easiest integrator to convert to quadruple precision.

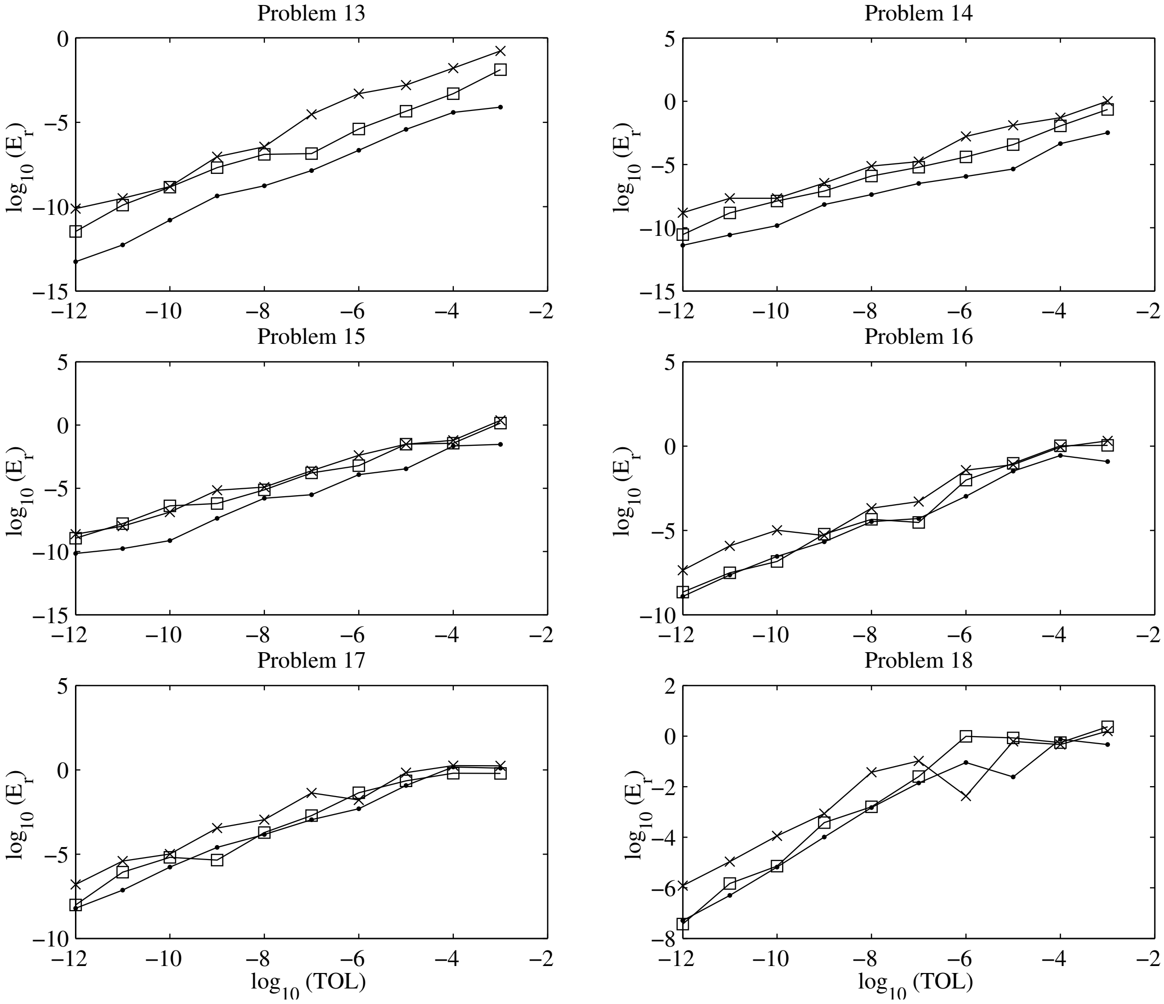

Figure 3: $\log_{10} - \log_{10}$ graphs of E_r against TOL for Problems 13 to 18: `DXRK8` ($\cdot$–$\cdot$), `ODEX` ($\times$–$\times$), `ATOMFT` ($\square$–$\square$).

TOL	10^{-10}	10^{-11}	10^{-12}	10^{-13}	10^{-14}	10^{-15}
Problem 25	1.62	.371	.472	.867	.127	.0173
Problem 26	2.01	9.62	1.93	.165	.00900	.00281

Table 6: E_{rv} for `ATOMFT` applied to Problems 25 and 26 for $\mathrm{TOL} = 10^{-i}$, $i = 10, \ldots, 15$.

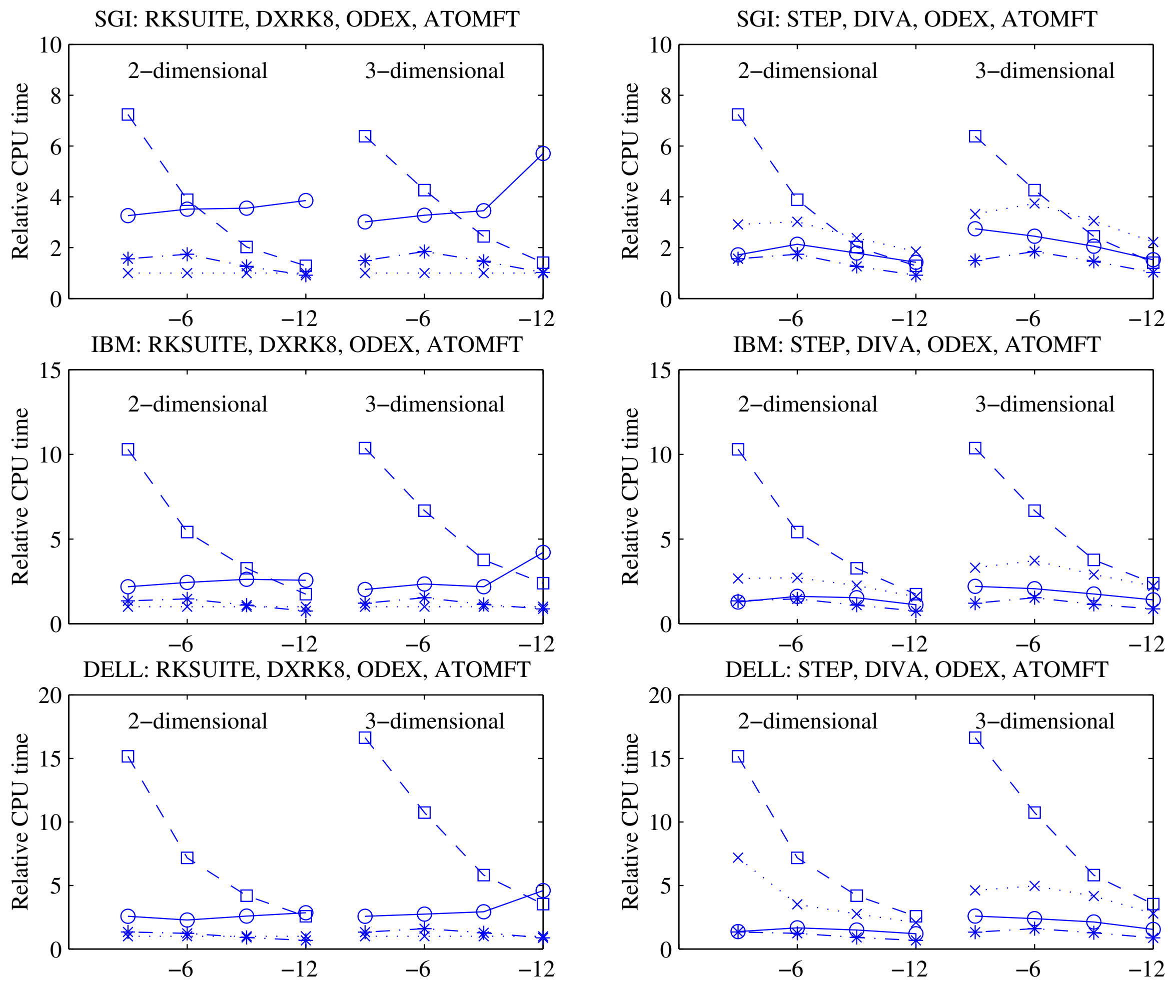

Figure 4: The relative CPU time on a 500 MHz SGI processor (top), a 700 MHz IBM xSeries 350 processor (middle) and a 2.2 GHz Dell PE2650 processor (bottom). `RKSUITE` (○–○), `DXRK8` (×–×), `ODEX` (⋆–⋆), `ATOMFT` (□–□), `STEP` (○–○), `DIVA` (×–×).

other integrators. `RKSUITE` failed on six of the 12 integrations, the stepsize becoming too small in each case, but it is interesting that on the six integrations `RKSUITE` completed, the global error was smaller than for the other integrators for the same tolerance. With the exception of the integrations for TOL $= 10^{-15}$, the integrators showed better tolerance proportionality than on G1 to G4, possibly because the truncation error was dominating the round-off error, As we could expect from the results for G1 to G4, the Adams integrators used fewer derivative evaluations than the ERK integrators.

5 Summary

We presented a diverse collection of 28 symmetric and asymmetric, stable and unstable periodic orbits for the two- and three-dimensional restricted three-body problem and

TOL	RKSUITE	DXRK8	STEP	DIVA	ODEX	ATOMFT
10^{-10}	3.86e-4, 9948	2.04e-1, 6536	1.12, 3197	1.19, 3998	3.66e-1, 6689	.339
10^{-11}	2.03e-5, 13085	1.59e-2, 8456	3.79e-1, 3841	3.10e-2, 4515	1.14e-2, 7609	7.62e-2
10^{-12}	3.11e-5, 17351	1.17e-3, 11279	1.31e-2, 4619	1.29e-2, 5161	1.59e-2, 8833	5.72e-3
10^{-13}	3.59e-6, 25211	1.23e-4, 14807	2.36e-4, 5533	1.31e-3, 5926	1.91e-3, 10055	5.80e-4
10^{-14}	h too small	2.60e-5, 19680	6.82e-5, 7062	2.53e-5, 6507	2.08e-3, 11724	4.29e-5
10^{-15}	h too small	1.45e-5, 26316	1.58e-5, 9645	7.29, 6611	3.56e-3, 13783	9.38e-6

Table 7: E_{rv} and the number of derivative evaluations for RKSUITE, DXRK8, STEP, DIVA, ODEX and ATOMFT applied to Problem 27 for TOL $= 10^{-i}$, $i = 10, \ldots, 15$.

TOL	RKSUITE	DXRK8	STEP	DIVA	ODEX	ATOMFT
10^{-10}	3.58e-2, 3739	1.08e-1, 2357	3.98, 1424	9.74e-2, 1448	2.36, 2683	1.44
10^{-11}	5.62e-3, 5602	2.76e-2, 2998	2.51e-1, 1591	5.66e-2, 1714	9.22e-1, 3719	.111
10^{-12}	h too small	3.00e-3, 4031	1.27e-1, 1929	3.30e-3, 1979	8.26e-2, 3511	3.67e-3
10^{-13}	h too small	3.11e-4, 5374	1.89e-2, 2264	1.14e-2, 1984	1.38e-3, 4104	7.84e-4
10^{-14}	h too small	2.01e-5, 7127	3.59e-3, 2915	8.34e-6, 2294	2.14e-3, 4577	1.56e-4
10^{-15}	h too small	1.18e-4, 9624	3.54e-4, 3864	9.20e-5, 2171	3.46e-3, 5370	3.32e-4

Table 8: E_{rv} and the number of derivative evaluations for RKSUITE, DXRK8, STEP, DIVA, ODEX and ATOMFT applied to Problem 28 for TOL $= 10^{-i}$, $i = 10, \ldots, 15$.

compared the performance of the general purpose integrators RKSUITE, DXRK8, DIVA, STEP, ODEX, ATOMFT, ode113 and ode45 on the orbits. The comparisons were of the efficiency, measured using the number of derivative evaluations and the CPU time (on three different processors), and of qualitative factors such as the number of completed integrations. The collection was constructed so there was a large variation in μ, the period T and the magnitude λ_M of the largest eigenvalue of the monodromy matrix among the orbits. Most orbits are new and were found after extensive searching using continuation on μ, T and the initial conditions.

Our work has several possible implications for the selection of an integrator for the restricted three-body problem. If λ_M is not very large, say less than 10^5, and round-off error is insignificant, the ERK integrator DXRK8 usually requires the least CPU time of the integrators to achieve a prescribed global error, followed by the Adams integrator STEP and the extrapolation integrator ODEX. DXRK8 may flag stiffness at lax tolerances but the integration can be continued efficiently using the integrator.

If λ_M is very large, say $O(10^{12})$, and the calculations must be done in double precision, the Taylor series integrator ATOMFT can be used to advantage, although the global error will not be small. A more reliable way to integrate such problems is to do the calculations in higher precision. This increased reliability will usually come at the cost of an increase in CPU time.

STEP uses compensated summation at very severe tolerances and potentially has an advantage over DIVA, RKSUITE, DXRK8, ODEX and ATOMFT when the round-off error is significant. However, our testing showed the gains are modest and hence it is difficult to recommend one integrator as having a clear advantage over the remaining ones when round-off error is significant.

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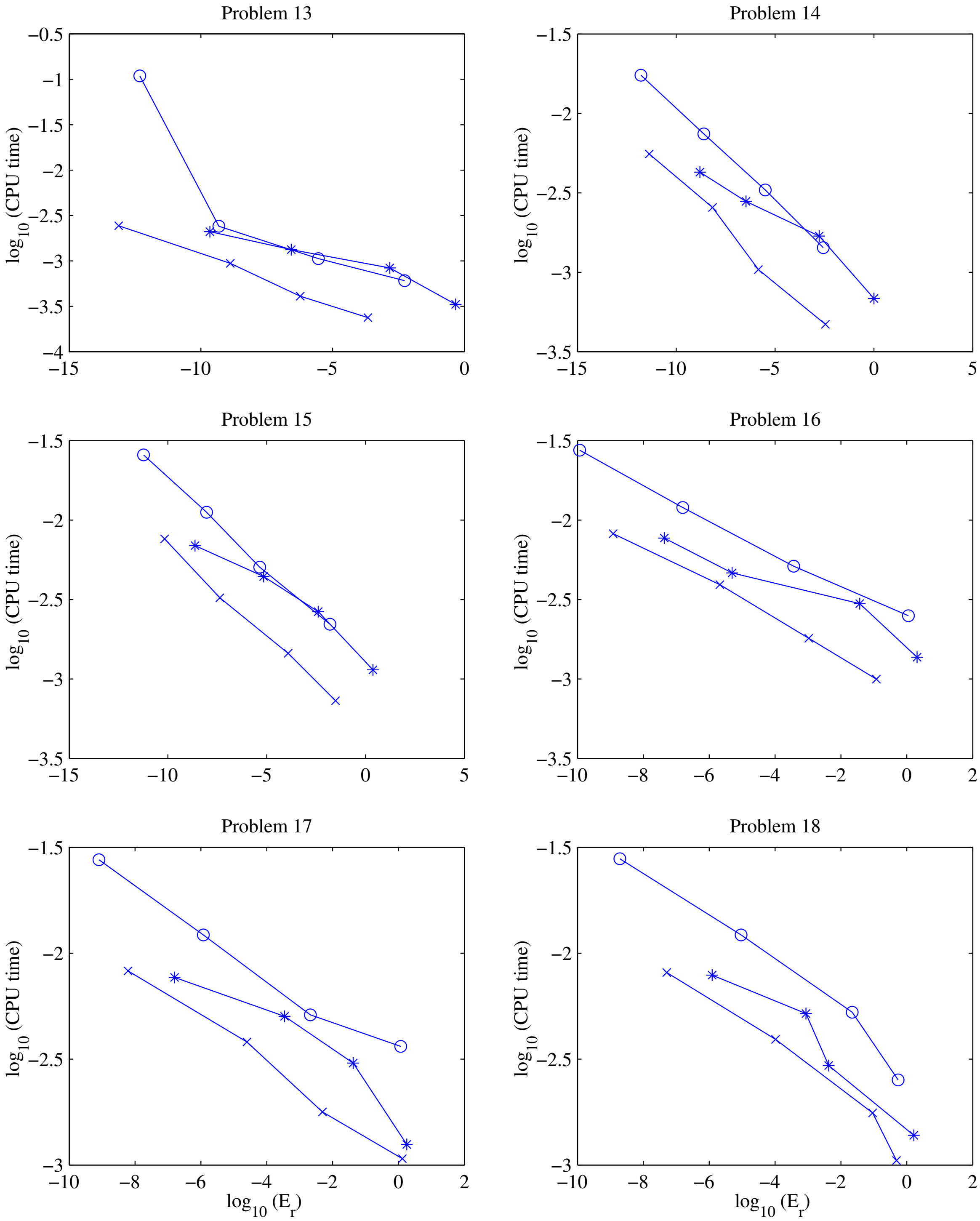

Figure 5: $\log_{10} - \log_{10}$ graphs of the CPU time in seconds against E_{rv} for Problems 13 to 18: `RKSUITE` ($\circ$–$\circ$), `DXRK8` ($\times$–$\times$), `ODEX` ($\star$–$\star$).

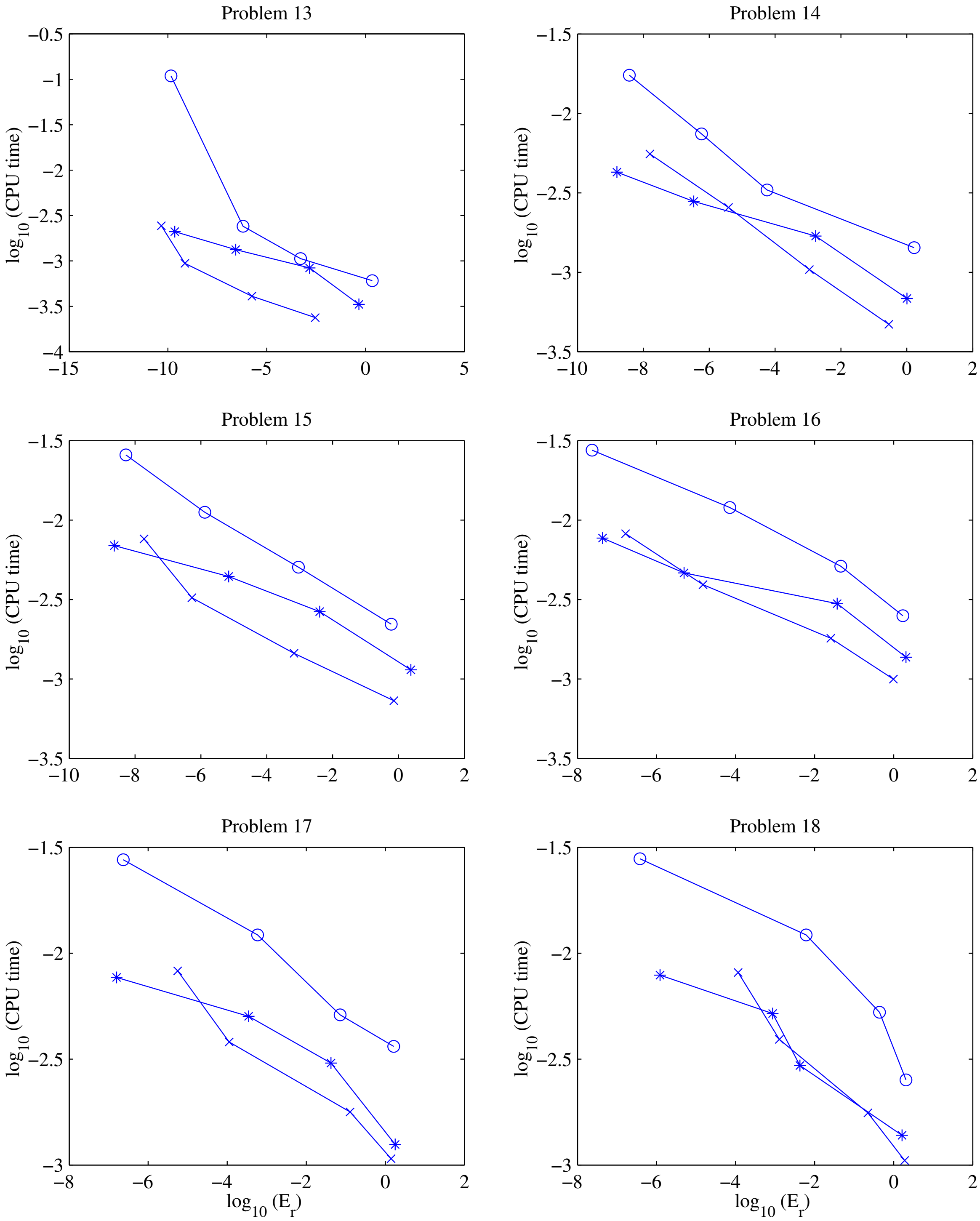

Figure 6: $\log_{10} - \log_{10}$ graphs of the CPU time in seconds against E_{rv} for Problems 13 to 18: `DIVA` ($\circ$–$\circ$), `STEP` ($\times$–$\times$), `ODEX` ($\star$–$\star$).